Effective irrationality measures for quotients of logarithms of rational numbers

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Abstract. *We establish uniform irrationality measures for the quotients of the logarithms of two rational numbers which are very close to* 1*. Our proof is based on a refinement in the theory of linear forms in logarithms which goes back to a paper of Shorey.*

1. Introduction and result

Let ξ be an irrational real number. The real number μ is an irrationality measure of ξ if, for every positive ε, there are a positive number $C(\xi, \varepsilon)$ and at most finitely many rational numbers p/q with $q \geq 1$ and

$$\left| \xi - \frac{p}{q} \right| < \frac{C(\xi, \varepsilon)}{q^{\mu+\varepsilon}}.$$

If, moreover, the constant $C(\xi, \varepsilon)$ is effectively computable for every positive ε, then μ is an effective irrationality measure of ξ. We denote by $\mu(\xi)$ (resp. $\mu_{\mathrm{eff}}(\xi)$) the infimum of the irrationality measures (resp. effective irrationality measures) of ξ. It follows from the theory of continued fractions that $\mu(\xi) \geq 2$ for every irrational real number ξ and an easy covering argument shows that there is equality for almost all ξ, with respect to the Lebesgue measure.

The following statement is a straightforward consequence of Baker's theory of linear forms in logarithms (see e.g. [10] and the references therein). By definition, two positive rational numbers are multiplicatively independent if the quotient of their logarithms is irrational.

Theorem 1.1. *Let a_1, a_2, b_1, b_2 be positive integers with $a_1 > a_2$ and $b_1 > b_2$. Assume that a_1/a_2 and b_1/b_2 are multiplicatively independent. There exists an absolute, effectively computable, constant C such that*

$$\mu_{\mathrm{eff}}\Big(\frac{\log(a_1/a_2)}{\log(b_1/b_2)} \Big) \leq C (\log a_1) (\log b_1).$$

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The purpose of this note is to show how a known refinement in the theory of linear forms in logarithms in the special case where the rational numbers are very close to 1, which goes back to Shorey's paper [7], allows one to considerably improve Theorem 1.1 in this special case. Several spectacular applications to Diophantine problems and to Diophantine equations of this idea of Shorey have already been found; see for example [8, 10] and the survey [1]. Quite surprisingly, it seems that it has not yet been noticed that it can be used to give uniform upper bounds for irrationality measures of roots of rational numbers (see [2]) and of quotients of logarithms of rational numbers, under some suitable assumptions.

Our main result is the following.

Theorem 1.2. *Let a_1, a_2, b_1, b_2 be positive integers such that*

$$\max\{16, a_2\} < a_1 < 6a_2/5 \quad \textit{and} \quad \max\{16, b_2\} < b_1 < 6b_2/5. \tag{1.1}$$

Define η by $a_1 - a_2 = a_1^{1-\eta}$ and ν by $b_1 - b_2 = b_1^{1-\nu}$. If a_1/b_1 and a_2/b_2 are multiplicatively independent, then we have

$$\mu_{\text{eff}}\Big(\frac{\log(a_1/a_2)}{\log(b_1/b_2)}\Big) \le 1 + 27638\,\frac{(\log a_1)(\log b_1)}{\min\{\eta \log a_1, \nu \log b_1\}^2}. \tag{1.2}$$

We display an immediate corollary of Theorem 1.2 which deals with the case $\eta > 1/2$ and $\nu > 1/2$. It illustrates the strength of the theory of linear forms in logarithms.

Corollary 1.3. *Let a_1, a_2, b_1, b_2 be positive integers such that*

$$36 \le a_2 < a_1 < a_2 + \sqrt{a_1}, \quad 36 \le b_2 < b_1 < b_2 + \sqrt{b_1}, \quad \textit{and} \quad \sqrt{b_1} < a_1 < b_1^2.$$

If a_1/b_1 and a_2/b_2 are multiplicatively independent, then we have

$$\mu_{\text{eff}}\Big(\frac{\log(a_1/a_2)}{\log(b_1/b_2)}\Big) \le 221105. \tag{1.3}$$

It is apparent from the proof of Theorem 1.2 that the numerical constants in (1.2) and (1.3) can be reduced (roughly, divided by 3) if a_1 and b_1 are sufficiently large. No particular significance has to be attached to the numerical constant $6/5$ in (1.1).

Let a, b and d be positive integers with $a \ne b$ and $\max\{a, b\} < d$. Under certain conditions, Rhin [5] (see also [6]) obtained explicit upper bounds for

$$\mu_{\text{eff}}\Big(\frac{\log(1 + a/d)}{\log(1 + b/d)}\Big).$$

His approach, which gives better numerical results than ours, heavily uses the fact that the two rational numbers a/d and b/d have the same denominator. It seems to us that Theorem 1.2, which applies without any specific restriction on the denominators a_2 and b_2 of the rational numbers, is new and cannot be straightforwardly derived from the methods of [5, 6].

2. Proof of Theorem 1.2

We reproduce with some simplification Corollary 2.4 of Gouillon [3] in the special case where the algebraic numbers involved are rational numbers. We replace his assumption $E \geq 2$ by $E \geq 15$, to avoid trouble with the quantity $\log\log\log E$ occurring in the definition of E^* in Corollary 2.4 of [3], which is not defined if E is too small.

Theorem G. *Let a_1, a_2, b_1, b_2 be positive integers such that a_1/a_2 and b_1/b_2 are multiplicatively independent and greater than* 1. *Let A and B be real numbers such that*

$$A \geq \max\{a_1, \mathrm{e}\}, \quad B \geq \max\{b_1, \mathrm{e}\}.$$

Let x and y be positive integers and set

$$X' = \frac{x}{\log A} + \frac{y}{\log B}.$$

Set

$$E = 1 + \min\left\{\frac{\log A}{\log(a_1/a_2)}, \frac{\log B}{\log(b_1/b_2)}\right\}$$

and

$$\log X = \max\{\log X' + \log E, 265 \log E, 600 + 150 \log E\}.$$

Assume furthermore that $15 \leq E \leq \min\{A^{3/2}, B^{3/2}\}$. *Then,*

$$\log|y \log(a_1/a_2) - x \log(b_1/b_2)| \geq -8550 (\log A)(\log B)(\log X)(4 + \log E)(\log E)^{-3}.$$

It is crucial for our proof that the dependence on X comes through the factor $(\log X)$ and not through $(\log X)^2$, as in [4]. Instead of Theorem G we could use an earlier result of Waldschmidt [9] (see also Theorem 9.1 of [10]), but the numerical constants in (1.2) and (1.3) would then be slightly larger.

Proof of Theorem 1.2. Our aim is to estimate from below the quantity

$$\left|\frac{\log(a_1/a_2)}{\log(b_1/b_2)} - \frac{x}{y}\right|,$$

for large positive integers x, y. We will establish a lower bound of the form

$$\log|y \log(a_1/a_2) - x \log(b_1/b_2)| \geq -C_1 (C_2 + \log \max\{x, y\}), \tag{2.1}$$

for some quantities C_1, C_2 which depend at most on a_1, a_2, b_1 and b_2. This will show that $1 + C_1$ is an irrationality measure for $\log(a_1/a_2)/\log(b_1/b_2)$.

We apply Theorem G and set $A = a_1$ and $B = b_1$.

Observe that

$$\log(a_1/a_2) = \log(1 + ((a_1 - a_2)/a_2)) \leq (a_1 - a_2)/a_2 = a_1^{1-\eta} a_2^{-1},$$

thus

$$\frac{\log a_1}{\log(a_1/a_2)} \geq (\log a_1)\frac{a_2}{a_1}a_1^\eta \geq \max\{14, a_1^\eta\},$$

since $a_1 \geq 17$ and $a_1^\eta = a_1/(a_1 - a_2) \geq 6$. Likewise, we check that

$$\frac{\log b_1}{\log(b_1/b_2)} \geq \max\{14, b_1^\nu\}.$$

Furthermore, since $a_2 \geq 14$ and $a_1 \geq 17$, we get

$$1 + \frac{\log a_1}{\log(a_1/a_2)} \leq 1 + \frac{\log a_1}{\log(1 + 1/a_2)} \leq 1 + 1.1a_2 \log a_1 \leq a_1^{3/2},$$

and a similar upper bound holds for $1 + (\log b_1)/(\log(b_1/b_2))$, thus we have proved that

$$E := 1 + \min\left\{\frac{\log a_1}{\log(a_1/a_2)}, \frac{\log b_1}{\log(b_1/b_2)}\right\}$$

satisfies

$$15 \leq E \leq \min\{A^{3/2}, B^{3/2}\}.$$

Note also that the quantity

$$E' = \min\{a_1^\eta, b_1^\nu\}$$

satisfies $6 \leq E' \leq E$, thus $(4 + \log E')/(\log E') \leq (4 + \log 6)/(\log 6)$. It then follows from Theorem G that

$$\log|y\log(a_1/a_2) - x\log(b_1/b_2)| \geq -27638(\log A)(\log B)(\log\max\{x, y\} + \log E)(\log E')^{-2},$$

when $\max\{x, y\}$ is sufficiently large. This lower bound is of the form (2.1), with explicit values for C_1 and C_2, thus we have proved that

$$\mu_{\mathrm{eff}}\left(\frac{\log(a_1/a_2)}{\log(b_1/b_2)}\right) \leq 1 + 27638\,\frac{(\log a_1)(\log b_1)}{\min\{\eta\log a_1, \nu\log b_1\}^2}.$$

This completes the proof of Theorem 1.2. $\square$

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