On integer polynomials with multiple roots

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Abstract — We establish a new lower bound for the distance between two roots of an integer polynomial, and a new upper bound for the distance between a given real number and the set of zeros of an integer polynomial. We apply the latter result to improve a metrical result in Diophantine approximation.

1. Introduction

Sharp estimates for the distance between two zeros of an integer polynomial or between a given complex number and the set of zeros of an integer polynomial have many applications in transcendence theory (see e.g. [9]). These questions were first systematically studied by Güting [7]. His results are essentially best possible in terms of the height H of the polynomial, but are somehow quite weak regarding the dependence on the degree n of the polynomial, when the latter has multiple roots. Improvements in terms of n on Güting's estimates have been obtained by Chudnovsky ([5], Lemma 1.12), Diaz & Mignotte [6], Amou [1, Lemma 2], and Amou & Bugeaud [3], see Appendix A from [4] for a survey. However, in all these works the dependence on H is not as good as in [7]. The purpose of the present note is to establish further refinements. As an application of our second result, we improve a metrical statement from [3] related to a conjecture of Sprindžuk in Diophantine approximation.

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2. On the distance between two roots of an integer polynomial

We begin by quoting the lower bound for the distance between two distinct roots of an integer polynomial, established by Amou & Bugeaud [3]: this is Theorem A.2 from [4], which is essentially Theorem 1 from [3].

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Theorem AB. *Let $P(X)$ be an integer polynomial of degree $n \geq 2$. Denote by α a zero of $P(X)$ of order s and by β a zero of $P(X)$ of order t. Assuming that $\alpha \neq \beta$, we have*

$$|\alpha - \beta| \geq 2^{-n/t} \, n^{-n/t-3n/(2st)} \, \mathrm{H}(P)^{-2n/(st)} \, \max\{1, |\alpha|\} \, \max\{1, |\beta|\} \tag{1}$$

if $t > s$, while we have

$$|\alpha - \beta| \geq 2^{-n/s} \, n^{-n(2s+3)/(4s^2)} \, \mathrm{H}(P)^{-n/s^2+1/(2s)} \, \max\{1, |\alpha|\}^{3/2} \, \max\{1, |\beta|\}^{3/2}$$

if $s = t$.

Since a factor $2^{-n^2/(st)} \, \mathrm{H}(P)^{-(n-\min\{s,t\})/(st)}$ occurs in Theorem 7' from [7], we see that Theorem AB considerably strengthens Güting's estimate in terms of n. However, it is unfortunately weaker in terms on $\mathrm{H}(P)$.

In the present Section, we provide a similar improvement on Güting's result in terms of n, but without loosing too much in terms of $\mathrm{H}(P)$.

Theorem 1. *Let $P(X)$ be an integer polynomial of degree $n \geq 2$. Denote by α a zero of $P(X)$ of order s and by β a zero of $P(X)$ of order t. Assuming that $\alpha \neq \beta$ and $t \geq s$, we have*

$$|\alpha - \beta| \geq 2^{-n/(2t)} \, (n+1)^{-n/(2s)-3n/(4st)} \, \mathrm{H}(P)^{-n/(st)} \, \max\{1, |\alpha|\} \, \max\{1, |\beta|\}. \tag{2}$$

Theorem 1 improves upon Theorem AB in most cases, but not in terms of all parameters. Indeed, when t is much larger than s, the exponent of $n+1$ is much smaller in (2) than the exponent of n in (1). However, since we are mainly interested in the dependence on the degree and on the height of the polynomial $P(X)$, we can honestly consider that Theorem 1 strengthens Theorem AB.

The proof of Theorem 1 is very close to that of Theorem 1 from [3]: the additional idea is a 'symmetrization' of the argument.

Remark. The factor $\mathrm{H}(P)^{-n/(st)}$ occurring in (2) cannot be replaced by a factor $\mathrm{H}(P)^{-cn/(st)}$ with $c < 1$, as is shown by the following example, given in [6] and already reproduced in [3]. Let $a \geq 2$ and $d \geq 2$ be integers and set $Q(X) := X^d + aX - 1$. Since $Q(1/a)$ and $Q(1/a - 1/a^{d+1})$ have opposite signs, $Q(X)$ has a root β satisfying $|1/a - \beta| < a^{-d-1}$. Let s and t be positive integers and set $P_{s,t}(X) = Q^s(X) \cdot (aX-1)^t$. The polynomial $P_{s,t}(X)$ has degree $n := ds + t$ and its height satisfies $\mathrm{H}(P) \leq 2a^{s+t}$ when a is large enough. Consequently, the roots $1/a$ and β of $P_{s,t}(X)$ satisfy

$$|1/a - \beta| < \big(\mathrm{H}(P)/2\big)^{-(d+1)/(s+t)} = \big(\mathrm{H}(P)/2\big)^{-(n-t+s)/(s^2+st)}.$$

Taking a, d, s, t large with $s \ll t \ll d$, this shows that (2) is essentially best possible as far as the dependence on $\mathrm{H}(P)$ is concerned.

Proof. In view of Theorem AB, we may assume that $s \neq t$. Let $Q(X) = a(X - \alpha_1)\dots(X - \alpha_d)$ and $R(X) = b(X - \alpha_{d+1})\dots(X - \alpha_{d+d'})$ be the minimal defining polynomials of $\alpha =: \alpha_1$ and $\beta := \alpha_{d+1}$, respectively. Observe that $Q(X)^s\, R(X)^t$ divides $P(X)$ in $\mathbf{Z}[X]$. Since the resultant of $Q(X)$ and $P^{(s)}(X)/s!$ is a non-zero integer, we get

$$1 \leq |a|^{n-s} \prod_{1 \leq i \leq d} \frac{|P^{(s)}(\alpha_i)|}{s!}.$$

Similarly, we have

$$1 \leq |b|^{n-t} \prod_{d+1 \leq j \leq d+d'} \frac{|P^{(t)}(\alpha_j)|}{t!}.$$

Consequently, we get

$$1 \leq \Lambda := |a|^{s(n-s)}\, |b|^{t(n-t)} \prod_{1 \leq i \leq d} \left(\frac{|P^{(s)}(\alpha_i)|}{s!} \right)^s \prod_{d+1 \leq j \leq d+d'} \left(\frac{|P^{(t)}(\alpha_j)|}{t!} \right)^t. \tag{3}$$

For any integers $i = 2, \dots, d$ and $j = d+2, \dots, d+d'$ we obtain

$$\frac{|P^{(s)}(\alpha_i)|}{s!} \leq \mathrm{H}(P) \sum_{k=s}^{n} \binom{k}{s} \max\{1, |\alpha_i|\}^{n-k} \leq \binom{n+1}{s+1} \mathrm{H}(P) \max\{1, |\alpha_i|\}^{n-s} \tag{4}$$

and

$$\frac{|P^{(t)}(\alpha_j)|}{t!} \leq \binom{n+1}{t+1} \mathrm{H}(P) \max\{1, |\alpha_j|\}^{n-t}. \tag{5}$$

Denoting by a_n the leading coefficient of $P(X)$, we have

$$\begin{aligned} \frac{|P^{(s)}(\alpha)|}{s!} &= |a_n| \prod_{\substack{\gamma \neq \alpha \\ P(\gamma)=0}} |\alpha - \gamma| \\ &\leq 2^{n-s-t}\, |a_n| \cdot |\alpha - \beta|^t \cdot \max\{1, |\alpha|\}^{n-s-t} \cdot \prod_{\substack{\gamma \neq \alpha, \gamma \neq \beta \\ P(\gamma)=0}} \max\{1, |\gamma|\} \end{aligned} \tag{6}$$

and

$$\frac{|P^{(t)}(\beta)|}{t!} \leq 2^{n-t-s}\, |a_n| \cdot |\alpha - \beta|^s \cdot \max\{1, |\beta|\}^{n-t-s} \cdot \prod_{\substack{\gamma \neq \beta, \gamma \neq \alpha \\ P(\gamma)=0}} \max\{1, |\gamma|\}, \tag{7}$$

where the roots of $P(X)$ are counted with their multiplicities in the above products.

The combination of estimates (3) to (7) yields

$$\begin{aligned} |\alpha - \beta|^{2st} \geq\ & 2^{-(n-s-t)(s+t)}\, M(\alpha)^{-s(n-s)}\, M(\beta)^{-t(n-t)}\, M(P)^{-s-t}\, \mathrm{H}(P)^{-s(d-1)-t(d'-1)} \\ & \times \binom{n+1}{s+1}^{-s(d-1)} \binom{n+1}{t+1}^{-t(d'-1)} \max\{1, |\alpha|\}^{s(s+2t)} \max\{1, |\beta|\}^{t(t+2s)}. \end{aligned}$$

Observe that

$$\binom{n+1}{s+1} \le (n+1)\,\frac{n^s}{2^s}. \tag{8}$$

It follows from (8) and $ds^2 + d't^2 \le nt$ that

$$\binom{n+1}{s+1}^{-s(d-1)} \binom{n+1}{t+1}^{-t(d'-1)} \ge (n+1)^{-s(d-1)-t(d'-1)} \left(\frac{n}{2}\right)^{-nt}.$$

Since $M(\alpha)^s\, M(\beta)^t \le M(P)$ and

$$M(P) \le \sqrt{n+1}\,\mathrm{H}(P), \tag{9}$$

we then get

$$\begin{aligned} |\alpha - \beta|^{2st} \ge\ & 2^{-ns}\,(n+1)^{-sd-td'-(n-s-t)/2}\,n^{-nt} \\ & \times\ \mathrm{H}(P)^{-2n}\,\max\{1, |\alpha|\}^{s(s+2t)}\,\max\{1, |\beta|\}^{t(t+2s)}. \end{aligned}$$

Using now that $t > s$, we obtain after some simplification

$$|\alpha - \beta| \ge 2^{-n/(2t)}\,(n+1)^{-n/(2s)-3n/(4st)}\,\mathrm{H}(P)^{-n/(st)}\,\max\{1, |\alpha|\}\,\max\{1, |\beta|\},$$

as announced. □

3. On the distance between a given real number and the set of zeros of an integer polynomial

To begin this section, we consider the special case when the integer polynomial has only simple roots. We state inequality (A.24) from Lemma A.8 of [4], which is essentially due to Feldman.

Theorem F. *Let $P(X)$ be an integer polynomial of degree $n \ge 2$ with no multiple roots. Let ξ be a complex number and α be a root of $P(X)$ such that $|\xi - \alpha|$ is minimal. We then have*

$$|\xi - \alpha| \le \sqrt{2}\,(2n)^{n-3/2}\,\mathrm{H}(P)^{n-2}\,|P(\xi)|. \tag{10}$$

In the general case (that is, when the polynomial may have multiple roots), Chudnovsky ([5], Lemma 1.12) improved upon Theorem 4' of Güting [7], as far as the dependence on the degree of the polynomial is concerned. His result was subsequently slightly refined by Diaz & Mignotte [6], with a simpler proof: they replaced the use of an auxiliary semi-discriminant by that of a resultant. Their main result can be stated as follows.

Theorem DM. *Let $P(X)$ be an integer polynomial of degree $n \geq 2$. Let ξ be a complex number and α be a root of $P(X)$ such that $|\xi - \alpha|$ is minimal. Denote by s the multiplicity of α as a root of $P(X)$. We then have*

$$|\xi - \alpha|^s \leq n^{n+3n/(2s)}\, 2^{n-s}\, \mathrm{H}(P)^{2(n-s)/s}\, |P(\xi)|. \tag{11}$$

Probably, the exponent of $\mathrm{H}(P)$ in (11) is not best possible: it is likely that the factor 2 could be replaced by 1. We tried to get such an improvement using similar ideas as in the proof of Theorem 1, but unsuccesfully. The main problem comes from the fact that, unlike in Theorem 1, we have to deal with *all* the roots of $P(X)$. However, we are able to present a variant of Theorem DM, that appears to have an interesting application in Diophantine approximation, see Theorem 3 in Section 4.

Let ξ be a fixed real or complex number. Let $P(X)$ be an integer polynomial of degree $n \geq 2$ with $|P(\xi)| < 1$. Let α be a root of $P(X)$ such that $|\xi - \alpha|$ is minimal, and denote by s the multiplicity of α as root of $P(X)$. If α is the unique root satisfying $|\xi - \alpha| < 1$, then we have

$$|\xi - \alpha|^s \leq |P(\xi)|. \tag{12}$$

Our variant of Theorem DM involves a second root of $P(X)$ lying near ξ. In what follows we assume that there exists a root $\beta \neq \alpha$ of $P(X)$ satisfying $|\xi - \beta| < 1$. We take such a β satisfying $|\xi - \beta| \leq |\xi - \gamma|$ for any root $\gamma \neq \alpha,\ \beta$ of $P(X)$, and denote by t the multiplicity of β as root of $P(X)$.

Theorem 2. *Under the above assumption, we have*

$$|\xi - \alpha|^{s(t+s)} \leq 2^{nt} n^{ns} (n+1)^{3n/2} H(P)^{2n} |P(\xi)|^{t+s}$$

if $s \geq t$, while we have

$$\min(\{|\xi - \alpha|^s, |\xi - \beta|^t\})^{2t} \leq 2^{ns} n^{nt} (n+1)^{3n/2} H(P)^{2n} |P(\xi)|^{t+s}$$

if $s \leq t$.

When $P(X)$ is the s-power of a polynomial with distinct roots, then Lemma A.7 from [4] is slightly better than Theorem 2. In many cases, including when $s \geq t$ and $s - t$ is small compared to s, Theorem 2 improves upon Theorem DM.

Proof. Keep the same notation as in the proof of Theorem 1. Let us start with (3) and set $\lambda := (|P^{(s)}(\alpha)|/s!)^s (|P^{(t)}(\beta)|/t!)^t$. Then, using (3), (4), (5), (8), (9), and $s^2 d + t^2 d' \leq \max\{s,t\} n$, we get

$$\Lambda/\lambda \leq (n+1)^n (n/2)^{\max\{s,t\}n} (H(P)M(P))^n \leq (n+1)^{3n/2} (n/2)^{\max\{s,t\}n} H(P)^{2n}.$$

Since $|\alpha_i - \alpha_j| \leq 2|\xi - \alpha_j|$ if $i < j$, we have

$$\frac{|P^{(s)}(\alpha)|}{s!} \leq 2^{n-s} |\xi - \alpha|^{-s} |P(\xi)|$$

and

$$\frac{|P^{(t)}(\beta)|}{t!} \leq 2^{n-t}|\xi-\alpha|^{-s}|\xi-\beta|^{s-t}|P(\xi)|.$$

Hence we get

$$\lambda \leq 2^{(n-s)s+(n-t)t}|\xi-\alpha|^{-s(t+s)}|\xi-\beta|^{-t(t-s)}|P(\xi)|^{t+s}.$$

Combining this with the upper bound for Λ/λ above we obtain

$$|\xi-\alpha|^{s(t+s)}|\xi-\beta|^{t(t-s)} \leq 2^{(s+t-\max\{s,t\})n}n^{\max\{s,t\}n}(n+1)^{3n/2}H(P)^{2n}|P(\xi)|^{t+s},$$

which clearly yields the desired inequalities. $\square$

The following is a direct consequence of Theorem 2, which will play an important role in the next section.

Corollary 1. *Under the assumption in Theorem 2, assume further that*

$$|P(\xi)| \leq \exp\{-an\log H - bn\log n\} \qquad (n = \deg P,\ H = H(P))$$

with some positive constants a and b. Then we have

$$|\xi-\alpha|^s \leq \exp\left\{-n\left(\left(a-\frac{2}{t+s}\right)\log H + \left(b-\frac{2s+3}{2(t+s)}\right)\log n - 2\right)\right\}$$

if $s \geq t$, while we have

$$\min\{|\xi-\alpha|^s,\ |\xi-\beta|^t\} \leq \exp\{-n(A(s,t,a)\log H + B(s,t,b)\log n - 2)\} \tag{13}$$

with

$$A(s,t,a) = \frac{(t+s)a}{2t} - \frac{1}{t}, \qquad B(s,t,b) = \frac{(t+s)b}{2t} - \frac{1}{2} - \frac{3}{4t} \tag{14}$$

if $s \leq t$.

4. Application to a question from metric number theory

The purpose of this section is to establish the following result.

Theorem 3. *Let ε be a positive real number. Then, for almost all real numbers ξ, there exists a positive constant $c(\xi,\varepsilon)$, depending only on ξ and on ε, such that*

$$|P(\xi)| > \exp\{-(2+\varepsilon)n\log H - (2.5+\varepsilon)n\log n\} \tag{15}$$

for all integer polynomials $P(X)$ of degree n and height H satisfying $\max\{n,H\} \geq c(\xi,\varepsilon)$.

Theorem 3 improves Théorème 2 of Amou & Bugeaud [3] (which is Theorem 8.3 in [4]), where the latter has the constants $3+\varepsilon$ and $4+\varepsilon$ instead of $2+\varepsilon$ and $2.5+\varepsilon$, respectively, in the inequality given in Theorem 3. This improvement comes mainly from Theorem 2 and a refined argument in counting certain algebraic numbers (see Lemma 1 below). We refer the reader to [3] for a discussion on the earlier literature.

It is likely that Theorem 3 still holds with (15) replaced by

$$|P(\xi)| > \exp\{-(1+\varepsilon)n\log H - C(\varepsilon,n)\},$$

where ε is an arbitrary positive number and $C(\varepsilon,n)$ is a suitable function depending only on ε and n. However, it seems to us that such a result (which implies the Mahler conjecture, solved by Sprindžuk [8]) would be very difficult to prove. Even replacing the constant $2+\varepsilon$ in (15) by some real number smaller than 2 seems to require totally new ideas. We stress that in our problem the height H and the degree n have to vary simultaneously. This is not the case for the Mahler conjecture, where the degree is fixed.

The general strategy for proving Theorem 3 is the same as that in [3]. Let ξ be a real number. With each integer polynomial $P(X)$ of degree n and height H, we associate its root (or one of its roots) α nearest to ξ. We get an upper bound for $|\xi-\alpha|$ in terms of $|P(\xi)|$. Since there are at most $n(2H+1)^{n+1}$ algebraic numbers of degree n and height H, we conclude by using the Borel–Cantelli lemma. If we restrict our attention to the polynomials with only simple roots, then we can apply (10) to easily get essentially the same statement as Theorem 3, with the constants $2+\varepsilon$ and $2.5+\varepsilon$ in (15) replaced with $2+\varepsilon$ and $1+\varepsilon$, respectively. There is no such restriction in Theorem 3, thus (10) cannot be used and a delicate analysis of polynomial with multiple roots is needed.

Proof. Let I be a real interval of length 1, and let $\mathcal{E}$ be the set of real numbers ξ in I for which there exist infinitely many integer polynomials $P(X)$ of degree n and height H satisfying

$$|P(\xi)| \le \exp\{-(2+\varepsilon)n\log H - (2.5+\varepsilon)n\log n\}. \tag{16}$$

We prove the theorem by showing that $\mathcal{E}$ is a null set. To this end, as is explained in [3], we may impose with the help of Sprindžuk's Theorem ([4], Theorem 4.2) an additional condition $n \ge n_0$ in the above, where n_0 can be chosen as large as we want whenever it depends only on ε. In the sequel we shall take a suitable n_0 in each necessary occasion, but without mentioning it. For $P(X)$ satisfying (16) let us denote by s the multiplicity of a root of $P(X)$ nearest to ξ. We consider $\mathcal{E}$ as a union of the two subsets $\mathcal{E}_1$ and $\mathcal{E}_2$, where for the former (*resp.* latter) an additional condition $s=1$ (*resp.* $s>1$) is imposed to infinitely many $P(X)$ satisfying (16). We first prove that $\mathcal{E}_1$ is a null set.

Let n, t, and H be positive integers with $1 \le t \le n$. We denote by $\mathcal{P}(n,H)$ the set of integer polynomials $P(X)$ of the degree n and the height H, and by $\mathcal{A}(n,H,t)$ the set of algebraic numbers γ which is a root of some $P(X) \in \mathcal{P}(n,H)$ with multiplicity t. Let

$$A(t) := A(1,t,2+\varepsilon) = 1 + \frac{\varepsilon}{2} + \frac{\varepsilon}{2t} \tag{17}$$

and

$$B(t) := B(1,t,2.5+\varepsilon) = \frac{3}{4} + \frac{\varepsilon(t+1)}{2t} + \frac{1}{2t} \tag{18}$$

with functions A and B given in (14), and let

$$\rho(n,H,t) := \exp\left\{-\frac{n}{t}(A(t)\log H + B(t)\log n - 2)\right\}. \tag{19}$$

Then, for any t and for any $\alpha \in \mathcal{A}(n,H,1)$, we define $\mathcal{E}_{1,1}(n,H,t,\alpha)$ to be the set of real numbers $\xi \in I$ for which

$$|\xi - \alpha| \le \rho(n,H,t)^t = \exp\{-n(A(t)\log H + B(t)\log n - 2)\}, \tag{20}$$

and define further

$$\mathcal{E}_{1,1}(n,H) := \bigcup_{1\le t\le n}\ \bigcup_{\alpha\in\mathcal{A}(n,H,1)} \mathcal{E}_{1,1}(n,H,t,\alpha). \tag{21}$$

Also, for any $\beta \in \mathcal{A}(n,H,t)$ with $t \ge 2$, we define $\mathcal{E}_{1,2}(n,H,t,\beta)$ to be the set of real numbers $\xi \in I$ for which

$$|\xi - \beta| \le \rho(n,H,t),$$

and define further

$$\mathcal{E}_{1,2}(n,H) := \bigcup_{2\le t\le n}\ \bigcup_{\beta\in\mathcal{A}(n,H,t)} \mathcal{E}_{1,2}(n,H,t,\beta). \tag{22}$$

In view of (12) and (13) with (14), where $s = 1$, each $\xi \in \mathcal{E}_1$ belongs to infinitely many $\mathcal{E}_{1,1}(n,H)$'s or $\mathcal{E}_{1,2}(n,H)$'s. Let $\mathcal{E}_{1,i}$ $(i = 1,2)$ be the set of $\xi \in \mathcal{E}_1$ belonging to infinitely many $\mathcal{E}_{1,i}(n,H)$'s. We show that $\mathcal{E}_{1,1}$ is a null set. Let us denote by $\lambda(\cdot)$ the Lebesgue measure on the real line. Since

$$\mathrm{Card}\mathcal{A}(n,H,1) \le 2^{3n}H^n$$

by Lemma 1 in [3], we combine (17), (18), (20), and (21) to get that

$$\lambda(\mathcal{E}_{1,1}) \le 2\mathrm{Card}\mathcal{A}(n,H,1)\cdot \max_{1\le t\le n}\rho(n,H,t)^t \le (nH)^{-\varepsilon n/2},$$

which gives $\lambda(\mathcal{E}_{1,1}) \le (nH)^{-2}$. Thus the Borel–Cantelli lemma ([4], Lemma 1.3) implies that $\mathcal{E}_{1,1}$ is a null set.

To treat $\mathcal{E}_{1,2}$ we separate $\mathcal{E}_{1,2}(n,H)$ into the three parts $\mathcal{E}_{1,2,i}(n,H)$ $(i = 1,2,3)$ where t varies in the ranges

$$2 \le t \le n(\log n)^{-1/2}; \quad n(\log n)^{-1/2} < t \le n/2; \quad n/2 < t \le n$$

in the first union on the right-hand side of (22), respectively. Then we consider the set $\mathcal{E}_{1,2,i}$ $(i = 1,2,3)$ consisting those $\xi \in \mathcal{E}_{1,2}$ which belongs to infinitely many $\mathcal{E}_{1,2,i}(n,H)$'s. We use the following result in order to estimate $\lambda(\mathcal{E}_{1,2,i}(n,H))$ $(i = 1,2)$ from above, which refines Lemma 2 in [3].

Lemma 1. *Let $\tilde{\mathcal{A}}(n,H,t)$ denote the set of elements $\gamma \in \mathcal{A}(n,H,t)$ whose distance from the interval I is less than $\eta(n,H,t)$ where*

$$\eta(n,H,t) := 2^{-1-n/t}(n+1)^{-3n/(2t^2)}H^{-2n/t^2}. \tag{23}$$

Then we have

$$\mathrm{Card}\tilde{\mathcal{A}}(n,H,t) \le \mathrm{Card}\left(\bigcup_{k=t}^{n} \tilde{\mathcal{A}}(n,H,k)\right) \le \frac{n}{t}\eta(n,H,t)^{-1}.$$

Proof. For any $P(X) \in \mathcal{P}(n,H)$ let us denote by P° the set of roots of $P(X)$ belonging to $\tilde{\mathcal{A}} := \tilde{\mathcal{A}}(n,H,t) \cup \ldots \cup \tilde{\mathcal{A}}(n,H,n)$. Then we can construct inductively finite sequences $\{P_i(X)\}_{1\le i\le \ell}$ and $\{\alpha_i\}_{1\le i\le \ell}$ with $P_i(X) \in \mathcal{P}(n,H)$ and $\alpha_i \in P_i^\circ$ such that

$$\alpha_{i+1} \notin P_1^\circ \cup \ldots \cup P_i^\circ \qquad (1 \le i \le \ell - 1)$$

and that $P_1^\circ \cup \ldots \cup P_\ell^\circ = \tilde{\mathcal{A}}$. Since the former conditions imply that $P_i(\alpha_j) \neq 0$ whenever $i < j$, by Theorem A.1 in [4], we have

$$|\alpha_i - \alpha_j| \ge 4 \cdot \eta(n,H,t)$$

whenever $i \neq j$. Thus we have $[(2 \cdot \eta(n,H,t))^{-1}] + 2$ as an upper bound for ℓ. This with $\mathrm{Card}P_i^\circ \le n/t$ for each i completes the proof of the lemma. $\square$

Since, for any $t \ge 2$,

$$A(t) - \frac{2}{t} > \frac{\varepsilon}{2} \quad \text{and} \quad B(t) - \frac{3}{2t} > \frac{1}{4} + \frac{\varepsilon}{2}$$

by (17) and (18), comparing (19) with (23), we obtain

$$\frac{\rho(n,H,t)}{\eta(n,H,t)} \le n^{-n/(4t)}(nH)^{-\varepsilon n/(3t)}, \tag{24}$$

in particular, $\rho(n,H,t) < \eta(n,H,t)$. Hence, in the second union on the right-hand side of (22), we may replace the range $\mathcal{A}(n,H,t)$ with $\tilde{\mathcal{A}}(n,H,t)$. Then, by Lemma 1 with (24), we have

$$2\sum_{t=2}^{[n(\log n)^{-1/2}]} \frac{n}{t} \cdot \frac{\rho(n,H,t)}{\eta(n,H,t)} \le n^{7/3}(\log n)^{-1/2}(nH)^{-(\varepsilon/3)(\log n)^{1/2}} \le (nH)^{-2}$$

as an upper bound for $\lambda(\mathcal{E}_{1,2,1}(n,H))$. Thus $\mathcal{E}_{1,2,1}$ is a null set by the Borel–Cantelli lemma.

Since, for any $n(\log n)^{-1/2} < t \le n/2$,

$$\rho(n, H, t) \le (nH)^{-(3+\varepsilon)/2}$$

by (19) with (17) and (18), we apply Lemma 1 to get

$$(\log n)^{1/2}\eta(n, H, n(\log n)^{-1/2})^{-1} \max_{n(\log n)^{-1/2} < t \le n/2} \rho(n, H, t) \le (nH)^{-3/2}$$

as an upper bound for $\lambda(\mathcal{E}_{1,2,2}(n, H))$. Thus $\mathcal{E}_{1,2,2}$ is also a null set.

We now deduce from the following result that $\mathcal{E}_{1,2,3}$ is a null set as well.

Lemma 2. *Let $\mathcal{E}_0(\kappa)$ with a positive number κ be the set of real numbers ξ for which there exist infinitely many pairs $(P(X), \gamma)$ of an integer polynomial $P(X)$ of degree greater than $6/\kappa$ and a root γ of $P(X)$ with multiplicity greater than $(\deg P)/2$ such that*

$$|\xi - \gamma| \le (nH)^{-\kappa} \qquad (n = \deg P,\ H = H(P)).$$

Then $\mathcal{E}_0(\kappa)$ is a null set.

Proof. Let ξ be an element of $\mathcal{E}_0(\kappa)$. We first claim that any γ having the property given in the lemma should be a rational number. In fact, denoting by s the multiplicity of γ as root of $P(X)$, we have $\deg \gamma \le n/s < 2$, which implies that $\deg \gamma = 1$. Then we have relatively prime integers p and $q > 0$ satisfying $\gamma = p/q$. Since the leading coefficient of $P(X)$ is divisible by q^s, we have $q^s \le H$, which implies that

$$\left|\xi - \frac{p}{q}\right| \le q^{-\kappa s} \le q^{-3}.$$

Letting the product nH tend to infinity, we see that this inequality holds for infinitely many p/q whenever ξ is irrational. Once again, we conclude by the Borel–Cantelli lemma. □

In view of (19) with (17) and (18) the set $\mathcal{E}_{1,2,3}$ is contained in $\mathcal{E}_0(3/4)$ given in Lemma 2, which implies that $\mathcal{E}_{1,2,3}$ is a null set, as desired. Thus we have shown that $\mathcal{E}_{1,2}$ is a null set.

We next prove that $\mathcal{E}_2$ is a null set. By the definition, for each $\xi \in \mathcal{E}_2$, there exist infinitely many pairs $(P(X), \alpha)$ of an integer polynomial $P(X)$ and a root α of $P(X)$ nearest to ξ with multiplicity $s > 1$ satisfying (16). According to Theorem DM, for such a pair $(P(X), \alpha)$, we have

$$|\xi - \alpha| \le \exp\left\{-\frac{n}{s}\left(\left(2 + \varepsilon - \frac{2}{s}\right)\log H + \left(\frac{3}{2} + \varepsilon - \frac{3}{2s}\right)\log n - 1\right)\right\}.$$

We see easily that this upper bound is less than $\rho(n, H, s)$ except when $s = 2$, and that, for the exceptional case, it is less than $n^{n/8}\rho(n, H, 2)$. Therefore, in view of (24), the above argument for proving $\lambda(\mathcal{E}_{1,2,2}) = 0$ remains valid to show $\lambda(\mathcal{E}_2) = 0$. Thus we have shown that $\mathcal{E}_2$ is a null set.

Combining the results above we see that $\mathcal{E}$ is a null set, which completes the proof of the theorem. $\square$

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