Around the Littlewood conjecture in Diophantine approximation

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Abstract. The Littlewood conjecture in Diophantine approximation claims that

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0$$

holds for all real numbers α and β, where $\| \cdot \|$ denotes the distance to the nearest integer. Its p-adic analogue, formulated by de Mathan and Teulié in 2004, asserts that

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_p = 0$$

holds for every real number α and every prime number p, where $| \cdot |_p$ denotes the p-adic absolute value normalized by $|p|_p = p^{-1}$. We survey the known results on these conjectures and highlight recent developments.

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A famous open problem in simultaneous Diophantine approximation, called the Littlewood conjecture, claims that, for every given pair (α, β) of real numbers, we have

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0,$$

where $\|\cdot\|$ denotes the distance to the nearest integer. According to Montgomery **[27]**, the first occurrence of the Littlewood conjecture appeared in 1942 in a paper by Spencer **[34]**, a student of Littlewood.

Since 2000, there has been much activity on and around the Littlewood conjecture, including the formulation by de Mathan and Teulié **[26]** of a closely related open problem, called the mixed Littlewood conjecture. The purpose of the present survey is to highlight recent results and developments on these questions. We make the choice to state more than twenty theorems and to give only a single proof.

Section 1 is devoted to the Littlewood conjecture itself, while the mixed and the p-adic (a special case of the mixed) Littlewood conjectures are addressed in Section 2. The reader will observe that the state-of-the-art regarding the Littlewood and the p-adic Littlewood conjectures is essentially the same. The proof of one result from [**5**] is given in Section 3. We conclude in Section 4 by mentioning recent results and open questions on inhomogeneous variations of the Littlewood and the mixed Littlewood conjectures.

The number of papers just appeared, being submitted or in preparation shows that there is currently a lot of activities on this topic.

Throughout, we assume that the reader is familiar with the theory of continued fractions and 'almost all' (or 'almost every') always refers to the Lebesgue measure.

1. The Littlewood conjecture

Let α and β be real numbers. Clearly,

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0 \tag{1.1}$$

always holds when $1, \alpha, \beta$ are linearly dependent over the rationals or when α or β has unbounded partial quotients in its continued fraction expansion. Thus, we may assume that α and β belong to the set **Bad** of badly approximable real numbers, where

$$\mathbf{Bad} = \{\alpha \in \mathbb{R} : \inf_{q \geq 1} q \cdot \|q\alpha\| > 0\}.$$

The set **Bad** is the set of real numbers whose sequence of partial quotients is bounded. It has zero Lebesgue measure and full Hausdorff dimension (that is, Hausdorff dimension one).

In 1955 Cassels and Swinnerton-Dyer [**14**] made the first significant contribution on the Littlewood conjecture in showing that (1.1) holds when α and β belong to the same cubic field. Note that it is still not known whether or not cubic real numbers belong to **Bad**.

Pollingon and Velani [**32**] showed in 2000 that, for every badly approximable real number α, there exist uncountably many badly approximable real numbers β such that a strong form of (1.1) holds for the pair (α, β).

Theorem 1. — *For every real number α in* **Bad**, *there exists a subset $G(\alpha)$ of* **Bad** *with full Hausdorff dimension such that, for any β in $G(\alpha)$, there exist* *arbitrarily large integers q satisfying*

$$q \cdot (\log q) \cdot \|q\alpha\| \cdot \|q\beta\| \leq 1.$$

For an alternative proof of a slightly weaker form of Theorem 1, together with some additional interesting results, the reader is referred to **[22]** ; see also Theorem 29 in Section 4.

Einsiedler, Katok, and Lindenstrauss **[16]** (see also **[35]**) established that the set of exceptions to the Littlewood conjecture is very small.

Theorem 2. — *The set of pairs (α, β) of real numbers such that*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| > 0$$

has Hausdorff dimension zero. Furthermore, it is contained in a countable union of compact sets of box dimension zero.

Furthermore, Lindenstrauss **[24]** stressed that one may deduce from the techniques of **[16]** an explicit, sufficient criterion for a real number α in order that the Littlewood conjecture holds for every pair (α, β), where β is an arbitrary real number.

To present his result (and also subsequent results given in Section 2), we adopt a point of view from combinatorics on words. We look at the continued fraction expansion of a given real number α as an infinite word. For an infinite word $\mathbf{w} = w_1 w_2 \ldots$ and an integer $n \geq 1$, we denote by $p(n, \mathbf{w})$ the number of distinct blocks of n consecutive letters occurring in $\mathbf{w}$, that is,

$$p(n, \mathbf{w}) := \mathrm{Card}\{w_{\ell+1} \ldots w_{\ell+n} : \ell \geq 0\}.$$

The function $n \mapsto p(n, \mathbf{w})$ is called the *complexity function* of $\mathbf{w}$. For an irrational real number $\alpha = [a_0; a_1, a_2, \ldots]$, we set

$$p(n, \alpha) := p(n, a_1 a_2 \ldots), \quad n \geq 1.$$

Clearly, for positive integers n, n' we have

$$p(n + n', \alpha) \leq p(n, \alpha) \cdot p(n', \alpha).$$

This inequality implies that the sequence $(\log p(n, \alpha))_{n \geq 1}$ is subadditive, thus, $((\log p(n, \alpha)/n)_{n \geq 1}$ converges.

Definition 3. — *The entropy of a real number α is the quantity*

$$E(\alpha) = \lim_{n \to +\infty} \frac{\log p(n, \alpha)}{n}.$$

It is an easy exercise to show that the set of real numbers α such that $E(\alpha) = 0$ has Hausdorff dimension zero.

With the above notation, the result alluded to below Theorem 2 and stated as Theorem 5 in **[24]** can be formulated as follows.

Theorem 4. — *If the real number α satisfies $E(\alpha) > 0$, then, for every real number β, we have*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0.$$

Theorem 1 is a metrical result and, as such, does not tell us how to associate explicitly to some given badly approximable number α a badly approximable number β such that $1, \alpha$ and β are linearly independent over the integers and (1.1) holds for the pair (α, β). This problem has been addressed in [**1**] (see [**25**] for a weaker previous result).

Theorem 5. — *Let φ be a positive, non-increasing function defined on the set of positive integers and satisfying $\varphi(1) = 1$, $\lim_{q \to +\infty} \varphi(q) = 0$ and $\lim_{q \to +\infty} q\varphi(q) = +\infty$. Given α in* **Bad**, *there exists an uncountable subset $B_\varphi(\alpha)$ of* **Bad** *such that, for any β in $B_\varphi(\alpha)$, there exist infinitely many positive integers q with*

$$q \cdot \|q\alpha\| \cdot \|q\beta\| \leq \frac{1}{q \cdot \varphi(q)}. \tag{1.2}$$

In particular, the Littlewood conjecture holds for the pair (α, β) for any β in $B_\varphi(\alpha)$. Furthermore, the set $B_\varphi(\alpha)$ can be effectively constructed.

The proof of Theorem 5 rests on the theory of continued fractions. For given α and φ, we construct inductively the sequence of partial quotients of a suitable real number β such that (1.2) holds for the pair (α, β).

Going back to metrical results, the following theorem of Gallagher [**19**] shows that (1.1) can be improved for almost all pairs (α, β) of real numbers.

Theorem 6. — *Let $\psi : \mathbb{Z}_{\geq 1} \to \mathbb{R}$ be a non-negative decreasing function. Then, for almost every pair (α, β) of real numbers, the inequality*

$$\|q\alpha\| \cdot \|q\beta\| \leq \psi(q)$$

has infinitely (resp. finitely) many integer solutions $q \geq 1$ if the series $\sum_{q \geq 1} \psi(q) \log q$ diverges (resp. converges). In particular, for almost every pair (α, β) of real numbers, we have

$$\inf_{q \geq 2} q \cdot (\log q)^2 \cdot \|q\alpha\| \cdot \|q\beta\| = 0.$$

Since we are, at present, not able to confirm nor to deny the Littlewood conjecture, we may search for pairs (α, β) of real numbers for which there exists a slowly growing function φ such that

$$\liminf_{q \to +\infty} q \cdot \varphi(q) \cdot \|q\alpha\| \cdot \|q\beta\| > 0. \tag{1.3}$$

In view of Theorem 6, a first non-trivial step is to show the existence of pairs (α, β) for which (1.3) holds with the function $q \mapsto \varphi(q) = (\log q)^2$. This has been done in 2011 in [**12**], by means of a method introduced by Peres and Schlag [**31**]. This result has been subsequently considerably improved by Badziahin [**3**], who used an intricate Cantor-type construction to establish the following theorem.

Theorem 7. — *For every real number α in* **Bad**, *the set of real numbers β such that*

$$\inf_{q\geq 3} q\cdot \log q\cdot \log\log q\cdot \|q\alpha\|\cdot\|q\beta\|>0$$

has full Hausdorff dimension. In particular, the set of pairs (α,β) of real numbers satisfying

$$\inf_{q\geq 3} q\cdot \log q\cdot \log\log q\cdot \|q\alpha\|\cdot\|q\beta\|>0$$

has full Hausdorff dimension in $\mathbb{R}^2$.

It remains an open problem to show the existence of pairs (α,β) of real numbers for which inequality (1.3) holds with the function $\varphi : q\mapsto \log q$.

2. The mixed and the p-adic Littlewood conjectures

In 2004 de Mathan and Teulié [**26**] proposed a *mixed Littlewood conjecture* which can be stated as follows. Let $\mathcal{D}=(d_k)_{k\geq 1}$ be a sequence of integers greater than or equal to 2. Set $e_0=1$ and, for $n\geq 1$,

$$e_n=\prod_{1\leq k\leq n} d_k.$$

For an integer q, set

$$w_{\mathcal{D}}(q)=\sup\{n\geq 0 : q\in e_n\mathbf{Z}\}$$

and

$$|q|_{\mathcal{D}}=1/e_{w_{\mathcal{D}}(q)}=\inf\{1/e_n : q\in e_n\mathbf{Z}\}.$$

When $\mathcal{D}$ is the constant sequence equal to p, where p is a prime number, then $|\cdot|_{\mathcal{D}}$ is the usual p-adic value $|\cdot|_p$ normalized by $|p|_p=p^{-1}$. In analogy with the Littlewood conjecture, de Mathan and Teulié formulated the following conjecture.

Conjecture 8. — *(Mixed Littlewood Conjecture.) For every real number α and every sequence $\mathcal{D}$ as above, we have*

$$\inf_{q\geq 1} q\cdot\|q\alpha\|\cdot|q|_{\mathcal{D}}=0. \tag{2.1}$$

Obviously, (2.1) holds if α is rational or has unbounded partial quotients. Thus, we only consider the case when α is an element of the set **Bad** defined in Section 1.

By Lemme 3.1 of [**26**], if $(\alpha, \mathcal{D})$ does not satisfy (2.1), then there exists a real number M such that all the partial quotients of the real numbers $\{e_n \alpha\}$, $n \geq 0$, are less than M. Here and below, $\{\cdot\}$ denotes the fractional part function.

De Mathan and Teulié proved that (2.1) and even the stronger statement

$$\liminf_{q \to +\infty} q \cdot \log q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} < +\infty$$

holds for every real quadratic number α, provided that the sequence $\mathcal{D}$ is bounded; see [**5, 23**] for alternative proofs when $\mathcal{D}$ is the constant sequence equal to a prime number, a particular case which deserves to be highlighted.

Conjecture 9. — *(p-adic Littlewood Conjecture.) For every real number α and every prime number p, we have*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_p = 0. \tag{2.2}$$

Einsiedler and Kleinbock [**17**] showed that a slightly weaker form of the p-adic Littlewood conjecture, namely Theorem 11 below, can easily be deduced from the following theorem of Furstenberg [**18**].

Theorem 10. — *Let r and s be multiplicatively independent integers. Then, for every irrational number α, the set of real numbers $\{\alpha r^m s^n\}$, where m and n run through the set of non-negative integers, is dense in $[0,1]$.*

An alternative proof of Theorem 10 was given by Boshernitzan [**8**] and is reproduced in the monograph [**13**].

Theorem 11. — *Let p_1 and p_2 be distinct prime numbers. Then,*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_{p_1} \cdot |q|_{p_2} = 0$$

holds for every real number α.

Bourgain, Lindenstrauss, Michel and Venkatesh [**9**] established a quantitative version of Theorem 11.

Theorem 12. — *Let p_1 and p_2 be distinct prime numbers. There exists a positive real number c such that, for any real number α, we have*

$$\inf_{q \geq 16} q \cdot (\log\log\log q)^c \cdot \|q\alpha\| \cdot |q|_{p_1} \cdot |q|_{p_2} = 0.$$

Harrap and Haynes [**21**] managed recently to extend Theorem 11. We quote below their Corollary 1. For an integer $a \geq 2$ and for $\mathcal{D}$ being the infinite sequence $a, a, \ldots$, we write $|\cdot|_a$ instead of $|\cdot|_{\mathcal{D}}$.

Theorem 13. — *Let $a \geq 2$ be an integer and $\mathcal{D}$ be a bounded sequence of integers coprime to a and greater than or equal to 2. Then,*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_a \cdot |q|_{\mathcal{D}} = 0$$

holds for every real number α.

The proof of Theorem 13 is a nice combination of ideas from [**9, 15**] and lower bounds for linear forms in logarithms of algebraic numbers (Baker's theory).

Einsiedler and Kleinbock [**17**] established that the set of possible exceptions to the p-adic Littlewood conjecture is very small from the metric point of view.

Theorem 14. — *Let p be a prime number. The set of real numbers α such that*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_p > 0$$

has Hausdorff dimension zero.

Theorem 14 is the analogue of Theorem 2. Einsiedler and Kleinbock also explained how to modify their proof to get an analogous result when $\mathcal{D}$ is the constant sequence equal to an integer $a \geq 2$ (not necessarily prime).

The analogue of Theorem 4 was very recently proved in [**5**].

Theorem 15. — *If the real number α satisfies $E(\alpha) > 0$, then for every prime number p we have*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_p = 0.$$

Theorem 15 asserts that the complexity function of the continued fraction expansion of every potential counterexample to the p-adic Littlewood conjecture cannot grow exponentially fast.

We present now various explicit examples of real numbers α in **Bad** for which (2.2) and even (2.1) hold. First, we need some classical results and definitions from combinatorics on words.

A well-known result of Morse and Hedlund [**28, 29**] asserts that $p(n, \mathbf{w}) \geq n + 1$ for $n \geq 1$, unless $\mathbf{w}$ is ultimately periodic (in which case there exists a constant C such that $p(n, \mathbf{w}) \leq C$ for $n \geq 1$). Infinite words $\mathbf{w}$ satisfying $p(n, \mathbf{w}) = n + 1$ for every $n \geq 1$ do exist and are called *Sturmian words*. We start with a classical definition (see e.g. [**2**]).

Definition 16. — *An infinite word $\mathbf{w}$ is recurrent if every finite block occurring in $\mathbf{w}$ occurs infinitely often.*

Classical examples of recurrent infinite words include periodic words, Sturmian words, the Thue–Morse word, etc.

Theorem 17. — *Let $(a_k)_{k\geq 1}$ be a sequence of positive integers. If there exists an integer $m \geq 0$ such that the infinite word $a_{m+1}a_{m+2}\ldots$ is recurrent, then, for every sequence $\mathcal{D}$ of integers greater than or equal to 2, the real number $\alpha := [0; a_1, a_2, \ldots]$ satisfies*

$$\inf_{q\geq 1} q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} = 0.$$

The proof of Theorem 17, given in Section 3, is elementary, in the sense that it uses only the theory of continued fractions.

As a particular case, Theorem 17 asserts that (2.1) holds for every quadratic number α and every (bounded or unbounded) sequence $\mathcal{D}$ of integers greater than or equal to 2.

As shown in [**5**], Theorem 17 implies a non-trivial lower bound for the complexity function of the continued fraction expansion of a putative counterexample to (2.1).

Corollary 18. — *Let α be a real number such that*

$$\lim_{n\to+\infty} p(n,\alpha) - n < +\infty.$$

Then, for every sequence $\mathcal{D}$ of integers greater than or equal to 2, we have

$$\inf_{q\geq 1} q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} = 0.$$

The next corollary of Theorem 17 deals with a special family of infinite recurrent words. A finite word $w_1 \ldots w_n$ is called a *palindrome* if $w_{n+1-h} = w_h$ for $h = 1, \ldots, n$.

Corollary 19. — *Let $(a_k)_{k\geq 1}$ be a sequence of positive integers. If there exists an increasing sequence $(n_j)_{j\geq 1}$ of positive integers such that $a_1 \ldots a_{n_j}$ is a palindrome for $j \geq 1$, then, for every sequence $\mathcal{D}$ of integers greater than or equal to 2, the real number $\alpha := [0; a_1, a_2, \ldots]$ satisfies*

$$\inf_{q\geq 1} q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} = 0.$$

To derive Corollary 19 from Theorem 17, it is sufficient to note that, if $a_1 \ldots a_n$ and $a_1 \ldots a_{n'}$ are palindromes with $n' > 2n$, then

$$a_{n'-n+1} \ldots a_{n'} = a_n \ldots a_1 = a_1 \ldots a_n.$$

The corollary then follows from Theorem 17 applied with $m = 0$.

The next result asserts that the mixed Littlewood conjecture holds for every prime number p and every real number α whose sequence of partial quotients contains arbitrarily long concatenations of a given finite block.

Theorem 20. — *Let $\alpha = [a_0; a_1, a_2, \ldots]$ be a real number. Let $T \geq 1$ be an integer and $b_1, \ldots, b_T$ be positive integers. If there exist two sequences $(m_k)_{k\geq 1}$ and $(h_k)_{k\geq 1}$ of positive integers with $(h_k)_{k\geq 1}$ being unbounded and*

$$a_{m_k+j+nT} = b_j, \quad \text{for every } j = 1, \ldots, T \text{ and every } n = 0, \ldots, h_k - 1,$$

then, for every prime number p, we have

$$\inf_{q\geq 1} q \cdot \|q\alpha\| \cdot |q|_p = 0.$$

The following consequence of Theorem 20 deserves to be pointed out. Let α be a real number having exactly m distinct partial quotients in its continued fraction expansion. If $E(\alpha) = \log m$, then for every prime number p we have

$$\inf_{q\geq 1} q \cdot ||q\alpha|| \cdot |q|_p = 0.$$

Clearly, this has been superseded by Theorem 15.

The assumption of Theorem 20 can be restated as follows. To an irrational real number $\alpha := [a_0; a_1, a_2, \ldots]$ we associate the set

$$\mathrm{Adh}(\alpha) := \overline{\{[0; a_m, a_{m+1}, \ldots] : m \geq 1\}},$$

which is the closure of the set composed of the iterates of $\{\alpha\}$ under the Gauss transformation. Then, Theorem 20 asserts that the mixed Littlewood conjecture holds for every irrational real number α such that $\mathrm{Adh}(\alpha)$ contains a quadratic number.

In the case of the p-adic Littlewood conjecture, Badziahin [**4**] established a common extension to Corollary 18 and Theorem 20.

Theorem 21. — *Let α be an irrational real number. If the set* $\mathrm{Adh}(\alpha)$ *contains a real number α' satisfying*

$$\lim_{n\to+\infty} p(n, \alpha') - n < +\infty,$$

then

$$\inf_{q\geq 1} q \cdot ||q\alpha|| \cdot |q|_p = 0$$

holds for every prime number p.

Badziahin's paper [**4**] contains further new results, which show that the continued fraction expansion, viewed as an infinite word, of a putative counterexample to the p-adic Littlewood conjecture must satisfy various strong combinatorial properties.

Metric considerations in the same spirit as in Gallagher's paper [**19**] can be found in [**11, 7**]. We state below Theorem 1 from [**11**].

Theorem 22. — *Let $p_1, \ldots, p_k$ be distinct prime numbers and let $\psi : \mathbb{Z}_{\geq 1} \to \mathbb{R}$ be a non-negative decreasing function. Then, for almost every real number α the inequality*

$$\|q\alpha\| \cdot |q|_{p_1} \cdots |q|_{p_k} \leq \psi(q)$$

has infinitely (resp. finitely) many integer solutions q if the series

$$\sum_{q \geq 1} (\log q)^k \psi(q)$$

diverges (resp. converges). In particular, for almost all real numbers α, we have

$$\inf_{q \geq 2} q \cdot (\log q)^2 \cdot \|q\alpha\| \cdot |q|_p = 0,$$

for every prime number p.

It is proved in **[12]** that the set of real numbers α in **Bad** such that, for every prime number p, we have

$$\inf_{q \geq 2} q \cdot (\log q)^2 \cdot \|q\alpha\| \cdot |q|_p > 0$$

has full Hausdorff dimension. This was considerably improved by Badziahin and Velani **[6]**, by means of a subtle Cantor-type construction.

Theorem 23. — *For every sequence $\mathcal{D}$ of integers greater than or equal to 2, the set of real numbers α such that*

$$\inf_{q \geq 3} q \cdot \log q \cdot \log\log q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} > 0$$

has full Hausdorff dimension. Moreover, if $\mathcal{D}$ denotes the sequence $(2^{2^n})_{n \geq 1}$, then the set of real numbers α such that

$$\inf_{q \geq 16} q \cdot \log\log q \cdot \log\log\log q \cdot \|q\alpha\| \cdot |q|_{\mathcal{D}} > 0$$

has full Hausdorff dimension.

Theorem 23 was proved shortly before Theorem 7.

3. Proof of Theorem 17

Without any loss of generality, we consider real numbers in $(0, 1)$. We associate to every real irrational number $\alpha := [0; a_1, a_2, \ldots]$ the infinite word $\mathbf{a} := a_1 a_2 \ldots$ formed by the sequence of its partial quotients. Set

$$p_{-1} = q_0 = 1, \quad p_0 = q_{-1} = 0,$$

and

$$\frac{p_n}{q_n} = [0; a_1, \ldots, a_n], \quad \text{for } n \geq 1.$$

By the theory of continued fractions, we know that

$$\frac{q_n}{q_{n-1}} = [a_n; a_{n-1}, \ldots, a_1].$$

This is one of the key tools of our proof.

For simplicity, we establish Theorem 17 only in the case $m = 0$.

Assume that the infinite word $a_1 a_2 \ldots$ is recurrent. Then, there exists an increasing sequence of positive integers $(n_j)_{j \geq 1}$ such that

$$a_1 a_2 \ldots a_{n_j} \text{ is a suffix of } a_1 a_2 \ldots a_{n_{j+1}}, \text{ for } j \geq 1.$$

Said differently, there are finite words $V_1, V_2, \ldots$ such that

$$a_1 a_2 \ldots a_{n_{j+1}} = V_j a_1 a_2 \ldots a_{n_j}, \text{ for } j \geq 1.$$

Actually, these properties are equivalent.

Let $\ell \geq 2$ be an integer. Let $k \geq \ell^2 + 1$ be an integer. By Dirichlet's *Schubfachprinzip*, there exist integers i, j with $1 \leq i < j \leq k$ such that

$$q_{n_i} \equiv q_{n_j} \pmod{\ell}, \quad q_{n_i - 1} \equiv q_{n_j - 1} \pmod{\ell}$$

and j is minimal with this property.

Setting

$$Q := |q_{n_i} q_{n_j - 1} - q_{n_i - 1} q_{n_j}|,$$

we observe that

$$\ell \text{ divides } Q \tag{3.1}$$

and that

$$\frac{q_{n_i - 1}}{q_{n_i}} = [0; a_{n_i}, a_{n_i - 1}, \ldots, a_1]$$

is a convergent of

$$\frac{q_{n_j - 1}}{q_{n_j}} = [0; a_{n_j}, a_{n_j - 1}, \ldots, a_1].$$

Consequently, we get

$$\begin{aligned} 0 < Q &= q_{n_i} q_{n_j} \left| \frac{q_{n_j - 1}}{q_{n_j}} - \frac{q_{n_i - 1}}{q_{n_i}} \right| \\ &\leq q_{n_i} q_{n_j} q_{n_i}^{-2} = q_{n_i}^{-1} q_{n_j}. \end{aligned}$$

Since

$$||Q\alpha|| \leq 2 q_{n_i} q_{n_j}^{-1},$$

we finally obtain

$$Q \cdot ||Q\alpha|| \leq 2. \tag{3.2}$$

It then follows from (3.1) and (3.2) that

$$Q \cdot ||Q\alpha|| \cdot |Q|_\ell \leq 2\ell^{-1},$$

where $|Q|_\ell$ is equal to ℓ^{-a} if ℓ^a divides Q but ℓ^{a+1} does not. Since ℓ is arbitrary, this proves Theorem 17 when $m = 0$.

Exactly the same idea works for $m \geq 1$ and there is no extra difficulty, just a little more care is needed in the various estimates.

4. Inhomogeneous approximation

The Littlewood conjecture and its p-adic analogue can be extended in a natural way to inhomogeneous approximation.

Problem 24. — *Let α, β be real numbers such that $1, \alpha, \beta$ are linearly independent over the rationals. Is it true that, for all real numbers α_0, β_0, we have*

$$\liminf_{q \to +\infty} q \cdot \|q\alpha - \alpha_0\| \cdot \|q\beta - \beta_0\| = 0?$$

The assumption that $1, \alpha, \beta$ are linearly independent over the rationals is clearly necessary.

Shapira **[33]** established that the answer to Problem 24 is positive for almost all pairs (α, β), including all pairs (α, β) of cubic real numbers in a same cubic field.

Theorem 25. — *Almost every pair (α, β) of real numbers satisfies*

$$\liminf_{q \to +\infty} q \cdot \|q\alpha - \alpha_0\| \cdot \|q\beta - \beta_0\| = 0, \tag{4.1}$$

for all real numbers α_0, β_0. Moreover, if $1, \alpha, \beta$ forms a basis of a real cubic field, then (4.1) holds for all real numbers α_0, β_0.

Gorodnik and Vishe **[20]** established recently a quantitative version of Theorem 25.

Theorem 26. — *There exists a positive constant c such that almost every pair (α, β) of real numbers satisfies*

$$\liminf_{q \to +\infty} q \cdot (\log\log\log\log\log q)^c \cdot \|q\alpha - \alpha_0\| \cdot \|q\beta - \beta_0\| = 0, \tag{4.2}$$

for all real numbers α_0, β_0. Moreover, if $1, \alpha, \beta$ forms a basis of a real cubic field, then (4.2) holds for all real numbers α_0, β_0.

We highlight the following problem, which can be viewed as the p-adic analogue of Problem 24.

Problem 27. — *Let p be a prime number. Let α be an irrational real number. Is it true that, for every integer q_0 and every irrational α_0, we have*

$$\liminf_{q \to +\infty} q \cdot \|q\alpha - \alpha_0\| \cdot |q - q_0|_p = 0? \tag{4.3}$$

Examples of real numbers α for which (4.3) holds with $\alpha_0 = 0$ are given in [**10**]. For metrical results related to Problem 27, see [**22**].

Gorodnik and Vishe [**20**] proved the p-adic analogue of their Theorem 26.

Theorem 28. — *Let p be a prime number. There exists a positive constant c such that almost every real number α satisfies*

$$\liminf_{q \to +\infty} q \cdot (\log\log\log\log\log q)^{c} \cdot \|q\alpha - \alpha_0\| \cdot |q - q_0|_p = 0, \tag{4.4}$$

for every real number α_0 and every integer q_0. Moreover, every quadratic real number α satisfies (4.4) for every real number α_0 and every integer q_0.

Haynes, Jensen and Kristensen [**22**] have obtained several metrical results related to the inhomogeneous Littlewood conjecture and its p-adic analogue. One of their results is the following theorem.

Theorem 29. — *Let ε be a positive real number. Let $(\alpha_i)_{i \geq 1}$ be a countable sequence of badly approximable numbers. There exists a subset G of* **Bad** *with full Hausdorff dimension such that, for every β in G, every $i \geq 1$ and every real number β_0, there exist arbitrarily large integers q satisfying*

$$q \cdot (\log q)^{1/2-\varepsilon} \cdot \|q\alpha_i\| \cdot \|q\beta - \beta_0\| \leq 1.$$

In view of Theorem 11, we may ask whether, for some integer $d \geq 3$, we have

$$\inf_{q \geq 1} q \cdot \|q\alpha_1\| \cdots \|q\alpha_d\| = 0,$$

for all badly approximable real numbers $\alpha_1, \ldots, \alpha_d$. Except the following result of Peck [**30**], nothing more is known on this question than on the Littlewood conjecture.

Theorem 30. — *Let $d \geq 2$ be an integer and $1, \alpha_1, \ldots, \alpha_d$ be a basis of a real number field of degree $d+1$. Then, we have*

$$\liminf_{q \to +\infty} q \cdot (\log q) \cdot \|q\alpha_1\| \cdots \|q\alpha_d\| < +\infty,$$

thus, in particular,

$$\inf_{q \geq 1} q \cdot \|q\alpha_1\| \cdots \|q\alpha_d\| = 0.$$

We do not know whether the algebraic numbers $\alpha_1, \ldots, \alpha_d$ in the statement of Theorem 30 are badly approximable. Theorem 30 extends and improves the result of Cassels and Swinnerton-Dyer [**14**] mentioned in Section 1. The second statements of Theorem 25 and of Theorem 26 can be viewed as inhomogeneous analogues of Theorem 30 when $d = 2$. This motivates the following open problem.

Problem 31. — *Let $d \geq 3$ be an integer and $1, \alpha_1, \ldots, \alpha_d$ be a basis of a real number field of degree $d+1$. Is it true that*

$$\liminf_{q \to +\infty} q \cdot \| q\alpha_1 - \alpha'_1 \| \cdots \| q\alpha_d - \alpha'_d \| = 0$$

holds for all real numbers $\alpha'_1, \ldots, \alpha'_d$?

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