On a mixed Littlewood conjecture in fields of power series

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Abstract. *In a recent paper, de Mathan and Teulié asked whether* $\liminf_{q \to +\infty} q \cdot \|q\alpha\| \cdot |q|_p = 0$ *holds for every badly approximable real number* α *and every prime number* p. *After a survey of the known results on this open problem, we study the analogous question in fields of power series.*

1. Introduction

A famous open problem in simultaneous Diophantine approximation is the Littlewood conjecture [17]. It claims that, for every given pair (α, β) of real numbers, we have

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot \|q\beta\| = 0, \tag{1.1}$$

where $\| \cdot \|$ denotes the distance to the nearest integer. The first significant contribution to this question goes back to Cassels and Swinnerton-Dyer [8] who showed that (1.1) holds when α and β belong to the same cubic field. Further explicit examples of pairs (α, β) of real numbers satisfying (1.1) have been given in [18, 1]. Despite some recent remarkable progress [21, 13] the Littlewood conjecture remains an open problem.

In analogy with (1.1), de Mathan and Teulié [20] proposed recently a 'mixed Littlewood conjecture'. For any prime number p, we normalize the usual p-adic value $| \cdot |_p$ in such a way that $|p|_p = p^{-1}$.

De Mathan–Teulié conjecture. *For every real number* α *and every prime number* p, *we have*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_p = 0. \tag{1.3}$$

Obviously, the above conjecture holds if α is rational or has unbounded partial quotients. Thus, we only consider the case when α is an element of the set $\boldsymbol{Bad}_{\mathbf{R}}$ of badly approximable real numbers, where

$$\boldsymbol{Bad}_{\mathbf{R}} = \{\alpha \in \mathbf{R} : \inf_{q \geq 1} q \cdot \|q\alpha\| > 0\}.$$

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We briefly survey the main results from [20, 14, 6].

Next theorem, proved in [6], asserts that (1.3) holds for every pair (α, p) provided that the sequence of partial quotients of α is, in some sense, quasi-periodic.

Theorem BDM. *Let α be in $\boldsymbol{Bad}_{\mathbf{R}}$ with continued fraction expansion*

$$\alpha = [a_0; a_1, a_2, \ldots].$$

Let $T \geq 1$ be an integer and $b_1, \ldots, b_T$ be positive integers. If there exist two sequences $(m_k)_{k\geq 1}$ and $(h_k)_{k\geq 1}$ of positive integers with $(h_k)_{k\geq 1}$ being unbounded and

$$a_{m_k+j+nT} = b_j, \quad \text{for every } j = 1, \ldots, T \text{ and every } n = 0, \ldots, h_k - 1,$$

then

$$\inf_{q\geq 1} q \cdot \|q\alpha\| \cdot |q|_p = 0$$

for every prime number p.

An immediate consequence of Theorem BDM is that (1.3) holds for every prime p and every quadratic number α, a result already proved in [20]. Further explicit examples of pairs (α, p) for which (1.3), and even the stronger inequality

$$\liminf_{q\to+\infty} q^2 \cdot \|q\alpha\| \cdot |q|_p \leq 1 \tag{1.6}$$

holds can be found in [6].

Einsiedler and Kleinbock [14] established that the set of possible exceptions to the de Mathan–Teulié conjecture is, from the metric point of view, very small.

Theorem EK1. *Let p be a prime number. The set of real numbers α which do not satisfy (1.3) has Hausdorff dimension zero.*

Furthermore, Einsiedler and Kleinbock [14] showed that a slight modification of the de Mathan–Teulié conjecture easily follows from a theorem of Furstenberg [15, 5].

Theorem EK2. *Let p_1 and p_2 be distinct prime numbers. Then*

$$\inf_{q\geq 1} q \cdot \|q\alpha\| \cdot |q|_{p_1} \cdot |q|_{p_2} = 0$$

holds for every real number α.

The purpose of the present note is to investigate the analogue of the de Mathan–Teulié conjecture in fields of power series. Given an arbitrary field $\mathbf{k}$ and an indeterminate X, we define a norm $|\cdot|$ on the field $\mathbf{k}((X^{-1}))$ by setting $|0| = 0$ and, for any non-zero power series $F = F(X) = \sum_{h=-m}^{+\infty} f_h X^{-h}$ with $f_{-m} \neq 0$, by setting $|F| = 2^m$. We write $||F||$ to denote the norm of the fractional part of F, that is, of the part of the series which comprises only the negative powers of X. Let p be a monic, irreducible polynomial in $\mathbf{k}[X] \setminus \mathbf{k}$. Let q be an arbitrary polynomial in $\mathbf{k}[X]$. If a denotes the largest non-negative integer such that p^a divides q, we define the norm $|\cdot|_p$ by $|q|_p = |p|^{-a}$. In analogy with (1.3), we ask whether

$$\inf_{q \in \mathbf{k}[X]\setminus\{0\}} |q| \cdot \|q\Theta\| \cdot |q|_p = 0 \tag{1.7}$$

holds for any given Θ in $\mathbf{k}((X^{-1}))$.

When the field $\mathbf{k}$ is infinite, a negative answer to this question has been obtained by de Mathan and Teulié [20] for the polynomial $p = X$. They also proved that, if $\mathbf{k}$ is finite, then (1.7) holds for every quadratic power series Θ and every monic, irreducible polynomial p.

To our knowledge, these are the only known results on (1.7). In the present note, we discuss the analogues of the above-mentioned statements in fields of power series. Our main results are stated in Section 2 and proved in Sections 3 to 6. Furthermore, Section 7 is devoted to open questions related to the (real form of the) de Mathan–Teulié conjecture.

2. Results

Since (1.7) clearly holds as soon as Θ has unbounded partial quotients, we restrict our attention in all what follows to power series Θ with bounded partial quotients, that is, belonging to the set

$$\boldsymbol{Bad} = \{\Theta \in \mathbf{k}((X^{-1})) : \inf_{q \in \mathbf{k}[X]\setminus\{0\}} |q| \cdot \|q\Theta\| > 0\}.$$

In particular, we always assume that

$$\inf_{q \in \mathbf{k}[X]\setminus\{0\}} |q|^2 \cdot \|q\Theta\| \cdot |q|_p > 0. \tag{2.1}$$

We stress that, in Sections 2 to 4, the letters p and q denote non-zero elements of $\mathbf{k}[X]$.

De Mathan and Teulié [20], Théorème 4.3, established that the analogue of their conjecture does not hold in fields of power series when the field $\mathbf{k}$ is infinite. We extend their result by showing that, in the same setting, the analogue of Theorem EK2 does not hold neither.

Theorem 1. *Let $p_1, \ldots, p_r$ be distinct irreducible, monic, non-constant polynomials. If the field $\mathbf{k}$ is infinite, then there exist power series Θ such that*

$$|q| \cdot \|q\Theta\| \cdot |q|_{p_1} \cdots |q|_{p_r} \geq 1/|X|, \quad \textit{for all } q \in \mathbf{k}[X] \setminus \{0\}.$$

Our next result, obtained by means of the Folding Lemma (like Theorem 5 from [6]), shows that there are power series Θ in $\boldsymbol{Bad}$ for which (2.1) cannot be replaced by a stronger inequality. In particular, for any given non-constant polynomial p, it provides us with explicit examples of power series satisfying the analogue of the de Mathan–Teulié conjecture.

Theorem 2. *For any irreducible, monic polynomial p in $\mathbf{k}[X] \setminus \mathbf{k}$, there exist uncountably many power series Θ in **Bad** such that*

$$\liminf_{\deg q \to +\infty} |q|^2 \cdot \|q\Theta\| \cdot |q|_p = 1/|p|^h, \tag{2.2}$$

where $h = 2$ if $\mathbf{k} = \mathbf{F}_2$, and $h = 1$ otherwise.

The proof of Theorem 2 is given in Section 4. If the characteristic of $\mathbf{k}$ is 2, then our construction yields that the power series $\sum_{j \geq 1} p^{-2^j+1}$ is in ***Bad*** and satisfies (2.2). This power series is quadratic and it satisfies the equation $\Theta^2 + p\Theta + 1 = 0$.

Furthermore, for any finite field $\mathbf{k}$, there exist quadratic power series with (2.2).

Theorem 3. *If $\mathbf{k}$ is a finite field, then there are quadratic power series Θ such that*

$$\liminf_{\deg q \to +\infty} |q|^2 \cdot \|q\Theta\| \cdot |q|_X = 1/|X|.$$

Theorem 3, established in Section 5, shows that the analogue of Ridout's Theorem [23] is far from being true in positive characteristic.

Davenport and Lewis [11] proved that the analogue of the Littlewood conjecture does not hold in fields of power series when the ground field is infinite. We refer to [3] for further results on the Littlewood conjecture in fields of power series. We only mention that, for $\mathbf{k} = \mathbf{R}$, Baker [4] showed that

$$|q| \cdot ||qe^{1/X}|| \cdot ||qe^{2/X}|| \geq 2^{-5} \quad \text{for all } q \in \mathbf{R}[X] \setminus \{0\},$$

a result subsequently extended by several authors [7, 9, 10, 16].

As for the de Mathan–Teulié conjecture, we have a similar explicit statement.

Theorem 4. *If the field $\mathbf{k}$ is the real field, then*

$$|q| \cdot \|qe^{1/X}\| \cdot |q|_X \geq 1/|X|, \quad \textit{for all } q \in \mathbf{R}[X] \setminus \{0\}.$$

3. Proof of Theorem 1

We begin with an auxiliary lemma.

Lemma 1. *Let $\Theta = \sum_{h=m}^{+\infty} a_h X^{-h}$ be in $\mathbf{k}((X^{-1}))$, where $m \geq 1$. We have*

$$|q| \cdot \|q\Theta\| \geq 1/|X|, \quad \textit{for all } q \in \mathbf{k}[X] \setminus \{0\}, \tag{3.1}$$

if, and only if, for all $n \geq 1$, the determinant

$$H_n(a_1, \ldots, a_{2n-1}) = \begin{vmatrix} a_1 & a_2 & \ldots & a_n \\ a_2 & a_3 & \ldots & a_{n+1} \\ \ldots & \ldots & \ldots & \ldots \\ a_n & a_{n+1} & \ldots & a_{2n-1} \end{vmatrix} \tag{3.2}$$

is non-zero.

For sake of completeness, we give below a proof of Lemma 1, although we could refer to [20].

Proof. Condition (3.1) means that, for any positive integer n, the inequality

$$\|q\Theta\| \leq |X|^{-n-1} \tag{3.3}$$

has no solution with q being a non-zero polynomial of degree less than n. Writing q as

$$q = x_0 + \ldots + x_{n-1}X^{n-1},$$

we see that (3.3) holds if, and only if, the system

$$x_0 a_\ell + x_1 a_{\ell+1} + \ldots + x_{n-1} a_{\ell+n-1} = 0, \quad 1 \leq \ell \leq n,$$

has a solution. Consequently, Condition (3.1) means that this system has no non-trivial solution, that is, that its determinant is non-zero. This proves the lemma. □

Lemma 2. *Let $\mathcal{P}$ be a set of non-zero polynomials in $\mathbf{k}[X]$ such that, for any positive integer d, there exist only finitely many polynomials $p \in \mathcal{P}$ of degree d. Then, there exists $\Theta \in \mathbf{k}((X^{-1}))$ such that, for any $p \in \mathcal{P}$, the power series $p\Theta$ satisfies*

$$|q| \cdot \|qp\Theta\| \geq 1/|X|, \quad \textit{for all } q \in \mathbf{k}[X] \setminus \{0\}.$$

Proof. By successive applications of Lemma 1, we construct inductively $a_1, \ldots, a_n, \ldots$ in $\mathbf{k}$ such that the power series

$$\Theta = \sum_{n=1}^{+\infty} a_n X^{-n}$$

satisfies the conclusion of the lemma.

For a polynomial

$$p = b_0 + \ldots + b_d X^d$$

of degree d, let $c_1(p), \ldots, c_n(p), \ldots$ be the elements of $\mathbf{k}$ defined by

$$p \cdot \Theta - \sum_{h=1}^{+\infty} c_h(p) X^{-h} \in \mathbf{k}[X]. \tag{3.4}$$

Clearly,

$$c_h(p) = \sum_{k=0}^{d} a_{k+h} b_k, \quad \text{for } h \geq 1.$$

We proceed by induction. Let n be a positive integer and assume that there exist $a_1, \ldots, a_n$ such that, for every $p \in \mathcal{P}$ and every positive integer m with

$$2m - 1 + \deg p \leq n,$$

we have

$$H_m(c_1(p), \ldots, c_{2m-1}(p)) \neq 0,$$

where H_m is given by (3.2). This is clearly the case for $n = 1$ as soon as we choose $a_1 \neq 0$ (the condition is empty if $\mathcal{P}$ does not contain a constant polynomial). To prove that the assumption holds for the integer $n+1$, we have to select a_{n+1} such that

$$H_m(c_1(p), \ldots, c_{2m-1}(p)) \neq 0$$

for every $p \in \mathcal{P}$ and every positive integer m such that

$$2m - 1 + \deg p = n + 1,$$

the condition being already satisfied when $2m-1+\deg p \leq n$ since $H_m(c_1(p), \ldots, c_{2m-1}(p))$ is then a polynomial in $a_1, \ldots, a_n$. For each pair (m, p) composed of a positive integer m and a polynomial $p \in \mathcal{P}$ such that $2m - 1 + \deg p = n + 1$, we see, by expanding the determinant $H_m(c_1(p), \ldots, c_{2m-1}(p))$ along the last column, that, for $m \geq 2$, this polynomial in $a_1, \ldots, a_{n+1}$ is of the form

$$H_m(c_1(p), \ldots, c_{2m-1}(p)) = c_{2m-1}(p) H_{m-1}(c_1(p), \ldots, c_{2m-3}(p)) + R(a_1, \ldots, a_n),$$

thus

$$H_m(c_1(p), \ldots, c_{2m-1}(p)) = b_d a_{n+1} H_{m-1}(c_1(p), \ldots, c_{2m-3}(p)) + S(a_1, \ldots, a_n),$$

where R and S are polynomials in $a_1, \ldots, a_n$. If $m = 1$, then $d = n$, and we simply have

$$H_1(c_1(p)) = c_1(p) = \sum_{k=0}^{d} a_{k+1} b_k = b_d a_{n+1} + b_{d-1} a_n + \ldots + b_0 a_1.$$

By induction, we already have

$$H_{m-1}(c_1(p), \ldots, c_{2m-3}(p)) \neq 0.$$

Consequently, to get

$$H_m(c_1(p), \ldots, c_{2m-1}(p)) \neq 0$$

it is sufficient to choose

$$a_{n+1} \neq \lambda_{m,p}$$

where

$$\lambda_{m,p} = -\frac{S(a_1, \ldots, a_n)}{b_d H_{m-1}(c_1(p), \ldots, c_{2m-3}(p))}$$

if $m \geq 2$, and

$$\lambda_{1,p} = -\frac{b_{d-1} a_n + \ldots + b_0 a_1}{b_d}.$$

But there are only finitely many pairs (m, p) composed of a positive integer m and a polynomial $p \in \mathcal{P}$ such that $2m - 1 + \deg P = n + 1$. Since the field $\mathbf{k}$ is infinite, we can select a_{n+1} such that

$$a_{n+1} \neq \lambda_{m,p},$$

for each of these pairs. To summarize, we have constructed Θ such that

$$H_m(c_1(p), \ldots, c_{2m-1}(p)) \neq 0, \quad \text{for all } m \geq 1 \text{ and all } p \in \mathcal{P}.$$

By (3.4) and Lemma 1, this shows that, for every $p \in \mathcal{P}$, the power series $p\Theta$ satisfies

$$|q| \cdot \|qp\Theta\| \geq 1/|X|, \quad \text{for all } q \in \mathbf{k}[X] \setminus \{0\}.$$

This completes the proof of the lemma. □

Proof of Theorem 1. We apply Lemma 2 with $\mathcal{P}$ being the set of polynomials of the form $p_1^{m_1} \ldots p_r^{m_r}$, where $m_i \geq 0$ for $i = 1, \ldots, r$. Consequently, for all non-negative integers $m_1, \ldots, m_r$, we have

$$|q| \cdot \|qp_1^{m_1} \ldots p_r^{m_r}\Theta\| \geq 1/|X|, \quad \text{for all } q \in \mathbf{k}[X] \setminus \{0\},$$

that is,

$$|q| \cdot \|q\Theta\| \cdot |q|_{p_1} \ldots |q|_{p_r} \geq 1/|X|, \quad \text{for all } q \in \mathbf{k}[X] \setminus \{0\}.$$

This completes the proof of Theorem 1. □

4. Proof of Theorem 2

Our proof rests on the Folding Lemma for continued fraction expansions of formal power series (see Propositions 2 and 3 of [22]), recalled below.

Lemma F. *If $p_n/q_n = [0; a_1, a_2, \ldots, a_n]$ with $\deg a_i \geq 1$ for $i = 1, \ldots, n$, then, for every non constant polynomial t, we have*

$$\frac{p_n}{q_n} + \frac{(-1)^n}{tq_n^2} = [a_0; a_1, a_2, \ldots, a_n, t, -a_n, -a_{n-1}, \ldots, -a_2, -a_1].$$

Proof of Theorem 2. Let p be an irreducible, monic, non constant polynomial. Observe that the continued fraction of $1/p$ reads

$$\theta_1 := \frac{1}{p} = [0; p].$$

Let $(a_j)_{j \geq 2}$ be a sequence of nonzero elements of $\mathbf{k}$. By repeated applications of Lemma F with $t = \lambda p$, for suitable $\lambda \in \mathbf{k}^\times$, we see that, for any $j \geq 2$, the continued fraction of the rational function

$$\theta_j := \frac{1}{p} + \frac{a_2}{p^3} + \frac{a_3}{p^{2^3-1}} + \ldots + \frac{a_j}{p^{2^j-1}}$$

has all its partial quotients equal to p times an element of $\mathbf{k}^\times$.

Set

$$\Theta = \lim_{j \to +\infty} \theta_j = \frac{1}{p} + \sum_{j \geq 2} \frac{a_j}{p^{2^j - 1}}.$$

By construction, the degree of every partial quotients of Θ is equal to the degree of p, hence Θ is in ***Bad***. Furthermore, it is easily checked that, for any $j \geq 2$, we have

$$|p^{2^j}| \cdot ||p^{2^j - 1}\Theta|| = 1.$$

Since $|p|_p = |p|^{-1}$, this implies that

$$\liminf_{q \to +\infty} |q|^2 \cdot \|q\Theta\| \cdot |q|_p = 1/|p|, \tag{4.1}$$

as asserted.

Clearly, this gives uncountably many power series Θ with (4.1), unless $\mathbf{k}$ is the field $\mathbf{F}_2$. In the latter case, to get uncountably many power series satisfying

$$\liminf_{q \to +\infty} |q|^2 \cdot \|q\Theta\| \cdot |q|_p = 1/|p|^2,$$

we proceed as above, except that we use repeated applications of Lemma F with $t = p$ or $t = p^2$ (in such a way that $t = p^2$ is taken infinitely often). Observe also that, if the characteristic of $\mathbf{k}$ is 2, then the power series

$$\Theta = \sum_{j \geq 1} \frac{1}{p^{2^j - 1}}$$

satisfies (4.1) and the quadratic equation

$$\Theta^2 + p\Theta + 1 = 0.$$

If $\mathbf{k}$ is a finite field of characteristic different from 2, then it is possible to prove that any power series Θ as above is transcendental. The same conclusion follows from the analogue of the Schmidt Subspace Theorem if $\mathbf{k}$ has characteristic zero. $\square$

5. Further examples in positive characteristic

As noted after the statement of Theorem 2, Theorem 3 for finite fields of characteristic 2 has been established in the course of the proof of Theorem 2.

Throughout this section, p is an odd prime number and $q = p^s$ is a power of p. We consider power series over the field $\mathbf{k} = \mathbf{F}_q$.

Lemma 3. *Let a be in $\mathbf{F}_q^*$, and set $\Theta = \left(\frac{X}{X+a}\right)^{1/2}$ with $|\Theta - 1| \le 1/|X|$. Then we have*

$$\left|\Theta - \left(\frac{X+a}{X}\right)^{(p^n-1)/2}\right| = |X|^{-p^n}.$$

Proof. Observe that

$$\begin{aligned} &\left|\left(\frac{X+a}{X}\right)^{(p^{n+1}-1)/2} - \left(\frac{X+a}{X}\right)^{(p^n-1)/2}\right| \\ &= \left|\frac{X+a}{X}\right|^{(p^n-1)/2} \cdot \left|\left(\frac{X+a}{X}\right)^{p^n(p-1)/2} - 1\right|. \end{aligned} \tag{5.1}$$

We infer from $\left|\frac{X+a}{X}\right| = 1$ and $\gcd(p, (p-1)/2) = 1$ that

$$\left|\left(\frac{X+a}{X}\right)^{(p-1)/2} - 1\right| = \left|\frac{X+a}{X} - 1\right| = \frac{1}{|X|},$$

which, combined with (5.1), gives

$$\left|\left(\frac{X+a}{X}\right)^{(p^{n+1}-1)/2} - \left(\frac{X+a}{X}\right)^{(p^n-1)/2}\right| = \frac{1}{|X|^{p^n}}.$$

Thus, the sequence $((X+a)/X)^{(p^n-1)/2}$ converges to a limit Θ such that $|\Theta - 1| \le 1/|X|$ and $\Theta^2 = \lim_{n\to+\infty}((X+a)/X)^{p^n-1} = X/(X+a)$. □

We point out a consequence of Lemma 3. We keep the assumption of that lemma. Let a and b be distinct elements of $\mathbf{F}_q^*$. The power series

$$\Theta = \left(\frac{X}{X+a}\right)^{1/2} \quad \text{and} \quad \Phi = \left(\frac{X}{X+b}\right)^{1/2}$$

are algebraic of degree two, hence they are badly approximable by rational fractions. Furthermore, 1, Θ and Φ are linearly independent over $\mathbf{F}_q(X)$. Indeed, Φ does not belong to $\mathbf{F}_q(\Theta)$, since $\frac{X+a}{X+b}$ is not a square in $\mathbf{F}_q(X)$. However, for every positive integer n, the polynomial $R_n(X) = X^{(p^n-1)/2}$ satisfies

$$|R_n| \cdot ||R_n\Theta|| = |R_n| \cdot ||R_n\Phi|| = \frac{1}{|X|},$$

thus

$$|R_n|^2 \cdot ||R_n\Theta|| \cdot ||R_n\Phi|| = \frac{1}{|X|^2}.$$

We further have

$$|R_n|_X \cdot |R_n|^2 \cdot ||R_n \Theta|| = \frac{1}{|X|},$$

i.e., the power series Θ satisfies (2.2).

We refer the reader to [24] for further similar results.

6. Proof of Theorem 4

Since

$$e^{1/X} = 1 + \sum_{k \geq 1} \frac{1}{k!} X^{-k},$$

it is sufficient by Lemma 1 to prove that, for any positive integers k and n, the determinant

$$D_{n,k} = \begin{vmatrix} 1/k! & 1/(k+1)! & \dots & 1/(k+n)! \\ 1/(k+1)! & 1/(k+2)! & \dots & 1/(k+n+1)! \\ \dots & \dots & \dots & \dots \\ 1/(k+n)! & 1/(k+n+1)! & \dots & 1/(k+2n)! \end{vmatrix}$$

is non-zero. We do this by induction on n. Observe first that $D_{0,k}$ is nonzero for every positive integer k.

Let n and k be positive integers. Assume that $D_{n-1,\ell}$ is nonzero for every positive integer ℓ. Observe that

$$D_{n,k} = \frac{1}{(k+n)! \dots (k+2n)!} \, \delta_{n,k},$$

where

$$\delta_{n,k} = \begin{vmatrix} (k+n)\dots(k+1) & (k+n)\dots(k+2) & \dots & k+n & 1 \\ (k+n+1)\dots(k+2) & (k+n+1)\dots(k+3) & \dots & k+n+1 & 1 \\ \dots & \dots & \dots & \dots & \\ (k+2n)\dots(k+n+1) & (k+2n)\dots(k+n+2) & \dots & k+2n & 1 \end{vmatrix}$$

For $j = n+1, \dots, 2$, we subtract the $(j-1)$th line from the jth line to get that

$$\delta_{n,k} = \begin{vmatrix} (k+n)\dots(k+1) & (k+n)\dots(k+2) & \dots & k+n & 1 \\ n(k+n)\dots(k+2) & (n-1)(k+n)\dots(k+3) & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \\ n(k+2n-1)\dots(k+n+1) & (n-1)(k+2n-1)\dots(k+n+2) & \dots & 1 & 0 \end{vmatrix}$$

$$= (-1)^n n! \times \delta_{n-1,k+1} = (-1)^n n! \times (k+n)! \dots (k+2n-1)! \times D_{n-1,k+1}.$$

Since, by assumption, $D_{n-1,k+1}$ is nonzero, we have established that $D_{n,k}$ is also nonzero. This concludes the proof of the theorem.

7. Open questions

Throughout this section, p denotes a prime number. We begin with open problems motivated by the (usual) de Mathan–Teulié conjecture, and then we continue with questions suggested by its analogue in fields of power series.

Theorem EK2 motivates the following problems.

Problem 1. *Let $r \geq 2$ be an integer and let $p_1, \ldots, p_r$ be distinct prime numbers. Does there exist a non-decreasing function $\varepsilon_r : q \mapsto \varepsilon_r(q)$ that tends to infinity with q and satisfies*

$$\liminf_{q \to +\infty} q \cdot \|q\alpha\| \cdot |q|_{p_1} \cdots |q|_{p_r} \cdot \varepsilon_r(q) = 0$$

for every real number α?

Dirichlet's Theorem asserts that if α is a given real number, then, for any positive integer Q, there exists an integer q satisfying

$$1 \leq q \leq Q \quad \text{and} \quad Q \cdot ||q\alpha|| < c,$$

with $c = 1$. This result holds true for a real number α with a constant c strictly less than 1 if, and only if, α is badly approximable (see Davenport and Schmidt [12] for a precise statement). In view of this result, we propose the following open question.

Problem 2. *Let p be a prime number. Do there exist a real number α and a positive real number ε such that, for any sufficiently large positive integer Q, there exists an integer q satisfying*

$$1 \leq q \leq Q \quad \text{and} \quad Q \cdot ||q\alpha|| \cdot |q|_p < Q^{-\varepsilon}\,?$$

If the latter statement does not hold, then characterize the real numbers α such that, for any positive integer Q, there exist an integer q satisfying

$$1 \leq q \leq Q \quad \text{and} \quad Q \cdot ||q\alpha|| \cdot |q|_p < c,$$

with some $c < 1$.

Now, we turn to the power series case.

Problem 3. *Does the analogue of Theorem EK2 hold for power series over finite fields?*

Problem 4. *Does the analogue of Theorem BDM hold for power series over an arbitrary field?*

As pointed out in [20], even particular cases of the analogue of the de Mathan–Teulié conjecture over a finite field remain mysterious. For instance, we do not know whether, for a prime power $q \geq 4$, there exists a sequence $(a_h)_{h \geq 1}$ of elements of $\mathbf{F}_q$ for which each determinant

$$\Delta(m,n) := \begin{vmatrix} a_m & a_{m+1} & \ldots & a_{m+n} \\ a_{m+1} & a_{m+2} & \ldots & a_{m+n+1} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m+n} & a_{m+n+1} & \ldots & a_{m+2n} \end{vmatrix} \tag{3.2}$$

is non-zero.

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