# Transcendence measures for continued fractions involving repetitive or symmetric patterns

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## 1. Introduction

It was observed long ago (see e.g., [29] or [18], page 62) that Roth's theorem [24] and its $p$-adic extension established by Ridout [25] can be used to prove the transcendence of real numbers whose expansion in some integer base contains repetitive patterns. This was properly written only in 1997, by Ferenczi and Mauduit [19], who adopted a point of view from combinatorics on words before applying the above mentioned theorems from Diophantine approximation to establish e.g., the transcendence of numbers with a low complexity expansion. Their combinatorial transcendence criterion was subsequently considerably improved in [9], where the authors used the multidimensional extension of Roth's theorem established by W. M. Schmidt, commonly referred to as the Schmidt Subspace Theorem [26]. As shown in [4], this powerful criterion has many applications and yields among other things the transcendence of irrational real numbers whose expansion in some integer base can be generated by a finite automaton. The latter result was generalized in [7], where we give transcendence measures for a large class of real numbers whose transcendence was proved in [4]. The key ingredient for the proof is then the Quantitative Subspace Theorem [28], and we describe in [7] a general method that allows us in principle to get transcendence measures for many real numbers that are proved to be transcendental by an application of Roth's or Schmidt's theorem.

Besides expansions in integer bases, a classical way to represent a real number is by its continued fraction expansion. Again by means of the Schmidt Subspace Theorem, new classes of transcendental continued fractions were constructed in [1, 5, 6, 8, 16]. It is the purpose of the present work to show how the Quantitative Subspace Theorem yields transcendence measures for (most of) these transcendental continued fractions $\xi$, following the approach from [7]. These measures allow us to locate $\xi$ in the classification of real numbers defined in 1932 by Mahler [20] and recalled below.

For every integer $d \geq 1$ and every real number $\xi$, we denote by $w_d(\xi)$ the supremum of the exponents $w$ for which

$$0 < |P(\xi)| < H(P)^{-w}$$

has infinitely many solutions in integer polynomials $P(X)$ of degree at most $d$. Here, $H(P)$ stands for the naïve height of the polynomial $P(X)$, that is, the maximum of the absolute values of its coefficients. Further, we set $w(\xi) = \limsup_{d\to\infty}(w_d(\xi)/d)$ and, according to Mahler [20], we say that $\xi$ is an

$$S\text{-number, if } w(\xi) < \infty;$$
$$T\text{-number, if } w(\xi) = \infty \text{ and } w_d(\xi) < \infty \text{ for any integer } d \geq 1;$$
$$U\text{-number, if } w(\xi) = \infty \text{ and } w_d(\xi) = \infty \text{ for some integer } d \geq 1.$$

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In the sense of the Lebesgue measure, almost all numbers are $S$-numbers. Liouville numbers are examples of $U$-numbers, but the existence of $T$-numbers remained an open problem during nearly forty years, until it was confirmed by Schmidt, see Chapter 3 of [15] for references and further results. The set of $U$-numbers can be further divided in countably many subclasses according to the value of the smallest integer $d$ for which $w_d(\xi)$ is infinite.

**Definition 1.1.** *Let $\ell \geq 1$ be an integer. A real number $\xi$ is a $U_\ell$-number if and only if $w_\ell(\xi)$ is infinite and $w_d(\xi)$ is finite $d = 1, \ldots, \ell - 1$.*

To give a flavour of the results proved in the present paper, we quote below a theorem established in 1962 by A. Baker [13].

**Theorem (A. Baker.)** *Consider a quasi-periodic continued fraction*

$$\xi = [a_0, a_1, \ldots a_{n_0-1}, \underbrace{a_{n_0}, \ldots, a_{n_0+k_0-1}}_{\lambda_0 \ \ times}, \underbrace{a_{n_1}, \ldots, a_{n_1+k_1-1}}_{\lambda_1 \ \ times}, \ldots],$$

*where the notation implies that $n_i = n_{i-1} + \lambda_{i-1} k_{i-1}$ and the $\lambda$'s indicate the number of times a block of partial quotients is repeated. Suppose that the sequences $(a_n)_{n\geq 0}$ and $(k_n)_{n\geq 0}$ are bounded by $M$. Set*

$$L = \limsup_{i\to+\infty} \lambda_i/\lambda_{i-1}, \quad \ell = \liminf_{i\to+\infty} \lambda_i/\lambda_{i-1}.$$

*If $L$ is infinite and $\ell > 1$, then $\xi$ is a $U_2$-number. Furthermore, there exists a (large) real number $C$, depending only on $M$, such that if $L$ is finite and $\ell > C$, then $\xi$ is either an $S$-number or a $T$-number.*

A. Baker's theorem shows that the above quasi-periodic continued fractions for which $\ell$ is sufficiently large cannot include $U_d$-numbers with $d \geq 3$, that is, there is a gap in the type of transcendental numbers given by them. **(J'ai recopié Baker!)**

In Corollary 3.2 below, we obtain the same conclusion as Baker, but with the assumption $\ell > C$ replaced by the much weaker one $\ell > 1$.

Besides the quasi-periodic continued fractions studied by Baker, our method also applies to continued fractions involving repetitive patterns and symmetric patterns. An emblematic example is given by the Thue–Morse continued fraction

$$\xi_{\mathbf{t}} = [1, 2, 2, 1, 2, 1, 1, 2, \ldots],$$

whose sequence of partial quotients is the Thue–Morse infinite word on the alphabet $\{1, 2\}$. The fact that $\xi_{\mathbf{t}}$ is transcendental was established by M. Queffélec [23] quite recently. We strengthen this result by proving that $\xi_{\mathbf{t}}$ is either an $S$-number or a $T$-number.

## 2. Transcendence measures for purely stammering continued fractions

Throughout the present text, we adopt the point of view from combinatorics on words. Let $\mathcal{A}$ be a given set, not necessarily finite. The length of a word $W$ on the alphabet $\mathcal{A}$, that is, the number of letters composing $W$, is denoted by $|W|$. For any positive integer $\ell$, we write $W^\ell$ for the word $W \ldots W$ ($\ell$ times repeated concatenation of the word $W$). We denote by $W^\infty$

the infinite word obtained by concatenation of infinitely many copies of $W$. For any positive rational number $x$, we denote by $W^x$ the word $W^{\lfloor x\rfloor}W'$, where $W'$ is the prefix of $W$ of length $\lceil (x-\lfloor x\rfloor)|W|\rceil$. Here, and in all what follows, $\lfloor y\rfloor$ and $\lceil y\rceil$ denote, respectively, the integer part and the upper integer part of the real number $y$.

Let $\mathbf{a}=(a_n)_{n\geq 0}$ be an infinite word. For every positive integer $n$, let $w_n$ be the largest rational number such that $(a_0\ldots a_n)^{w_n}$ is a prefix of $\mathbf{a}$. Then, the *initial critical exponent* of $\mathbf{a}$, denoted by $\mathrm{ice}(\mathbf{a})$, is by definition equal to the supremum of the sequence $(w_n)_{n\geq 1}$. Clearly, $\mathrm{ice}(\mathbf{a})$ is always at least equal to 1 and is infinite if $\mathbf{a}$ is a purely periodic sequence.

### 2.1. Main results

Let $w>1$ be a real number. We say that a sequence $\mathbf{a}=(a_n)_{n\geq 0}$ satisfies Condition $(*)_{\widehat{w}}$ if $\mathbf{a}$ is not eventually periodic and if there exists an infinite sequence of finite words $(V_n)_{n\geq 1}$ such that:

(i) for every integer $n\geq 1$, the word $V_n^w$ is a prefix of the word $\mathbf{a}$;

(ii) the sequence $(|V_n|)_{n\geq 1}$ is (strictly) increasing;

(iii) the sequence $(|V_{n+1}|/|V_n|)_{n\geq 1}$ is bounded.

We establish the following result.

**Theorem 2.1.** *Let $\mathbf{a}=(a_\ell)_{\ell\geq 0}$ be a sequence of positive integers satisfying Condition $(*)_{\widehat{w}}$ for some $w>1$. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ be the sequence of convergents to the real number*

$$\xi := [a_0, a_1, a_2, \ldots],$$

*and assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded. If $\mathrm{ice}(\mathbf{a})$ is finite, then $\xi$ is either an $S$-number or a $T$-number; otherwise, $\xi$ is a $U_2$-number. Moreover, if $\mathrm{ice}(\mathbf{a})$ is finite, then there exists a constant $c$ independent of $d$ such that*

$$w_d(\xi)\leq \exp(c\,(\log 3d)^3\,(\log\log 3d)^2), \quad (d\geq 1). \tag{2.1}$$

The fact that the real number $\xi$ is transcendental when the sequence $\mathbf{a}$ only satifies assumptions (i) and (ii) in the definition of Condition $(*)_{\widehat{w}}$ is the main result of [1].

In the case where $w\geq 2$ in Theorem 2.1, we are actually able to derive a better transcendence measure, namely to replace (2.1) by

$$w_d(\xi)\leq \exp(c\,(\log 3d)^2\,(\log\log 3d)), \quad (d\geq 1). \tag{2.2}$$

### 2.2. Palindromic continued fractions

Denote the mirror image of a finite word $W:=a_1\ldots a_n$ by $\overline{W}:=a_n\ldots a_1$. In particular, $W$ is a palindrome if and only if $W=\overline{W}$.

We say that a sequence $\mathbf{a}=(a_n)_{n\geq 0}$ satisfies Condition $(*)_{\widehat{sym}}$ if $\mathbf{a}$ is not eventually periodic and if there exist two sequences of finite words $(U_n)_{n\geq 1}$ and $(V_n)_{n\geq 1}$ such that:

(i) For any $n \geq 1$, the word $U_n V_n \overline{U}_n$ is a prefix of the word $\mathbf{a}$;
(ii) The sequence $(|V_n|/|U_n|)_{n\geq 1}$ is bounded;
(iii) The sequence $(|U_{n+1}|/|U_n|)_{n\geq 1}$ is bounded;
(iv) The sequence $(|U_n|)_{n\geq 1}$ is increasing.

We establish the following analog of Theorem 2.1 where occurrences of repetitive patterns arising from Condition $(*)_{\widehat{w}}$ are replaced by those of symmetric patterns arising from Condition $(*)_{\widehat{sym}}$. **(C'est plutôt un cas particulier du Th. 2.1 obtenu à l'aide de la dém. du Th. 4.6 de [3].)**

**Theorem 2.2.1.** *Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of positive integers satisfying Condition $(*)_{\widehat{sym}}$. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ be the sequence of convergents to the real number*

$$\xi := [a_0, a_1, a_2, \ldots],$$

*and assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded. If $\mathrm{ice}(\mathbf{a})$ is finite, then $\xi$ is either an $S$-number or a $T$-number; otherwise, $\xi$ is a $U_2$-number.*

**Theorem 2.2.2.** *Let $\mathbf{a} = (a_\ell)_{\ell\geq 0}$ be a non-periodic sequence of positive integers. Assume that $\mathbf{a}$ has a positive palindromic density. Then, the real number $\xi$ satisfies the transcendence measure given in (2.2) and is thus either an $S$-number or a $T$-number.*

### 2.3. Applications to Sturmian and morphic continued fractions

In this subsection, we point out some consequences of Theorem 2.1 to various classes of continued fractions.

Let $\alpha$ be an irrational real number in $(0,1)$. A sequence $(a_n)_{n\geq 0}$ on the alphabet $\{a, b\}$ is a Sturmian sequence of slope $\alpha$ is there exist a real number $\rho$ such that

$$a_n = a \text{ if } \lfloor (n+1)\alpha + \rho \rfloor - \lfloor n\alpha + \rho \rfloor = 0 \text{ and } a_n = b \text{ if } \lfloor (n+1)\alpha + \rho \rfloor - \lfloor n\alpha + \rho \rfloor = 1, \quad (n \geq 0),$$

or

$$a_n = a \text{ if } \lceil (n+1)\alpha + \rho \rceil - \lceil n\alpha + \rho \rceil = 0 \text{ and } a_n = b \text{ if } \lceil (n+1)\alpha + \rho \rceil - \lceil n\alpha + \rho \rceil = 1, \quad (n \geq 0).$$

**Theorem 2.3.1.** *Let $a$ and $b$ be two distinct positive integers. Let $\mathbf{a} = (a_\ell)_{\ell\geq 0}$ be a Sturmian sequence of slope $\alpha$ on the alphabet $\{a, b\}$. Then, the real number*

$$\xi := [a_0, a_1, a_2, \ldots]$$

*is a $U_2$-number if and only if $\alpha$ has unbounded partial quotients. In the case where $\alpha$ has bounded partial quotient, $\xi$ satisfies the transcendence measure given in (2.1) and is thus either an $S$-number or a $T$-number.*

**Pour ne pas trop rallonger la sauce, on peut se contenter de renvoyer à Acta. Math. pour les définitions des notions utilisées ci-dessous.**

We refer the reader to [1] for the definitions of the notions occurring in the remaining of this section. Since the initial critical exponent of a non-periodic fixed point of a recurrent (resp. binary, linearly recurrent) morphism is finite, next statements follow straightforwardly from Theorem 2.1.

**Theorem 2.3.2.** *Let $\sigma$ be a morphism defined over a finite subset of positive integers. Let $\mathbf{a} = (a_\ell)_{\ell \geq 0}$ be a non-periodic recurrent fixed point for $\sigma$ and*

$$\xi := [a_0, a_1, a_2, \ldots, a_\ell, \ldots].$$

*Then, $\xi$ satisfies the transcendence measure given in (2.1) and is thus either an $S$-number or a $T$-number.*

**Corollary 2.3.3.** *Let $a$ and $b$ be two distinct positive integers. Let $\sigma$ be a binary morphism defined over $\{a, b\}$. Let $\mathbf{a} = (a_\ell)_{\ell \geq 0}$ be a non-periodic fixed point for $\sigma$ and*

$$\xi := [a_0, a_1, a_2, \ldots, a_\ell, \ldots].$$

*Then, $\xi$ satisfies the transcendence measure given in (2.1) and is thus either an $S$-number or a $T$-number.*

As a particular case of Corollary 2.3.3, a Thue–Morse continued fraction is either an $S$-number or a $T$-number.

**Theorem 2.3.4.** *Let $\mathbf{a} = (a_\ell)_{\ell \geq 0}$ be a linearly recurrent sequence of positive integers. Then, the real number*

$$\xi := [a_0, a_1, a_2, \ldots, a_\ell, \ldots]$$

*satisfies the transcendence measure given in (2.1) and is thus either an $S$-number or a $T$-number.*

## 3. Transcendence measures for Maillet–Baker continued fractions

In this section, we are interested in a class of quasi-periodic continued fractions introduced by Maillet [21] and studied by Baker [13, 14]. We consider a real number $\xi := [a_0, a_1, a_2, \ldots]$, where $\mathbf{a} = (a_n)_{n \geq 0}$ is a non-eventually periodic sequence of positive integers satisfying the following assumption:

There exists an increasing sequence of positive integers $(n_k)_{k \geq 0}$, a sequence of positive integers $(\lambda_k)_{k \geq 0}$, a finite sequence of positive integers $b_1, b_2 \ldots, b_r$, such that for every positive integer $k$, the sequence $\mathbf{a}$ begins with

$$a_0 a_1 \ldots a_{n_k - 1} \underbrace{BB \ldots B}_{\lambda_k \text{ times}}, \tag{3.1}$$

where $B := b_1 b_2 \ldots b_r$. It is understood here that

$$a_{n_k - r} a_{n_k - r + 1} \ldots a_{n_k - 1} \neq B. \tag{3.2}$$

We establish the following generalization of Baker's result.

**Theorem 3.1.** *Let $\xi := [a_0, a_1, a_2, \ldots]$ be defined as above. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to $\xi$ and assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded. Assume further that*

$$\liminf_{k\to\infty} \frac{\lambda_k}{n_k} > 0 \tag{3.3}$$

*and*

$$\limsup_{k\to\infty} \frac{n_{k+1}}{n_k} < +\infty. \tag{3.4}$$

*Then, there exist a constant $c$ independent of $d$ such that*

$$w_d(\xi) \leq \exp(c\,(\log 3d)^3\,(\log\log 3d)^2), \tag{3.5}$$

*for every positive integer $d$. In particular, $\xi$ is either an $S$-number or a $T$-number.*

The proof of this result also rely on the quantitative version of the Subspace Theorem given in Section 5.

Let us remark that a Maillet–Baker continued fraction satisfying

$$\limsup_{k\to\infty} \frac{\lambda_k}{n_k} = +\infty$$

corresponds to a $U_2$-number for it is extremly well approximated by the quadratic numbers

$$\alpha_k := [a_0, a_1, a_2, \ldots, a_{n_k}, B^\infty].$$

We note that such a situation cannot occur in Theorem 3.1 because of (3.4) and (3.2).

**Corollary 3.2.** *Let us consider the quasi-periodic continued fraction*

$$\xi = [a_0, a_1, \ldots, a_{n_0-1}, \underbrace{a_{n_0}, \ldots, a_{n_0+r_0-1}}_{\lambda_0 \text{ times}}, \underbrace{a_{n_1}, \ldots, a_{n_1+r_1-1}}_{\lambda_1 \text{ times}}, \ldots],$$

*where the notation implies that $n_{k+1} = n_k + \lambda_k r_k$ and the $\lambda$'s indicate the number of times a block of partial quotients is repeated. Denote by $(p_n/q_n)_{n\geq 0}$ the sequence of the convergents to $\xi$. Assume that the sequences $(q_n^{1/n})_{n\geq 0}$ and $(r_k)_{k\geq 0}$ are bounded, that $(a_n)_{n\geq 0}$ is not ultimately periodic, and that*

$$\liminf_{k\to\infty} \frac{\lambda_{k+1}}{\lambda_k} > 1.$$

*If, moreover,*

$$\limsup_{k\to\infty} \frac{\lambda_{k+1}}{\lambda_k} < +\infty,$$

*then the real number $\xi$ is either an $S$-number or a $T$-number.*

## 4. Auxiliary results

For the reader convenience, we first recall some classical results from the theory of continued fractions, whose proofs can be found for example in the book of Perron [22].

**Lemma 4.1.** *Let $\alpha = [a_0, a_1, a_2, \ldots]$ and $\beta = [b_0, b_1, b_2, \ldots]$ be two real numbers. Assume that there exists a positive integer $\ell$ such that $a_i = b_i$ for any $i = 0, \ldots, \ell$. We then have $|\alpha - \beta| \le q_\ell^{-2}$, where $q_\ell$ denotes the denominator of the $\ell$-th convergent to $\alpha$.*

**Lemma 4.2.** *Let $\alpha = [a_0, a_1, a_2, \ldots]$ be a real number with convergents $(p_\ell/q_\ell)_{\ell \ge 1}$. Then, for any $\ell \ge 2$, we have*

$$\frac{q_{\ell-1}}{q_\ell} = [0, a_\ell, a_{\ell-1}, \ldots, a_1].$$

For positive integers $a_1, \ldots, a_\ell$, denote by $K_\ell(a_1, \ldots, a_\ell)$ the denominator of the rational number $[0, a_1, \ldots, a_\ell]$. It is commonly called a *continuant*.

**Lemma 4.3.** *For any positive integers $a_1, \ldots, a_\ell$ and any integer $k$ with $1 \le k \le \ell - 1$, we have*

$$K_\ell(a_1, \ldots, a_\ell) = K_\ell(a_\ell, \ldots, a_1)$$

*and*

$$\begin{aligned} K_k(a_1, \ldots, a_k) \cdot K_{\ell-k}(a_{k+1}, \ldots, a_\ell) &\le K_\ell(a_1, \ldots, a_\ell) \\ &\le 2\, K_k(a_1, \ldots, a_k) \cdot K_{\ell-k}(a_{k+1}, \ldots, a_\ell). \end{aligned}$$

**Lemma 4.4.** *Let $(a_\ell)_{\ell \ge 1}$ be a sequence of positive integers at most equal to $M$. For any positive integer $\ell$, we have*

$$2^{(\ell-1)/2} \le K_\ell(a_1, \ldots, a_\ell) \le (M+1)^\ell.$$

We will also make use of the following three auxiliary results.

**Lemma 4.5.** *Let $\alpha = [a_0, a_1, a_2, \ldots]$ and $\beta = [b_0, b_1, b_2, \ldots]$ be two real numbers whose convergents are respectively denoted by $(p_n/q_n)_{n \ge 1}$ and $(r_n/s_n)_{n \ge 1}$. Assume that both sequences $(q_n^{1/n})_{n \ge 1}$ and $(s_n^{1/n})_{n \ge 1}$ are bounded by a real number $M$. Assume that there exists a positive integer $\ell$ such that $a_i = b_i$ for any $i = 0, \ldots, \ell$ and $a_{\ell+1} \ne b_{\ell+1}$. Then, there exists a positive real number $\mu$, depending only on $M$, such that*

$$|\alpha - \beta| \ge \frac{1}{q_\ell^\mu}.$$

*If we assume moreover that the partial quotients of $\beta$ are bounded by $M$, then there exists a positive real number $c$, depending only on $M$, such that*

$$|\alpha - \beta| > \frac{c}{q_\ell^2}.$$

**Proof.** We follow the proof of Lemma 5 from [2]. The constants implicit in $\ll$ and $\gg$ below depend only on $M$. Set $\alpha' = [a_{\ell+1}, a_{\ell+2}, \ldots]$ and $\beta' = [b_{\ell+1}, b_{\ell+2}, \ldots]$. Since $a_{\ell+1} \ne b_{\ell+1}$, we have

$$|\alpha' - \beta'| \ge 1 - [0, 1, M+1] \gg M^{-\ell}. \tag{4.1}$$

Furthermore, our assumption on $(q_n^{1/n})_{n\geq 1}$ and $(s_n^{1/n})_{n\geq 1}$ implies that

$$\alpha' \ll M^\ell \quad \text{and} \quad \beta' \ll M^\ell. \tag{4.2}$$

The theory of continued fractions gives that

$$\alpha = \frac{p_\ell \alpha' + p_{\ell-1}}{q_\ell \alpha' + q_{\ell-1}} \quad \text{and} \quad \beta = \frac{p_\ell \beta' + p_{\ell-1}}{q_\ell \beta' + q_{\ell-1}},$$

since the first $\ell$-th partial quotients of $\alpha$ and $\beta$ are assumed to be the same. We thus obtain

$$|\alpha - \beta| = \left| \frac{p_\ell \alpha' + p_{\ell-1}}{q_\ell \alpha' + q_{\ell-1}} - \frac{p_\ell \beta' + p_{\ell-1}}{q_\ell \beta' + q_{\ell-1}} \right| = \left| \frac{\alpha' - \beta'}{(q_\ell \alpha' + q_{\ell-1})(q_\ell \beta' + q_{\ell-1})} \right|.$$

Together with (4.1) and (4.2), this yields

$$|\alpha - \beta| \gg M^{-3\ell}\, q_\ell^{-2}. \tag{4.3}$$

We infer from Lemma 4.4 that $q_\ell \geq 2^{(\ell-1)/2}$. Combined with (4.3), this gives the first assertion of our lemma.

For the second assertion, we proceed as above, noticing that

$$|\alpha' - \beta'| \geq 1 - [0, 1, M+1] = \frac{1}{M+2} \tag{4.4}$$

and

$$\left| \frac{\alpha' - \beta'}{q_\ell \alpha' + q_{\ell-1}} \right| \geq \frac{1}{4(M+2)^2}. \tag{4.5}$$

Since $\beta' \leq M+1$, inequalities (4.4) and (4.5) yield the second assertion of the lemma. □

**Lemma 4.6.** *Let $\xi$ be a real number and $(p_\ell/q_\ell)_{\ell\geq 1}$ be the sequence of convergents to $\xi$. If the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded, then $w_1(\xi)$ is finite, that is, $\xi$ is not a Liouville number.*

**Proof.** Let $A$ be a positive real number such that $q_\ell < A^\ell$ for every positive integer $\ell$. By Lemma 4.4 we have $q_\ell > \sqrt{2}^{\ell-1}$ and thus there exists a positive real number $w > 1$ such that

$$q_{\ell+1} < q_\ell^w,$$

for every positive integer $\ell$. By the theory of continued fractions,

$$\left| \xi - \frac{p_\ell}{q_\ell} \right| > \frac{1}{2 q_\ell q_{\ell+1}} > \frac{1}{2 q_\ell^{w+1}}$$

holds for every positive integer $\ell$. Since the best rational approximations to $\xi$ are in the sequence $(p_\ell/q_\ell)_{\ell\geq 1}$, we get $w_1(\xi) \leq w$. Consequently, $\xi$ is not a Liouville number. □

We further state a classical Liouville inequality which can be found for instance in [15].

**Lemma 4.7.** *Let $\beta$ be an algebraic number of degree $d$ and $P$ a non-constant integer polynomial of degree $r$. Then, if $P(\beta) \neq 0$, we have*

$$|P(\beta)| \geq \frac{1}{(r+1)^{d-1}(d+1)^{r/2}H(P)^{d-1}H(\beta)^r}.$$

*If $\alpha$ and $\beta$ are two non-zero distinct algebraic numbers of degree respectively equal to $n$ and $m$, then*

$$|\alpha - \beta| \geq \frac{2\max\{2^{-n}(n+1)^{-(m-1)/2},\, 2^{-m}(m+1)^{-(n-1)/2}\}}{(n+1)^{m/2}(m+1)^{n/2}\, H(\alpha)^m\, H(\beta)^n}.$$

Our last lemma gives a bound for the height of a quotient of algebraic numbers.

**Lemma 4.8.** *Let $\alpha$ be a real algebraic number of degree $d \geq 2$. Let $x_1, x_2, x_3, x_4$ be integers with $(x_3, x_4) \neq (0,0)$ and of absolute values at most $A$. Then, the height of the algebraic number $\beta = (x_1\alpha + x_2)/(x_3\alpha + x_4)$ is at most equal to $2^{3(d+1)}H(\alpha)^2\, A^{2d}$.*

**Proof.** Using classical inequalities between the naïve height $H$, the Mahler measure $M$ and the logarithmic Weyl height h, we get that

$$\begin{aligned} H(\beta) \leq 2^d\, M(\beta) &\leq 2^d\, e^{d\mathrm{h}(\beta)} \leq 2^d e^{d\mathrm{h}(x_1\alpha+x_2)}\, e^{d\mathrm{h}(x_3\alpha+x_4)} \\ &\leq 2^d M(x_1\alpha + x_2)\, M(x_3\alpha + x_4) \\ &\leq 2^d(d+1)H(x_1\alpha + x_2)H(x_3\alpha + x_4) \\ &\leq 2^{3(d+1)}H(\alpha)^2\, A^{2d}, \end{aligned}$$

by Lemma A.4 from [15]. □

## 5. The quantitative Subspace Theorem

The proofs of our result rely on the following quantitative version of the Schmidt Subspace Theorem. This statement is due to Evertse [17].

**Theorem Ev.** *Let $m \geq 2$, $H$ and $d$ be positive integers. Let $L_1, \ldots, L_m$ be linearly independent (over $\overline{\mathbf{Q}}$) linear forms in $m$ variables with algebraic coefficients. Assume that $H(L_i) \leq H$ and that the number field generated by all the coefficients of these linear forms has degree at most $d$. Let $\varepsilon$ be a real number with $0 < \varepsilon < 1$. Then, the primitive integer vectors $\mathbf{x}$ in $\mathbf{Z}^m$ with $H(\mathbf{x}) \geq H$ and such that*

$$\prod_{i=1}^{m} |L_i(\mathbf{x})| \; < \; |\det(L_1, L_2, \ldots, L_m)| H(\mathbf{x})^{-m-\varepsilon}$$

*lie in at most*

$$c_{m,\varepsilon}\, (\log 4d)\, (\log\log 4d) \tag{5.1}$$

*proper subspaces of $\mathbf{Q}^m$, where $c_{m,\varepsilon}$ is a constant which only depends on $m$ and $\varepsilon$.*

## 6. Proof of Theorem 2.1

This section is devoted to the proof of Theorem 2.1. We first show how the initial critical exponent allows us to control the approximation by quadratic numbers.

**Lemma 6.1.** *Let* $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ *be a non-periodic sequence of positive integers. Let*

$$\xi := [0, a_1, a_2, \ldots]$$

*be a real number and denote by* $(p_\ell/q_\ell)_{\ell \geq 1}$ *the sequence of convergents to* $\xi$. *Assume that the sequence* $(q_\ell^{1/\ell})_{\ell \geq 1}$ *is bounded and that* $\mathrm{ice}(\mathbf{a}) = +\infty$. *Then* $\xi$ *is a* $U_2$*-number.*

**Proof.** By assumption, the initial critical exponent of the sequence $\mathbf{a}$ is infinite. Consequently, for every positive integer $n$, there exists a finite word $V_n$ such that $\mathbf{a}$ begins with the word $V_n^n$. Set $\alpha_n = [0, V_n^\infty]$ and denote by $l_n$ the length of the word $V_n$. Since $\alpha_n$ is a root of the polynomial

$$P(X) := q_{l_n - 1} X^2 + (q_{l_n} - p_{l_n - 1}) X - p_{l_n}, \tag{6.1}$$

its height is at most equal to $q_{l_n}$. The first $nl_n$ partial quotients of $\xi$ and $\alpha_n$ being the same, we infer from Lemmas 4.1 and 4.3 that

$$|\xi - \alpha_n| < q_{nl_n}^{-2} \leq q_{l_n}^{-2n} \leq H(\alpha_n)^{-2n}. \tag{6.2}$$

Since $\mathbf{a}$ is a non-periodic sequence, the set $\{\alpha_n,\ n \geq 1\}$ is infinite, and (6.2) implies that $w_2(\xi) = +\infty$. By Lemma 4.6, this shows that $\xi$ is a $U_2$-number. □

Next lemma is essentially outlined at the end of [10]. **(Est-ce bien vrai?)**

**Lemma 6.2.** *Let* $\mathbf{a} = (a_\ell)_{\ell \geq 1}$ *be a sequence of positive integers satisfying Condition* $(*)_{\widehat{w}}$ *for some* $w > 1$. *Let* $(p_\ell/q_\ell)_{\ell \geq 1}$ *be the sequence of convergents to the real number*

$$\xi := [0, a_1, a_2, \ldots].$$

*Assume that the sequence* $(q_\ell^{1/\ell})_{\ell \geq 1}$ *is bounded and that* $\mathrm{ice}(\mathbf{a})$ *is finite. Then, the exponent* $w_2(\xi)$ *is finite.*

**Proof.** Assume that the sequence of finite words $(V_n)_{n \geq 1}$ and the real number $w$ arising from Condition $(*)_{\widehat{w}}$ are fixed. Let $M$ be a positive integer such that $\mathrm{ice}(\mathbf{a}) < M$ and $q_\ell \leq M^\ell$ for $\ell \geq 1$. Denote by $l_n$ the length of the word $V_n$ for $n \geq 1$.

As in the proof of Lemma 6.1, we set $\alpha_n = [0, V_n^\infty]$ and note that $\alpha_n$ is a quadratic number whose height is at most $q_{l_n}$. Since the sequence $(q_\ell^{1/\ell})_{\ell \geq 1}$ is bounded and the first $\lfloor w l_n \rfloor$ partial quotients of $\xi$ and $\alpha_n$ are the same, we infer from Lemmas 4.1 and 4.3 that

$$|\xi - \alpha_n| < q_{\lfloor w l_n \rfloor}^{-2} < q_{l_n}^{-2-\delta}, \tag{6.3}$$

for some positive real number $\delta$.

On the other hand, since $ice(\mathbf{a}) < M$, the numbers $\alpha_n$ and $\xi$ cannot have the same first $Ml_n$ partial quotients. The same observation applies for $\alpha_n$ and $\alpha_{n+1}$. Since the sequence $(q_\ell^{1/\ell})_{\ell \geq 1}$ is bounded by $M$, we infer from Lemmas 4.3 and 4.5 that there exists a positive real number $\lambda$, depending only on $M$, such that

$$|\xi - \alpha_n| > q_{l_n}^{-\lambda} \tag{6.4}$$

and

$$|\alpha_{n+1} - \alpha_n| > q_{l_n}^{-\lambda}. \tag{6.5}$$

Extracting if needed a subsequence of the sequence $(V_n)_{n\geq 1}$, we can assume that

$$q_{l_{n+1}} > \max\{\left(2q_{l_n}^{\lambda}\right)^{1/(2+\delta)}, q_{l_n}^2\} \tag{6.6}$$

for every positive integer $n$. We also observe that assertion (iii) in Condition $(*)_{\widehat{w}}$ and Lemma 4.3 ensure the existence of a positive integer $C$ such that

$$q_{l_n}^2 < q_{l_{n+1}} < q_{l_n}^C, \tag{6.7}$$

for every positive integer $n$. Combining (6.3) for $\alpha_{n+1}$ with (6.5) and (6.6), we get that $2|\xi - \alpha_{n+1}|$ is at most $|\alpha_n - \alpha_{n+1}|$ and the triangle inequality then yields

$$|\xi - \alpha_n| \geq \frac{1}{2}|\alpha_{n+1} - \alpha_n|. \tag{6.8}$$

Furthermore, setting $c = 1/(4 \cdot 3^{5/2})$, Lemma 4.7 and (6.8) give

$$|\xi - \alpha_n| \geq cH(\alpha_n)^{-2}\, H(\alpha_{n+1})^{-2} \geq cq_{l_n}^{-2} H(\alpha_{n+1})^{-2}.$$

We then infer from (6.3) and (6.7) that

$$H(\alpha_{n+1}) \geq c^{1/2} q_{l_n}^{\delta/2} > c^{1/2} q_{l_{n+1}}^{\delta/2C}.$$

For $n$ large enough, say $n \geq n_0$, we thus obtain

$$q_{l_n}^{\delta/3C} < H(\alpha_n) \leq q_{l_n}. \tag{6.9}$$

Set $j := \lceil \log(3C/\delta)/\log 2 \rceil$. Using Inequality (6.7), this leads to

$$q_{l_n}^{\delta/3C} < H(\alpha_n) \leq q_{l_n} < q_{l_{n+j}}^{\delta/3C} < H(\alpha_{n+j}) \leq q_{l_{n+j}} < q_{l_n}^{C^j} < H(\alpha_n)^{3C^{j+1}/\delta}.$$

We also deduce from (6.4) and (6.9) that

$$|\xi - \alpha_n| > \frac{1}{H(\alpha_n)^{3\lambda C/\delta}},$$

for every integer $n \geq n_0$.

Set $\sigma = 3\lambda C/\delta$, $\theta = 3C^{j+1}/\delta$ and $\alpha'_n = \alpha_{n_0+(n-1)j}$ for $n \geq 1$. The situation can be summarized as follows. We have shown the existence of a sequence of quadratic irrational numbers $(\alpha'_n)_{n\geq 1}$ satisfying

(i) $H(\alpha'_n)^{-\sigma} < |\xi - \alpha'_n| < H(\alpha'_n)^{-2-\delta}$;

(ii) $H(\alpha'_n) < H(\alpha'_{n+1}) < H(\alpha'_n)^{\theta}$.

At this point, we are ready to prove that $w_2(\xi)$ is finite. Let us consider a quadratic irrational number $\alpha$ whose height is large enough. More precisely, we will assume that

$$H(\alpha) > \big(cH(\alpha_1')^\delta\big)^{1/2}.$$

If $\alpha$ belongs to the set $\{\alpha_n',\ n \geq 1\}$, we simply use (i) to get that

$$|\xi - \alpha| > H(\alpha)^{-\sigma}. \tag{6.10}$$

Let us now assume that $\alpha$ does not belong to the set $\{\alpha_n',\ n \geq 1\}$. Then, there exists a unique integer $n > 1$ such that

$$H(\alpha_{n-1}') < c^{1/\delta} H(\alpha)^{2/\delta} < H(\alpha_n'). \tag{6.11}$$

In particular, Inequalities (ii) show that

$$H(\alpha_n') < c^{\theta/\delta} H(\alpha)^{2\theta/\delta}. \tag{6.12}$$

Since by assumption $\alpha$ and $\alpha_n'$ denote two distinct quadratic numbers, Lemma 4.7 implies

$$|\alpha - \alpha_n'| \geq 2cH(\alpha)^{-2} H(\alpha_n')^{-2}. \tag{6.13}$$

We thus infer from (6.11) and (i) that

$$|\alpha - \alpha_n'| > 2\,|\xi - \alpha_n'|\,,$$

hence,

$$|\xi - \alpha| \geq \big|\, |\xi - \alpha_n'| - |\alpha - \alpha_n'|\, \big| \geq \frac{1}{2}\,|\alpha - \alpha_n'|\,.$$

We now infer from (6.12) and (6.13) that

$$|\xi - \alpha| \geq c^{1-2\theta/\delta} H(\alpha)^{-2-4\theta/\delta}.$$

Together with (6.10), the latter inequality shows that $w_2(\xi)$ is finite, concluding the proof. □

We are now ready to prove Theorem 2.1.

**Proof of Theorem 2.1.** We keep the notation of Theorem 2.1. Let $\mathbf{a} = (a_\ell)_{\ell\geq 1}$ be a sequence of positive integers satisfying Condition $(*)_{\widehat{w}}$ for some $w > 1$. Let $(p_\ell/q_\ell)_{\ell\geq 1}$ denote the sequence of convergents to the real number

$$\xi := [0, a_1, a_2, \ldots].$$

Assume that the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded by some constant $A$. By Lemma 4.6, $w_1(\xi)$ is finite. Thus, if ice($\mathbf{a}$) is infinite, then Lemma 6.1 implies that $\xi$ is a $U_2$-number, concluding the proof in that case.

From now on, we assume that ice($\mathbf{a}$) is finite. We first observe that $w_2(\xi)$ is finite, in virtue of Lemma 6.2. Consequently, we thus only have to study the approximations to $\xi$ by algebraic numbers of degree at least three. Let $d \geq 3$ be an integer. Let $\alpha$ be an algebraic number of degree $d$. At several places in the proof below, it is convenient to assume that the height of $\alpha$ is sufficiently large. Let $\chi$ be a positive real number such that

$$|\xi - \alpha| < H(\alpha)^{-\chi}.$$

Our aim is to find an upper bound for $\chi$ in terms of $d$. More precisely, we have to prove that

$$\chi < \exp(c\,(\log 3d)^3\,(\log\log 3d)^2) \tag{6.14}$$

for some constant $c$ which does not depend on $d$.

Extracting if necessary a subsequence of $(V_n)_{n\geq 1}$, we can assume that there exists a constant $C$ such that

(iii') $2|V_n| < |V_{n+1}| < C|V_n|$, for every positive integer $n$.

Set $s_n = |V_n|$ for every positive integer $n$. Let $\kappa$ be the unique positive integer such that

$$q_{s_\kappa} \leq H(\alpha) < q_{s_{\kappa+1}}. \tag{6.15}$$

Denote by $M_1$ the largest integer such that $q_{s_\kappa}^{\chi} > q^2_{\lfloor w s_{\kappa+M_1}\rfloor}$ and observe that

$$|\xi - \alpha| < q^{-2}_{\lfloor w s_{\kappa+h}\rfloor} \tag{6.16}$$

for every $1 \leq h \leq M_1$. By definition of $M_1$, we have

$$q_{s_\kappa}^{\chi} < q^2_{\lfloor w s_{\kappa+M_1+1}\rfloor}. \tag{6.16$a$}$$

Given a positive integer $\ell$, we have by assumption that $q_\ell \leq A^\ell$, while Lemma 4.4 ensures that $q_\ell \geq (\sqrt{2})^{\ell-1}$. We thus get from (6.16a) that

$$(\sqrt{2})^{\chi(s_\kappa - 1)} \leq A^{2w s_{\kappa+M_1+1}}$$

and, using (iii'), we obtain

$$(\sqrt{2})^{\chi(s_\kappa - 1)} \leq A^{2w s_\kappa C^{M_1+1}}.$$

Consequently, Inequality (6.14) holds if we have

$$M_1 < c_0(\log 3d)^3(\log\log 3d)^2$$

for some constant $c_0$ which does not depend on $d$.

We will argue by contradiction. From now on, we assume that

$$M_1 > c_1(\log 3d)^3(\log\log 3d)^2, \tag{6.17}$$

for some constant $c_1$, and we will derive a contradiction if $c_1$ is sufficiently large.

For every integer $n \geq 1$, set

$$\alpha_n = [0, V_n^\infty]$$

and observe as previously that $\alpha_n$ is a root of the quadratic polynomial

$$P_n(X) := q_{s_n-1}X^2 + (q_{s_n} - p_{s_n-1})X - p_{s_n}.$$

Rolle's Theorem and Lemma 4.1 give that

$$|P_n(\xi)| = |P_n(\xi) - P_n(\alpha_n)| \leq 3q_{s_n}\, |\xi - \alpha_n| < 3q_{s_n}\, q_{\lfloor ws_n \rfloor}^{-2}, \quad (n \geq 1), \tag{6.18}$$

since the condition (i) implies that the first $\lfloor ws_n \rfloor$ partial quotients of $\xi$ and $\alpha_n$ are the same. Furthermore, we infer from the theory of continued fractions that

$$|q_{s_n}\xi - p_{s_n}| < q_{s_n}^{-1} \quad \text{and} \quad |q_{s_n-1}\xi - p_{s_n-1}| < q_{s_n}^{-1}. \tag{6.19}$$

Using again Rolle's Theorem, Inequalities (6.16) and (6.18) imply that

$$|P_{\kappa+h}(\alpha)| < |P_{\kappa+h}(\xi)| + 3q_{s_{\kappa+h}}\, |\xi - \alpha| < 6q_{s_{\kappa+h}}\, q_{\lfloor ws_{\kappa+h} \rfloor}^{-2}, \quad (1 \leq h \leq M_1). \tag{6.20}$$

Inequalities (6.16) and (6.19) ensure that, for every $1 \leq h \leq M_1$,

$$|q_{s_{\kappa+h}}\alpha - p_{s_{\kappa+h}}| < 2q_{s_{\kappa+h}}^{-1} \quad \text{and} \quad |q_{s_{\kappa+h}-1}\alpha - p_{s_{\kappa+h}-1}| < 2q_{s_{\kappa+h}}^{-1}. \tag{6.21}$$

We are now going to apply Theorem Ev to the following system of linear forms:

$$\begin{aligned}
L_1(X_1, X_2, X_3, X_4) =& \alpha^2 X_2 + \alpha(X_1 - X_4) - X_3, \\
L_2(X_1, X_2, X_3, X_4) =& \alpha X_1 - X_3, \\
L_3(X_1, X_2, X_3, X_4) =& X_1, \\
L_4(X_1, X_2, X_3, X_4) =& X_2.
\end{aligned}$$

Observe first that these linear forms are independent and with algebraic coefficients. Set $\mathcal{N}_1 = \{s_{\kappa+h}, \quad 1 \leq h \leq M_1\}$ and $\mathcal{P}_1 = \{\mathbf{p}_n = (q_n, q_{n-1}, p_n, p_{n-1}), \;\; n \in \mathcal{N}_1\}$. Let $n$ be in $\mathcal{N}_1$. Evaluating these linear forms at the integer point $\mathbf{p}_n$, we infer from Inequalities (6.20) and (6.21) that

$$\prod_{1 \leq i \leq 4} |L_i(\mathbf{p}_n)| < 12 q_n^2\, q_{\lfloor wn \rfloor}^{-2}. \tag{6.22}$$

Using again that $\sqrt{2}^{\ell-1} \leq q_\ell \leq A^\ell$ holds for every non-negative integer $\ell$, we get that

$$q_{\lfloor wn \rfloor} > q_n^{1+2\eta},$$

for some positive real number $\eta$, depending only on $w$ and on $A$. Thus, if the height of $\alpha$ is large enough, then $s_\kappa$ is itself sufficiently large to guarantee that

$$\prod_{1 \leq i \leq 4} |L_i(\mathbf{p}_n)| < |\det(L_1, L_2, L_3, L_4)|\, q_n^{-\eta}.$$

On the other hand, all elements of the set $\mathcal{P}_1$ are primitive (since $p_n$ and $q_n$ are always relatively prime) and with height larger than the height of $\alpha$. Thus, $H(\mathbf{p}_n) > H(L_i)$ for

$i = 1, \ldots, 4$ and $n \in \mathcal{N}_1$. Furthermore, $[\mathbf{Q}(L_i) : \mathbf{Q}] \le d$ for $i = 1, \ldots, 4$. We can thus apply Theorem Ev with $m = 4$ and $\varepsilon = \eta$. Let $T_1$ be the upper bound given by (5.1) for the number of exceptional subspaces. Set

$$M_2 := \lfloor M_1 / T_1 \rfloor .$$

Since $\eta$ does not depend on the constant $c_1$, Inequality (6.17) ensures the existence of a constant $c_2$ such that

$$M_2 > c_2 (\log 3d)^2 (\log\log 3d). \tag{6.24}$$

By the pigeonhole principle, there exists a proper subspace of $\mathbf{Q}^4$ containing at least $M_2$ points of $\mathcal{P}_1$. Thus, there exist a non-zero integer vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$ and a set of integers $\mathcal{N}_2 \subset \mathcal{N}_1$ with cardinality $r \ge M_2$ such that

$$x_1 q_n + x_2 q_{n-1} + x_3 p_n + x_4 p_{n-1} = 0, \tag{6.25}$$

for every $n \in \mathcal{N}_2$. Let $l_1 < l_2 < \ldots < l_r$ denote the elements of $\mathcal{N}_2$ once ordered.

We are now going to make the following assumption that will be justified thereafter.

*Assumption $\mathcal{A}$:* There exist three integers $1 \le a < b < c \le \lfloor r/4 \rfloor$ such that the vectors $\mathbf{p}_{l_a}$, $\mathbf{p}_{l_b}$ and $\mathbf{p}_{l_c}$ are linearly independent.

This makes possible the choice of $\mathbf{x} = \mathbf{p}_{l_a} \wedge \mathbf{p}_{l_b} \wedge \mathbf{p}_{l_c}$ in Equality (6.25). Here, $\wedge$ denotes the vector product in $\mathbf{R}^4$. An easy computation shows that with this choice of $\mathbf{x}$ we have

$$h_1 := \max\{|x_1|,\ |x_2|,\ |x_3|,\ |x_4|\} \le 12 q_{l_c}^3. \tag{6.26}$$

Let us remark now that $(x_2, x_4) \ne (0,0)$ since the vectors $(q_{n_1}, p_{n_1})$ and $(q_{n_2}, p_{n_2})$ are not collinear. Furthermore, $\alpha$ being irrational, the real number

$$\beta := -\frac{x_1 + x_3\alpha}{x_2 + x_4\alpha}$$

is well-defined. It then follows from Lemma 4.8 and Inequality (6.26) that

$$H(\beta) \ll q_{l_c}^{6d}\, H(\alpha)^2. \tag{6.26$a$}$$

Since $l_c > s_\kappa$, we infer from (6.15) and (6.26a) that

$$H(\beta) < q_{l_c}^{9d}, \tag{6.27}$$

if the height of $\alpha$ has been chosen large enough. Following (6.25), we have

$$\left|\beta - \frac{q_{n-1}}{q_n}\right| = \left|\frac{x_1 + x_3\alpha}{x_2 + x_4\alpha} - \frac{x_1 + x_3 p_n/q_n}{x_2 + x_4 p_{n-1}/q_{n-1}}\right|,$$

for every $n \in \mathcal{N}_2$. Using (6.19), we thus obtain

$$\left|\beta - \frac{q_{n-1}}{q_n}\right| \ll \frac{h_1^2}{q_n^2\, |x_2 + x_4\alpha|^2}.$$

Then, Inequality (6.26) and Liouville's inequality given in Lemma 4.7 give that

$$\left|\beta - \frac{q_{n-1}}{q_n}\right| \ll h_1^{2d+2}\, H(\alpha)^2\, q_n^{-2} \ll q_{l_c}^{6(d+1)}\, H(\alpha)^2\, q_n^{-2}.$$

Once again, the height of $\alpha$ can be chosen large enough so that

$$\left|\beta - \frac{q_{n-1}}{q_n}\right| < q_{l_c}^{7(d+1)} q_n^{-2}, \tag{6.27a}$$

for every $n \in \mathcal{N}_2$. Since by assumption $c \leq \lfloor r/4 \rfloor$, $l_{j+1} > 2l_j$ and $r \geq M_2$, we deduce from (6.24) that

$$|q_{l_j}\beta - q_{l_j - 1}| < q_{l_j}^{-1/2}, \tag{6.28}$$

for every $j$ with $\lfloor r/2 \rfloor \leq j \leq r$. Set $\mathcal{N}_2' = \{n \in \mathcal{N}_2,\ n \geq l_{\lfloor r/2 \rfloor}\}$. Denote by $M_2'$ the cardinality of this set and observe that $M_2' \geq M_2/2$.

We are now going to apply again Theorem Ev. Let us consider the following new system of linear forms:

$$L_1'(Y_1, Y_2, Y_3) = \beta Y_1 - Y_2, \quad L_2'(Y_1, Y_2, Y_3) = \alpha Y_1 - Y_3, \quad L_3'(Y_1, Y_2, Y_3) = Y_1.$$

Observe that these linear forms have algebraic coefficients and are linearly independent. Furthermore, we have $\det(L_1', L_2', L_3') = 1$ and $[\mathbf{Q}(L_i') : \mathbf{Q}] \leq d$ for $i = 1, 2, 3$. Let us also remark that by Inequality (6.27), we have $H(L_i') \leq \max\{H(\beta),\ H(\alpha)\} < q_n$, for $i = 1, 2, 3$ and every $n \in \mathcal{N}_2'$.

Let $n$ be in $\mathcal{N}_2'$. Evaluating these linear forms at the primitive integer point $(q_n, q_{n-1}, p_n)$, we infer from Inequalities (6.19) and (6.28) that

$$\prod_{1 \leq i \leq 3} |L_i'(q_n, q_{n-1}, p_n)| < q_n^{-1/2}.$$

We can thus apply Theorem Ev with $\varepsilon = 1/2$ and $m = 3$. Let us denote by $T_2$ the upper bound given in (5.1) for the number of exceptional subspaces. Set $M_3 := \lfloor T_2/M_2' \rfloor$. Then, Inequality (6.24) implies the existence of a constant $c_3$ such that

$$M_3 > c_3 \log d. \tag{6.29}$$

By the pigeonhole principle, there exists a proper subspace of $\mathbf{Q}^3$ which contains at least $M_3$ points lying in the set $\mathcal{P}_2 := \{\mathbf{p}_n,\ n \in \mathcal{N}_2'\}$. Thus, there exist a non-zero integer vector $\mathbf{y} = (y_1, y_2, y_3)$ and a set of integers $\mathcal{N}_3 \subset \mathcal{N}_2'$ with cardinality $s \geq M_3$, and such that

$$y_1 q_n + y_2 q_{n-1} + y_3 p_n = 0, \tag{6.30}$$

for every $n \in \mathcal{N}_3$. Let $1 \leq m_1 < m_2 < \ldots < m_s$ denote the elements of $\mathcal{N}_3$ once ordered.

Observe that $\mathbf{p}_1' = (q_{m_1}, q_{m_1 - 1}, p_{m_1})$ and $\mathbf{p}_2' = (q_{m_2}, q_{m_2 - 1}, p_{m_2})$ are not collinear since $m_2 > m_1 > 1$. This make possible the choice of $\mathbf{y} = \mathbf{p}_1' \wedge \mathbf{p}_2'$ in Equality (6.30). Here and in the rest of the proof, $\wedge$ denote the vector product in $\mathbf{R}^3$. Then, an easy computation shows that

$$h_2 := \max\{|y_1|,\ |y_2|,\ |y_3|\} \le q_{m_2}^2. \tag{6.31}$$

For every $n \in \mathcal{N}_3$, we can rewrite Equality (6.30) as

$$y_1 + y_2\beta + y_3\alpha + y_2\left(\frac{q_{n-1}}{q_n} - \beta\right) + y_3\left(\frac{p_n}{q_n} - \alpha\right) = 0.$$

In particular, for $n = m_s$, Inequalities (6.19), (6.28) and (6.31) give

$$|y_1 + y_2\beta + y_3\alpha| < 2q_{m_2}^2 q_{m_s}^{-3/2}. \tag{6.32}$$

On the other hand, we have

$$|y_1 + y_2\beta + y_3\alpha| = \frac{\left|y_3x_4\alpha^2 + (x_2y_3 + x_3y_2 + x_4y_1)\alpha + (x_1y_2 + y_1x_2)\right|}{|x_2 + x_4\alpha|}.$$

Thus, if $y_1 + y_2\beta + y_3\alpha \neq 0$, we infer from Liouville's inequality given in Lemma 4.7 that

$$|y_1 + y_2\beta + y_3\alpha| \gg h_1^{-d}\, h_2^{-d+1}\, H(\alpha)^{-2}.$$

Using (6.15), (6.26), (6.31) and the fact that $m_2 > l_c > s_{\kappa+1}$, the previous inequality gives

$$|y_1 + y_2\beta + y_3\alpha| \gg q_{m_2}^{-5d}. \tag{6.33}$$

By (6.29), we have $s \ge M_3 > c_3 \log d$ and since by assumption $m_j > 2m_{j-1}$, we can secure that $q_{m_s} > q_{m_2}^{7d}$ and

$$|y_1 + y_2\beta + y_3\alpha| > q_{m_2}^2 q_{m_s}^{-1},$$

if $c_3$ is chosen large enough. But, then, (6.32) contradicts (6.33). Thereby, we obtain

$$y_1 + y_2\beta + y_3\alpha = 0. \tag{6.34}$$

In order to obtain the desired contradiction, we need another equation linking $\alpha$ and $\beta$. Actually, it is sufficient to slightly modify the previous system of linear forms. Indeed, consider the following linear forms:

$$L_1''(Z_1, Z_2, Z_3) = \beta Z_1 - Z_2, \quad L_2''(Z_1, Z_2, Z_3) = \alpha Z_2 - Z_3, \quad L_3''(Z_1, Z_2, Z_3) = Z_1.$$

We still have that $\det(L_1', L_2', L_3') = 1$, $[\mathbf{Q}(L_i') : \mathbf{Q}] \le d$ for $i = 1, 2, 3$, and $H(L_i') \le \max\{H(\beta),\ H(\alpha)\} < q_n$ for $i = 1, 2, 3$ and every $n \in \mathcal{N}_2'$. Given an integer $n \in \mathcal{N}_2'$ and evaluating these linear forms at the integer point $(q_n, q_{n-1}, p_{n-1})$, we infer from (6.19) and (6.28) that

$$\prod_{1 \le i \le 3} |L_i''(q_n, q_{n-1}, p_{n-1})| < q_n^{-1/2}.$$

We can apply Theorem Ev exactly as before and we will thus omit some details in what follows. The pigeon principle ensures the existence of a proper subspace of $\mathbf{Q}^3$ containing at least $M_3$

points of $\mathcal{P}_2'$. Consequently, there exist a non-zero integer vector $\mathbf{z} = (z_1, z_2, z_3)$ and a set $\mathcal{N}_4 \subset \mathcal{N}_2'$ with cardinality $t \geq M_3$, such that

$$z_1 q_n + z_2 q_{n-1} + z_3 p_{n-1} = 0,$$

for every $n \in \mathcal{N}_4$. Let $1 \leq n_1 < n_2 < \ldots < n_t$ denote the elements of $\mathcal{N}_4$ once ordered.

Then, Inequality (6.31) can be replaced by

$$\max\{|z_1|,\ |z_2|,\ |z_3|\} \leq q_{n_2}^2,$$

while (6.32) becomes

$$\left| \frac{z_1}{\beta} + z_2 + z_3\alpha \right| < 2 q_{n_2}^2 q_{n_t}^{-3/2}.$$

Using Lemma 4.7, we argue as above to show that

$$\frac{z_1}{\beta} + z_2 + z_3\alpha = 0 \tag{6.35}$$

if $c_3$ is sufficiently large.

Our last step is to show that Equalities (6.34) and (6.35) provide a contradiction. Indeed, since $\alpha$ is irrational, $y_2$ is not zero and these equalities imply

$$(z_3\alpha + z_2)(y_3\alpha + y_1) = y_2 z_1. \tag{6.36}$$

Now, if $y_3 z_3 = 0$, then (6.34) and (6.35) imply that $\beta$ is rational which gives a contradiction in virtue of (6.27) and (6.28). Consequently, we can assume that $y_3 z_3 \neq 0$. But, then we obtain from (6.36) that $\alpha$ is a quadratic irrational. This contradicts the fact that $\alpha$ is an algebraic number of degree at least three. We have proved that the constant $c_3$ cannot be taken arbitrarily large. Thus, the constant $c_1$ in (6.17) is bounded, as well. This ends the proof in the case where *Assumption* $\mathcal{A}$ is satisfied.

We turn now to the case where *Assumption* $\mathcal{A}$ is not satisfied. Since some steps are very close from the previous exposition, we just outline the proof.

We come back to the place where Assumption $\mathcal{A}$ has been introduced. Set $\mathcal{N}_2'' = \{n \in \mathcal{N}_2,\ l_{\lfloor r/8 \rfloor} \leq n \leq l_{\lfloor r/4 \rfloor}\}$ and $\mathcal{P}_2'' = \{\mathbf{p}_n,\ n \in \mathcal{N}_2''\}$. Since $\mathcal{A}$ is not satisfied, all points lying in $\mathcal{P}_2''$ belong to the subspace generated by $\mathbf{p}_{l_1}$ and $\mathbf{p}_{l_2}$. Indeed, these vectors are clearly not collinear. In particular, if we set

$$\begin{pmatrix} y_1' \\ y_2' \\ y_3' \end{pmatrix} := \begin{pmatrix} q_{l_1} \\ q_{l_1-1} \\ p_{l_1} \end{pmatrix} \wedge \begin{pmatrix} q_{l_2} \\ q_{l_2-1} \\ p_{l_2} \end{pmatrix},$$

we obtain that $(y_1', y_2', y_3')$ is a non-zero integer vector such that

$$y_1' q_n + y_2' q_{n-1} + y_3' p_n = 0$$

for every integer $n \in \mathcal{N}_2^{''}$. Moreover, an easy computation shows that

$$h_1' := \max\{|y_j'|,\ 1 \leq j \leq 3\} < q_{l_2}^2. \tag{6.36}$$

Then, we set

$$\beta' := -\frac{y_1' + y_3'\alpha}{y_2'}.$$

Note that $\beta$ is well-defined since it is easily checked that $y_2' \neq 0$. Using now (6.36) and Lemma 4.8, we obtain

$$H(\beta') < q_{l_2}^{3d}. \tag{6.37}$$

On the other hand, we can chose the height of $\alpha$ large enough to ensure that

$$|q_n\beta - q_{n-1}| < q_n^{-1/2},$$

for every $n \in \mathcal{N}_2^{''}$. Since $\mathcal{N}_2^{''}$ is a set with cardinality $M_2^{''} \geq M_2/8$, we infer from lemma 4.7 and (6.37) that

$$z_1' + z_2'\beta + z_3'\alpha = 0. \tag{6.38}$$

Setting

$$\begin{pmatrix} z_1' \\ z_2' \\ z_3' \end{pmatrix} := \begin{pmatrix} q_{l_1} \\ q_{l_1-1} \\ p_{l_1-1} \end{pmatrix} \wedge \begin{pmatrix} q_{l_2} \\ q_{l_2-1} \\ p_{l_2-1} \end{pmatrix},$$

the non-zero integer vector $(z_1', z_2', z_3')$ is such that

$$z_1'q_n + z_2'q_{n-1} + z_3'p_{n-1} = 0,$$

for every $n \in \mathcal{N}_2^{''}$. As previously, this leads to

$$\frac{z_1'}{\beta'} + z_2' + z_3'\alpha = 0. \tag{6.39}$$

We then obtain a contradiction using (6.38) and (6.39), exactly as in the case where Assumption $\mathcal{A}$ is satisfied. This concludes the proof of the theorem. □

## 7. Proof of Theorem 3.1

This section is devoted to the proof of Theorem 3.1.

**Proof of Theorem 3.1.** We keep the notation of Theorem 3.1 and denote by $A$ an upper bound for the sequence $(q_\ell^{1/\ell})_{\ell \geq 1}$.

Let us consider the quadratic real number $\beta$ whose continued fraction expansion is purely periodic with period $\overline{B} = b_r, b_{r-1}, \ldots, b_1$, that is,

$$\beta := [b_r, b_{r-1}, \ldots, b_1, b_r, b_{r-1}, \ldots, b_1, \ldots] = [\overline{B}, \overline{B}, \ldots, \overline{B}, \ldots].$$

Let $(r_\ell/s_\ell)_{\ell\geq 1}$ denote the sequence of convergents to $\beta$. For every positive integer $k$, set

$$P_k := p_{n_k+r\lambda_k},\; Q_k := q_{n_k+r\lambda_k},\; P'_k := p_{n_k+r\lambda_k-1},\; Q'_k := q_{n_k+r\lambda_k-1} \text{ and } S_k := s_{r\lambda_k}.$$

Taking if necessary a subsequence of the sequence $(n_k)_{k\geq 1}$, we can assume that there exists a positive integer $C$ such that

$$Q_k^2 < Q_{k+1} < Q_k^C, \tag{7.1}$$

for every positive integer $k$.

We also infer from (i) and (ii) that there exists a positive real number $\nu$ such that

$$\nu < \lambda_k/n_k < 1/\nu, \tag{7.2}$$

for every positive integer $k$.

On the one hand, the theory of continued fractions gives

$$|Q_k\xi - P_k| < \frac{1}{Q_k} \text{ and } |Q'_k\xi - P'_k| < \frac{1}{Q'_k}, \tag{7.3}$$

while, on the other hand, the assumption made on the sequence $\mathbf{a}$ implies that

$$\frac{P_k}{Q_k} = [0, a_1, \cdots, a_{n_k-1}, \underbrace{B, B, \ldots, B}_{\lambda_k}].$$

By Lemma 4.2, we obtain

$$\frac{Q'_k}{Q_k} = [0, \underbrace{\overline{B}, \overline{B}, \ldots, \overline{B}}_{\lambda_k}, a_{n_k-1}, \cdots, a_1]$$

and, following Lemma 4.1, we get that

$$|Q'_k\beta - Q_k| < \frac{Q'_k}{S_k^2}\cdot \tag{7.4}$$

Furthermore, we infer from (3.2) that $a_{n_k-1}\ldots a_{n_k-r} \neq \overline{B}$. Thus, Lemmas 4.3 and 4.5 imply that

$$\left|\beta - \frac{Q_k}{Q'_k}\right| > \frac{c_\beta}{S_k^2}, \tag{7.5}$$

where $c_\beta$ is a positive constant depending only on $\beta$.

Since the sequence $(q_\ell^{1/\ell})_{\ell\geq 1}$ is bounded, we infer from Lemma 4.6 that $w_1(\xi)$ is finite. In the sequel, we thus only have to deal with the approximations of $\xi$ by algebraic numbers of degree at least equal to two. Let $d \geq 2$ be an integer. Let $\alpha$ be an algebraic number of degree $d$, the height of which will be chosen large enough in the sequel. Let $\chi$ be a positive real number such that

$$|\xi - \alpha| < H(\alpha)^{-\chi}.$$

Our aim is to find an upper bound for $\chi$ as a function of $d$. More precisely, we have to prove that

$$\chi < \exp(c\,(\log 3d)^3\,(\log\log 3d)^2) \tag{7.6}$$

for some constant $c$ which does not depend on $d$.

Let us denote by $k_0$ the unique positive integer such that

$$Q_{k_0} \leq H(\alpha) < Q_{k_0+1}, \tag{7.7}$$

and by $M_1$ the largest integer such that $Q_{k_0}^{\chi} > Q_{k_0+M_1}^2$. Consequently,

$$|\xi - \alpha| < Q_{k_0+h}^{-2} \tag{7.8}$$

holds for every integer $h = 1, \ldots, M_1$. If

$$M_1 < c_0(\log 3d)^3(\log\log 3d)^2$$

holds for some constant $c_0$ which does not depend on $d$, then Inequality (7.6) also holds, as shown by an easy computation using (7.1) (7.7) and (7.8).

We will argue by contradiction. From now on, we assume that

$$M_1 > c_1(\log 3d)^3(\log\log 3d)^2, \tag{7.9}$$

where $c_1$ denotes a constant which could be chosen arbitrarily large in the sequel, and we aim at deriving a contradiction.

It follows from Inequalities (7.3) and (7.8) that, for every integer $h = 1, \ldots, M_1$,

$$|Q_{k_0+h}\alpha - P_{k_0+h}| < \frac{2}{Q_{k_0+h}} \quad \text{and} \quad |Q'_{k_0+h}\alpha - P'_{k_0+h}| < \frac{2}{Q'_{k_0+h}}\cdot \tag{7.10}$$

We are now going to apply Theorem Ev to the following system of linear forms:

$$\begin{aligned}
L_1(X_1, X_2, X_3, X_4) =& \alpha X_1 - X_3,\\
L_2(X_1, X_2, X_3, X_4) =& \alpha X_2 - X_4,\\
L_3(X_1, X_2, X_3, X_4) =& \beta X_1 - X_2,\\
L_4(X_1, X_2, X_3, X_4) =& X_1.
\end{aligned}$$

We first observe that these linear forms are linearly independent over $\overline{\mathbf{Q}}$ and all have algebraic coefficients. Set $\mathcal{N}_1 = \{k_0 + h,\ \ 1 \leq h \leq M_1\}$ and $\mathcal{P}_1 = \{\mathbf{p}_k = (Q_k, Q'_k, P_k, P'_k),\ \ k \in \mathcal{N}_1\}$. Let $k \in \mathcal{N}_1$. Evaluating these linear forms at the integer point $\mathbf{p}_k$, Inequalities (7.4) and (7.10) give

$$\prod_{1\leq i\leq 4} |L_i(\mathbf{p}_k)| < \frac{4}{S_k^2}\cdot \tag{7.11}$$

By assumption, $Q_k \leq A^{n_k + r\lambda_k}$, while Lemma 4.4 implies that $S_k \geq (\sqrt{2})^{r\lambda_k - 1}$. We then infer from Inequality (7.2) that

$$S_k \geq Q_k^{\eta}, \tag{7.12}$$

for some positive real number $\eta$, depending only on $r$, $\nu$ and $A$. If the height of $\alpha$ is large enough, we get from (7.11) and (7.12) that

$$\prod_{1\le i\le 4} |L_i(\mathbf{p}_k)| < |\det(L_1,L_2,L_3,L_4)|\, Q_k^{-\eta}.$$

On the other hand, every element in $\mathcal{P}_1$ is a primitive vector (since $P_k$ and $Q_k$ are relatively prime) whose height is at least equal to the height of $\alpha$. Consequently, $H(\mathbf{p}_k) > H(L_i)$ for $i = 1,\ldots,4$ and $k \in \mathcal{N}_1$. Furthermore, the degree of the number field generated by the coefficients of our linear forms is at most $2d$. We can thus apply Theorem Ev with $m = 4$ and $\varepsilon = \eta$. Let $T_1$ be the upper bound given in (5.1) for the number of exceptional subspaces. Set

$$M_2 := \lfloor M_1/T_1 \rfloor.$$

Since $\eta$ does not depend on the choice of the constant $c_1$, Inequality (7.9) ensures the existence of a constant $c_2$ which does not depend on $d$ and such that

$$M_2 > c_2(\log 3d)^2(\log\log 3d). \tag{7.13}$$

Moreover, the constant $c_2$ could be chosen arbitrarily large in the sequel. By the pigeonhole principle, there exists a proper subspace of $\mathbf{Q}^4$ containing at least $M_2$ points of $\mathcal{P}_1$. That is, there exist a non-zero integer vector $\mathbf{x} = (x_1,x_2,x_3,x_4)$ and a set of integers $\mathcal{N}_2 \subset \mathcal{N}_1$ with cardinality $s \ge M_2$ such that

$$x_1Q_k + x_2Q'_k + x_3P_k + x_4P'_k = 0, \tag{7.14}$$

for every $k \in \mathcal{N}_2$. Let $l_1 < l_2 < \ldots < l_s$ denote the elements of $\mathcal{N}_2$ once ordered.

We are now going to make the following assumption that will be justified thereafter.

*Assumption* $\mathcal{A}$: we assume that there exist three integers $1 \le a < b < c \le \lfloor s/4 \rfloor$ such that the vectors $\mathbf{p}_{l_a}$, $\mathbf{p}_{l_b}$ and $\mathbf{p}_{l_c}$ are linearly independent. This makes possible the choice of $\mathbf{x} = \mathbf{p}_{l_a} \wedge \mathbf{p}_{l_b} \wedge \mathbf{p}_{l_c}$ in Equality (6.25). Here, $\wedge$ denotes the vector product in $\mathbf{R}^4$. As shown an easy computation, this choice of $\mathbf{x}$ guarantee that

$$h_1 := \max\{|x_1|,\ |x_2|,\ |x_3|,\ |x_4|\} \le 12Q_{l_c}^3. \tag{7.15}$$

It thus follows from Inequality (7.14) that

$$|x_4\alpha Q'_k + x_1Q_k + x_2Q'_k + x_3P_k| = |x_4(Q'_k\alpha - P'_k)| < \frac{|x_4|}{Q_k} < \frac{12Q_{l_c}^3}{Q_k}, \tag{7.16}$$

for every $k \in \mathcal{N}_2$.

Our next step consists in applying again Theorem Ev to the following new system of linear forms:

$$\begin{aligned}
L'_1(X_1,X_2,X_3) =& \alpha X_1 - X_3,\\
L'_2(X_1,X_2,X_3) =& x_1X_1 + (x_4\alpha + x_2)X_2 + x_3X_3,\\
L'_3(X_1,X_2,X_3) =& X_1.
\end{aligned}$$

We first observe that these linear forms have algebraic coefficients. We also deduce from Equality (7.14) that $(x_2, x_4) \neq (0,0)$. Consequently, $x_4\alpha + x_2 \neq 0$, since $\alpha$ is irrational. We then deduce that our linear forms are linearly independent. Furthermore, the number field generated by all the coefficients of these linear forms has degree $d$, and we have

$$\det(L_1', L_2', L_3') = x_4\alpha + x_2.$$

We also infer from Lemma 4.7, (7.15) and (7.16) that

$$|\det(L_1', L_2', L_3')| = |x_4\alpha + x_3| > Q_{l_c}^{-3d}, \tag{7.17}$$

if $H(\alpha)$ is large enough.

Since by assumption $c \leq \lfloor s/4 \rfloor$ and $s \geq M_2$, we infer from (7.1) and (7.13) that the constant $c_2$ can be chosen large enough so that

$$Q_k^{1/2} > 12\, Q_{l_c}^{3d+3}, \tag{7.18}$$

for every $\lfloor s/2 \rfloor \leq k \leq s$. For the same reason, we can assume that $H(L_i') < Q_k$, for $i = 1,2,3$ and every $\lfloor s/2 \rfloor \leq k \leq s$. Set $\mathcal{N}_2' = \{n \in \mathcal{N}_2,\ n \geq l_{\lfloor s/2 \rfloor}\}$. Let us denote by $M_2'$ the cardinality of this set and observe that

$$M_2' \geq M_2/2. \tag{7.19}$$

Given an integer $k \in \mathcal{N}_2'$ and evaluating these linear forms at the primitive integer point $\mathbf{p}_k' := (Q_k, Q_k', P_k)$, we infer from Inequalities (7.16), (7.17) and (7.18) that

$$\prod_{1 \leq i \leq 3} |L_i'(Q_k, Q_k', P_k)| < |\det(L_1', L_2', L_3')|\, Q_k^{-1/2}.$$

We can thus apply Theorem Ev with $\varepsilon = 1/2$ and $m = 3$. Let us denote by $T_2$ the upper bound given in (5.1) for the number of exceptional subspaces. Set $M_3 := \lfloor T_2/M_2' \rfloor$. Then, Inequalities (7.13) and (7.19) imply the existence of a constant $c_3$ such that

$$M_3 > c_3 \log d. \tag{7.20}$$

Furthermore, the constant $c_3$ can be chosen arbitrarily large. By the pigeonhole principle, there exists a proper subspace of $\mathbf{Q}^3$ which contains at least $M_3$ points lying in the set $\mathcal{P}_2 := \{\mathbf{p}_k',\ k \in \mathcal{N}_2'\}$. There thus exist a non-zero integer vector $\mathbf{y} = (y_1, y_2, y_3)$ and a set of integers $\mathcal{N}_3 \subset \mathcal{N}_2'$ of cardinality $t \geq M_3$, and such that

$$y_1 Q_k + y_2 Q_k' + y_3 P_k = 0, \tag{7.21}$$

for every $k \in \mathcal{N}_3$. Let $1 \leq m_1 < m_2 < \ldots < m_t$ denote the elements of $\mathcal{N}_3$ once ordered.

Observe that $\mathbf{p}_1'$ and $\mathbf{p}_2'$ are not collinear. This make possible the choice of $\mathbf{y} = \mathbf{p}_1' \wedge \mathbf{p}_2'$ in Equality (7.21). Here and in the rest of the proof, $\wedge$ denote the vector product in $\mathbf{R}^3$. Then, an easy computation shows that

$$h_2 := \max\{|y_1|,\ |y_2|,\ |y_3|\} \leq Q_{m_2}^2. \tag{7.22}$$

For every $k \in \mathcal{N}_3$, we can rewrite Equality (7.21) as

$$y_1 + y_2\beta + y_3\alpha + y_2\left(\frac{Q'_k}{Q_k} - \beta\right) + y_3\left(\frac{P_k}{Q_k} - \alpha\right) = 0. \tag{7.23}$$

Choosing $k = m_t$, we can argue as in Theorem 2.1 and use (7.20) to deduce from the Liouville inequality given in Lemma 4.7 that

$$y_1 + y_2\beta + y_3\alpha = 0. \tag{7.24}$$

As a particular instance of Equality (7.23), we thus obtain

$$y_2\left(\frac{Q'_{m_t}}{Q_{m_t}} - \beta\right) = y_3\left(\frac{P_{m_t}}{Q_{m_t}} - \alpha\right).$$

Remark now that, on the one hand, we have

$$\left|y_3\left(\frac{P_{m_t}}{Q_{m_t}} - \alpha\right)\right| < \frac{|y_3|}{Q_{m_t}^2} < \frac{|y_3|}{Q_{m_{t-1}}^2 S_{m_t}^2} < \frac{1}{Q_{m_{t-1}}} \cdot \frac{1}{S_{m_t}^2}, \tag{7.25}$$

while on the other hand, Inequality (7.5) shows that

$$\left|y_2\left(\frac{Q'_{m_t}}{Q_{m_t}} - \beta\right)\right| > \frac{c_\beta}{S_{m_t}^2}\cdot \tag{7.26}$$

Indeed, since $\alpha$ is irrational, we infer from (7.24) that $y_2 \neq 0$. The constant $c_\beta$ depending only on $\beta$, Inequality (7.25) and (7.26) provides us with a contradiction as soon as $c_3$ is chosen large enough. This ends the proof in the case where *Assumption* $\mathcal{A}$ is satisfied.

We turn now to outline the case where this assumption is not satisfied. We come back to the place where *Assumption* $\mathcal{A}$ has been introduced. Set $\mathcal{N}_2^{''} = \{n \in \mathcal{N}_2,\ l_{\lfloor r/8 \rfloor} \leq n \leq l_{\lfloor r/4 \rfloor}\}$ and $\mathcal{P}_2^{''} = \{\mathbf{p}_n,\ n \in \mathcal{N}_2^{''}\}$. Since *Assumption* $\mathcal{A}$ is not satisfied, all points lying in $\mathcal{P}_2^{''}$ belong to the vectorial plan generated by $\mathbf{p}_{l_1}$ and $\mathbf{p}_{l_2}$. Indeed, these vectors are clearly not collinear. In particular, if we set

$$\begin{pmatrix} y'_1 \\ y'_2 \\ y'_3 \end{pmatrix} := \begin{pmatrix} q_{l_1} \\ q_{l_1-1} \\ p_{l_1} \end{pmatrix} \wedge \begin{pmatrix} q_{l_2} \\ q_{l_2-1} \\ p_{l_2} \end{pmatrix},$$

we obtain that $(y'_1, y'_2, y'_3)$ is a non-zero integer vector such that

$$y'_1 Q_k + y'_2 Q'_k + y'_3 P_k = 0$$

for every integer $k \in \mathcal{N}_2^{''}$. Moreover, an easy computation shows that

$$h'_1 := \max\{|y'_j|,\ 1 \leq j \leq 3\} < Q_{l_2}^2.$$

We are now in the same situation as in (7.22). We can thus argue exactely as previously to conclude the proof of the theorem. □

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