# On a mixed problem in Diophantine approximation

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**Abstract.** *Let $d$ be a positive integer. Let $p$ be a prime number. Let $\alpha$ be a real algebraic number of degree $d+1$. We establish that there exist a positive constant $c$ and infinitely many algebraic numbers $\xi$ of degree $d$ such that $|\alpha - \xi| \cdot \min\{|\mathrm{Norm}(\xi)|_p, 1\} < cH(\xi)^{-d-1} (\log 3H(\xi))^{-1/d}$. Here, $H(\xi)$ and $\mathrm{Norm}(\xi)$ denote the naïve height of $\xi$ and its norm, respectively. This extends an earlier result of de Mathan and Teulié that deals with the case $d = 1$.*

## 1. Introduction

In analogy with the Littlewood conjecture, de Mathan and Teulié [7] proposed recently a 'mixed Littlewood conjecture'. For any prime number $p$, the usual $p$-adic absolute value $|\cdot|_p$ is normalized in such a way that $|p|_p = p^{-1}$. We denote by $\|\cdot\|$ the distance to the nearest integer.

**De Mathan–Teulié conjecture.** *For every real number $\alpha$ and every prime number $p$, we have*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_p = 0. \tag{1.1}$$

Obviously, the above conjecture holds if $\alpha$ is rational or has unbounded partial quotients in its continued fraction expansion. Thus, it only remains to consider the case when $\alpha$ is an element of the set $\boldsymbol{Bad}_1$ of badly approximable real numbers, where

$$\boldsymbol{Bad}_1 = \{\alpha \in \mathbf{R} : \inf_{q \geq 1} q \cdot \|q\alpha\| > 0\}.$$

De Mathan and Teulié [7] proved that (1.1) holds for every quadratic real number $\alpha$ (recall that such a number is in $\boldsymbol{Bad}_1$) but, despite several recent results [4, 3], the general conjecture is still unsolved.

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If we rewrite (1.1) under the form

$$\inf_{a,q\geq 1,\, \gcd(a,q)=1} q^2 \cdot \left|\alpha - \frac{a}{q}\right| \cdot |q|_p = 0\,,$$

then we have $|q|_p = \min\{|\mathrm{Norm}(q/a)|_p, 1\}$. Hence, upon replacing $\alpha$ by $1/\alpha$, the de Mathan–Teulié conjecture can be reformulated as follows: For every irrational real number $\alpha$, for every prime number $p$ and every positive real number $\varepsilon$, there exists a non-zero rational number $\xi$ satisfying

$$|\alpha - \xi| \cdot \min\{|\mathrm{Norm}(\xi)|_p, 1\} < \varepsilon H(\xi)^{-2}.$$

Throughout this paper, the height $H(P)$ of an integer polynomial $P(X)$ is the maximal of the absolute values of its coefficients. The height $H(\xi)$ of an algebraic number $\xi$ is the height of its minimal defining polynomial over the rational integers $a_0 + a_1X + \ldots + a_dX^d$, and the norm of $\xi$, denoted by $\mathrm{Norm}(\xi)$, is the rational number $(-1)^d a_0/a_d$.

This reformulation suggests us to ask the following question.

**Problem 1.** *Let $d$ be a positive integer. Let $\alpha$ be a real number that is not algebraic of degree less than or equal to $d$. For every prime number $p$ and every positive real number $\varepsilon$, does there exist a non-zero real algebraic number $\xi$ of degree at most $d$ satisfying*

$$|\alpha - \xi| \cdot \min\{|\mathrm{Norm}(\xi)|_p, 1\} < \varepsilon H(\xi)^{-d-1}?$$

The answer to Problem 1 is clearly positive, unless (perhaps) when $\alpha$ is an element of the set $\boldsymbol{Bad}_d$ of real numbers that are badly approximable by algebraic numbers of degree at most $d$, where

$$\begin{aligned}\boldsymbol{Bad}_d = \{&\alpha \in \mathbf{R} : \text{There exists } c > 0 \text{ such that } |\alpha - \xi| > cH(\xi)^{-d-1},\\ &\text{for all algebraic numbers } \xi \text{ of degree at most } d\}.\end{aligned}$$

For $d \geq 1$, the set $\boldsymbol{Bad}_d$ contains the set of algebraic numbers of degree $d+1$, but it remains an open problem to decide whether this inclusion is strict for $d \geq 2$; see the monograph [2] for more information. The purpose of the present note is to give a positive answer to Problem 1 for every positive integer $d$ and every real algebraic number $\alpha$ of degree $d+1$. This extends the result from [7] which deals with the case $d = 1$.

## 2. Results

Throughout this paper, for a prime number $p$, a number field $\mathbf{K}$, and a non-Archimedean place $v$ on $\mathbf{K}$ lying above $p$, we normalize the absolute value $|\cdot|_v$ in such a way that $|\cdot|_v$ and $|\cdot|_p$ coincide on $\mathbf{Q}$.

Our main result includes a positive answer to Problem 1 when $\alpha$ is a real algebraic number of degree $d+1$.

**Theorem 1.** *Let $d$ be a positive integer. Let $\alpha$ be a real algebraic number of degree $d+1$ and denote by $r$ the unit rank of $\mathbf{Q}(\alpha)$. Let $p$ be a prime number. There exist positive constants $c_1, c_2, c_3$ and infinitely many real algebraic numbers $\xi$ of degree $d$ such that*

$$|\alpha - \xi| < c_1 H(\xi)^{-d-1}, \tag{2.1}$$

$$|\xi|_v < c_2 (\log 3H(\xi))^{-1/(rd)}, \tag{2.2}$$

*for every absolute value $|\cdot|_v$ on $\mathbf{Q}(\xi)$ above the prime $p$, and*

$$|\alpha - \xi| \cdot \min\{|\mathrm{Norm}(\xi)|_p, 1\} < c_3 H(\xi)^{-d-1} (\log 3H(\xi))^{-1/r}. \tag{2.3}$$

Theorem 1 extends Théorème 2.1 of [7] that is only concerned with the case $d = 1$.

Under the assumptions of Theorem 1, Wirsing [10] established that there are infinitely many real algebraic numbers $\xi$ satisfying (2.1).

The proof of Theorem 1 is very much inspired by a paper of Peck [8] on simultaneous rational approximation to real algebraic numbers. Roughly speaking, we use a method dual to Peck's to construct integer polynomials $P(X)$ that take small values at $\alpha$, and we need an extra argument to ensure that our polynomials have a root $\xi$ very close to $\alpha$.

De Mathan [6] used the theory of linear forms in non-Archimedean logarithms to prove that Theorem 1 for $d = 1$ is best possible, in the sense that the absolute value of the exponent of $(\log 3H(\xi))$ in (2.2) cannot be too large. Next theorem extends this result to all values of $d$.

**Theorem 2.** *Let $p$ be a prime number, $d$ a positive integer and $\alpha$ a real algebraic number of degree $d+1$. Let $\lambda$ be a positive real number. There exists a positive real number $\kappa = \kappa(\lambda)$ such that for every non-zero real algebraic number $\xi$ of degree $d$ satisfying*

$$|\alpha - \xi| \le \lambda H(\xi)^{-d-1} \tag{2.4}$$

*we have*

$$|\xi|_v \ge (\log 3H(\xi))^{-\kappa}$$

*for at least one absolute value $|\cdot|_v$ on $\mathbf{Q}(\xi)$ above the prime $p$.*

As in [6], the proof of Theorem 2 rests on the theory of linear forms in non-Archimedean logarithms.

Let $d$ be a positive integer. We recall that it follows from the $p$-adic version of the Schmidt Subspace Theorem that for every algebraic number $\alpha$ of degree $d+1$ and for every positive real number $\varepsilon$, there are only finitely many non-zero integer polynomials $P(X) = a_0 + a_1 X + \ldots + a_d X^d$ of degree at most $d$, with $a_0 \neq 0$, that satisfy

$$|P(\alpha)| \cdot |a_0|_p < H(P)^{-d-\varepsilon}.$$

Let $\xi$ be a real algebraic number of degree at most $d$, and denote by $P(X) = a_0 + a_1 X + \ldots + a_d X^d$ its minimal defining polynomial over $\mathbf{Z}$. Then,

$$\min\{|\mathrm{Norm}(\xi)|_p, 1\} \ge |a_0|_p$$

and there exists a constant $c(\alpha)$, depending only on $\alpha$, such that

$$|P(\alpha)| \leq c(\alpha)\, H(\xi) \cdot |\xi - \alpha|.$$

Let $\varepsilon$ be a positive real number. Applying the above statement deduced from the $p$-adic version of the Schmidt Subspace Theorem to these polynomials $P(X)$, we deduce that

$$|\alpha - \xi| \cdot \min\{|\mathrm{Norm}(\xi)|_p, 1\} \geq H(P)^{-d-1-\varepsilon}$$

holds if $H(P)$ is sufficiently large. This implies that if $\xi$ satisfies (2.4) and if $H(\xi)$ is sufficiently large, then we have

$$|\mathrm{Norm}(\xi)|_p \geq H(\xi)^{-\varepsilon},$$

accordingly

$$\max_{v|p} |\xi|_v \geq H(\xi)^{-\varepsilon/d}.$$

The result of Theorem 2 is more precise, however we cannot obtain a good lower bound for $|\mathrm{Norm}(\xi)|_p$.

We conclude this section by pointing out that Einsiedler and Kleinbock [4] showed that a slight modification of the de Mathan–Teulié conjecture easily follows from a theorem of Furstenberg [5, 1].

**Theorem EK.** *Let $p_1$ and $p_2$ be distinct prime numbers. Then*

$$\inf_{q \geq 1} q \cdot \|q\alpha\| \cdot |q|_{p_1} \cdot |q|_{p_2} = 0$$

*holds for every real number $\alpha$.*

In view of Theorem EK, we formulate the following question, presumably easier to solve than Problem 1.

**Problem 2.** *Let $d$ be a positive integer. Let $\alpha$ be a real number that is not algebraic of degree less than or equal to $d$. For every distinct prime numbers $p_1$, $p_2$ and every positive real number $\varepsilon$, does there exist a non-zero real algebraic number $\xi$ of degree at most $d$ satisfying*

$$|\alpha - \xi| \cdot \min\{|\mathrm{Norm}(\xi)|_{p_1}, 1\} \cdot \min\{|\mathrm{Norm}(\xi)|_{p_2}, 1\} < \varepsilon H(\xi)^{-d-1}?$$

Theorem EK gives a positive answer to Problem 2 when $d = 1$.

The sequel of the paper is organized as follows. We gather several auxiliary results in Section 3, and Theorems 1 and 2 are established in Sections 4 and 5, respectively.

In the next sections, we fix a real algebraic number field $\mathbf{K}$ of degree $d+1$. The notation $A \ll B$ means, unless specific indications, that the implicit constant depends on $\mathbf{K}$. Furthermore, we write $A \asymp B$ if we have simultaneously $A \ll B$ and $B \ll A$.

## 3. Auxiliary lemmas

Let $\mathbf{K}$ be a real algebraic number field of degree $d+1$. Let $\mathcal{O}$ denote its ring of integers, and let $\alpha_0 = 1, \alpha_1, \ldots, \alpha_d$ be a basis of $\mathbf{K}$. Let $D$ be a positive integer satisfying

$$D(\mathbf{Z} + \alpha_1 \mathbf{Z} + \ldots + \alpha_d \mathbf{Z}) \subset \mathcal{O} \subset \frac{1}{D}(\mathbf{Z} + \alpha_1 \mathbf{Z} + \ldots + \alpha_d \mathbf{Z})$$

and the corresponding inequalities for the dual basis $\beta_0, \ldots, \beta_d$ defined by

$$\mathrm{Tr}(\alpha_i \beta_j) = \delta_{i,j},$$

where Tr is the trace and $\delta_{i,j}$ is the Kronecker symbol.

We denote by $\sigma_0 = \mathrm{Id}, \ldots, \sigma_d$, the complex embeddings of $\mathbf{K}$, numbered in such a way that $\sigma_0, \ldots, \sigma_{r_1 - 1}$ are real, $\sigma_{r_1}, \ldots, \sigma_d$ are imaginary and $\sigma_{r_1 + r_2 + j} = \overline{\sigma_{r_1 + j}}$ for $0 \leq j < r_2$. Set also $r = r_1 + r_2 - 1$, and let $\varepsilon_1, \ldots, \varepsilon_r$ be multiplicatively independent units in $\mathbf{K}$.

**Lemma 1.** *Let $\eta$ be a unit in $\mathcal{O}$ such that $-1 < \eta < 1$ and define the real number $N$ by $|\eta| = N^{-1}$. The conditions*

$$|\sigma_j(\eta)| \asymp N^{1/d}, \qquad 0 < j \leq d, \tag{3.1}$$

*and*

$$|\sigma_i(\eta)| \asymp |\sigma_j(\eta)|, \qquad 0 < i < j \leq d, \tag{3.2}$$

*are equivalent. Let $\gamma \neq 0$ be in $\mathbf{K}$ and let $\Delta$ be a positive integer such that $\Delta\gamma \in \mathcal{O}$. If $\eta$ satisfies (3.1) or (3.2), write*

$$\gamma\eta = a_0 + \ldots + a_d \alpha_d,$$

*with $a_0, \ldots, a_d$ in $\mathbf{Q}$. We have $D\Delta a_k \in \mathbf{Z}$ for $k = 0, \ldots, d$ and*

$$\max_{k=0,\ldots,d} |a_k| \asymp N^{1/d},$$

*where the implicit constants depend on $\gamma$.*

*Proof.* Since $\eta$ is a unit, we have

$$\prod_{0 \leq j \leq d} \sigma_j(\eta) = \pm 1.$$

and (3.1) and (3.2) are clearly equivalent. The formula

$$a_k = \mathrm{Tr}(\gamma\eta\beta_k) = \gamma\eta\beta_k + \sum_{j=1}^{d} \sigma_j(\eta)\sigma_j(\gamma\beta_k)$$

implies that if $\eta$ satisfies (3.1), then

$$|a_k| \ll N^{1/d}, \qquad 0 \leq k \leq d.$$

Combined with

$$\sigma_1(\gamma)\sigma_1(\eta) = a_0 + \ldots + a_d\sigma_1(\alpha_d)\,,$$

this shows that

$$N^{1/d} \asymp |\sigma_1(\eta)| \ll \max_{k=0,\ldots,d} |a_k|.$$

The proof of the lemma is complete. $\square$

Let $\alpha$ be a real algebraic number of degree $d+1$. We keep the above notation with the field $\mathbf{K} = \mathbf{Q}(\alpha)$ and the basis $1, \alpha, \ldots, \alpha^d$ of $\mathbf{K}$ over $\mathbf{Q}$, and we display an immediate consequence of Lemma 1.

**Corollary 1.** *Let $\eta$ be a unit in $\mathcal{O}$ such that $-1 < \eta < 1$ and set $N = |\eta|^{-1}$. Then*

$$D\Delta\gamma\eta = P(\alpha),$$

*where $P(X)$ is an integral polynomial of degree at most $d$ satisfying*

$$H(P) \asymp N^{1/d}, \quad |P(\alpha)| \asymp N^{-1}\,,$$

*and thus*

$$|P(\alpha)| \asymp H(P)^{-d}.$$

Denote by $\tau_j$, $j = 0, \ldots, d$ the embeddings of $\mathbf{K}$ into $\mathbf{C}_p$. Recall that the absolute value $|\cdot|_p$ on $\mathbf{Q}$ has a unique extension to $\mathbf{C}_p$, that we also denote by $|\cdot|_p$. In Lemmata 2 to 4 below we work in $\mathbf{C}_p$. Let $P(X)$ be an irreducible integer polynomial of degree $n \geq 1$. Let $\xi$ be a complex root of $P(X)$ and $\xi_1, \ldots, \xi_n$ be the roots of $P(X)$ in $\mathbf{C}_p$. We point out that the sets

$$\{|\xi|_v : v \text{ is above } p \text{ on } \mathbf{Q}(\xi)\}$$

and

$$\{|\xi_i|_p : 1 \leq i \leq n\}$$

coincide, since all the absolute values $|\cdot|_v$ and $|\cdot|_p$ coincide on $\mathbf{Q}$.

Keeping the notation of Lemma 1, we have the following auxiliary result.

**Lemma 2.** *Assume that $\gamma = \alpha_d$. Then*

$$|a_k|_p \ll \max_{0 \leq j \leq d} |\tau_j(\eta) - 1|_p\,, \qquad\qquad 0 \leq k < d\,,$$

*and*

$$|a_d - 1|_p \ll \max_{0 \leq j \leq d} |\tau_j(\eta) - 1|_p\,.$$

*Proof.* Since

$$\mathrm{Tr}(\alpha_d\beta_k) = 0\,, \quad \text{for } k = 0, \ldots, d-1,$$

we get

$$a_k = \mathrm{Tr}(\gamma\eta\beta_k) = \mathrm{Tr}(\alpha_d(\eta - 1)\beta_k) = \sum_{j=0}^{d} (\tau_j(\eta) - 1)\tau_j(\alpha_d\beta_k),$$

and deduce that

$$|a_k|_p \ll \max_{0\le j\le d} |\tau_j(\eta)-1|_p\,, \qquad\qquad 0\le k<d.$$

It follows from

$$\mathrm{Tr}(\alpha_d\beta_d)=1$$

that

$$a_d = 1+\mathrm{Tr}(\alpha_d\beta_d(\eta-1)) = 1+\sum_{j=0}^{d}(\tau_j(\eta)-1)\tau_j(\alpha_d\beta_d),$$

and we derive that

$$|a_d-1|_p \ll \max_{0\le j\le d} |\tau_j(\eta)-1|_p\,.$$

This concludes the proof. □

**Lemma 3.** *Let $0<\delta<1$. There exist arbitrarily large positive real numbers $H$ and units $\eta$ satisfying $\eta = H^{-d}$,*

$$\left|\frac{\sigma_j(\eta)}{\sigma_1(\eta)}-1\right| \le \delta, \qquad 2\le j\le d, \tag{3.3}$$

*and*

$$|\tau_j(\eta)-1|_p \ll (\log H)^{-1/r}, \qquad 0\le j\le d.$$

*Proof.* By taking suitable powers of the units $\varepsilon_1,\ldots,\varepsilon_r$, we can assume that they are all positive, as well as their real conjugates, and that $|\tau_j(\varepsilon_i)-1|_p < p^{-1/(p-1)}$ for $i=1,\ldots,r$ and $j=0,\ldots,d$. This is possible since $|\tau_j(\varepsilon_i)|_p = 1$ for $i=1,\ldots,r$ and $j=0,\ldots,d$. This allows us to consider the $p$-adic logarithms of each $\tau_j(\varepsilon_i)$. Our aim is to construct a suitable unit $\eta$ of the form

$$\eta = \varepsilon_1^{\mu_1 p^s}\ldots\varepsilon_r^{\mu_r p^s},$$

where $\mu_i\in\mathbf{Z}$. The conditions for (3.3) are then

$$p^s\left|\mu_1\log\frac{|\sigma_j(\varepsilon_1)|}{|\sigma_1(\varepsilon_1)|}+\ldots+\mu_r\log\frac{|\sigma_j(\varepsilon_r)|}{|\sigma_r(\varepsilon_r)|}\right| \le C_1, \qquad 2\le j\le r,$$

where $C_1=C_1(\delta)>0$ is a constant, and

$$\left\|\frac{p^s}{2\pi}\left(\mu_1\arg\sigma_j(\varepsilon_1)+\ldots+\mu_r\arg\sigma_j(\varepsilon_r)\right)\right\| \le C_2, \qquad r_1\le j\le r,$$

with $C_2=C_2(\delta)>0$. Set

$$Y_j = p^s\left(\mu_1\log\frac{|\sigma_1(\varepsilon_1)|}{|\sigma_j(\varepsilon_1)|}+\ldots+\mu_r\log\frac{|\sigma_1(\varepsilon_r)|}{|\sigma_j(\varepsilon_r)|}\right), \qquad 2\le j\le r,$$

and

$$Z_k = \frac{p^s}{2\pi}\left(\mu_1\arg\sigma_k(\varepsilon_1)+\ldots+\mu_r\arg\sigma_k(\varepsilon_r)\right)\in\mathbf{R}/\mathbf{Z}, \qquad r_1\le k\le r.$$

Taking $0 \le \mu_i < M$, we have $M^r$ points $(\mu_i)_{1\le i\le r}$. The $(Y_j, Z_k)_{2\le j\le r, r_1\le k\le r}$ are in the product of intervals $I_j$, $2 \le j \le r$, of lengths $O(Mp^s)$ and of $r_2$ factors identical to $\mathbf{R}/\mathbf{Z}$. This set can be covered by $C_3(Mp^s)^{r-1}$ sets of diameter at most $\max\{C_1, C_2\}$, where $C_3$ is a constant depending on $\delta$. By Dirichlet's *Schubfachprinzip*, choosing $M$ such that

$$C_3(Mp^s)^{r-1} < M^r ,$$

which can be done with

$$M \asymp p^{(r-1)s} ,$$

we get that there is $(\mu_1, \ldots, \mu_r) \in \mathbf{Z}^r \setminus \{0\}$ such that

$$\max_{1\le i\le r} |\mu_i| \ll M ,$$

$$|Y_j| \le C_1, \qquad 2 \le j \le r,$$

and

$$\|Z_k\| \le C_2, \qquad r_1 \le k \le r.$$

Set then

$$\eta = (\varepsilon_1^{\mu_1} \ldots \varepsilon_r^{\mu_r})^{p^s}$$

in such a way that $0 < \eta < 1$ (if needed, just consider $1/\eta$). This choice implies that

$$|\tau_i(\eta) - 1|_p = |\log_p \tau_i(\eta)|_p \le p^{-s}, \quad 0 \le i \le d,$$

and

$$|\log \eta| \ll p^s M \ll p^{rs},$$

and the lemma is proved. □

**Lemma 4.** *Let $P(X) \in \mathbf{C}_p[X]$ be a polynomial of degree $d$, and write*

$$P(X) = a_0 + \ldots + a_d X^d .$$

*Let $\xi_i$ $(1 \le i \le d)$ be the roots of $P(X)$ in $\mathbf{C}_p$. Let $c$ be a real number satisfying $0 \le c \le 1$. If*

$$|\xi_i|_p \le c , \qquad 1 \le i \le d,$$

*we get*

$$|a_k|_p \le c|a_d|_p , \qquad 0 \le k < d. \tag{3.4}$$

*Conversely, if (3.4) holds, then we have*

$$|\xi_i|_p \le c^{1/d}, \qquad 1 \le i \le d.$$

*Proof.* Since

$$P(X) = a_d \prod_{1\le i\le d} (X - \xi_i) ,$$

if $|\xi_i|_p \le c \le 1$ for $i = 1, \dots, d$, then we have

$$|a_k|_p \le c|a_d|_p\,, \quad \text{for } k = 0, \dots, d-1.$$

Conversely, if

$$|a_k|_p \le c|a_d|_p, \qquad 0 \le k < d,$$

and if $\xi \in \mathbf{C}_p$ is such that

$$a_d \xi^d + \ldots + a_0 = 0\,,$$

then, there exists $k$ with $0 \le k < d$ and

$$|a_k \xi^k|_p \ge |a_d \xi^d|_p,$$

thus,

$$|\xi|_p^d \le |\xi|_p^{d-k} \le c\,.$$

This completes the proof of the lemma. □

We conclude this section with two lemmas used in the proof of Theorem 2. The first of them was proved by Peck [8].

**Lemma 5.** *There exists a sequence $(\eta_m)_{m \ge 1}$ of positive units in $\mathcal{O}$ such that*

$$\eta_m \asymp e^{-dm}$$

*and*

$$|\sigma_j(\eta_m)| \asymp e^m\,, \qquad 1 \le j \le d\,.$$

*Proof.* Let us search the unit $\eta_m$ under the form

$$\eta_m = \varepsilon_1^{\mu_1} \ldots \varepsilon_r^{\mu_r},$$

with $\mu_i \in \mathbf{Z}$. We construct *real* numbers $\nu_1, \dots, \nu_r$ such that

$$\nu_1 \log \varepsilon_1 + \ldots + \nu_r \log \varepsilon_r = -dm \tag{3.5}$$

and

$$\nu_1 \log |\sigma_j(\varepsilon_1)| + \ldots + \nu_r \log |\sigma_j(\varepsilon_r)| = m\,, \qquad 1 \le j \le d\,. \tag{3.6}$$

Taking into account that, by complex conjugation, the equations (3.6) corresponding to an index $j$ with $r_1 \le j < r_1 + r_2$ and to the index $j + r_2$ are identical, and that the sum of (3.5) and equations (3.6) is zero, we simply have to deal with a Cramer system, since the matrix $(\sigma_j(\varepsilon_i))_{1 \le j \le r, 1 \le i \le r}$ is regular. We solve this system and then replace every $\nu_i$ by a rational integer $\mu_i$ such that $|\mu_i - \nu_i| \le 1/2$. □

**Lemma 6.** *Let $\lambda'$ be a positive real number. Let $(\eta_m)_{m \ge 1}$ be a sequence of positive units as in Lemma 5. There exists a finite set $\Gamma = \Gamma(\lambda')$ of non-zero elements of $\mathbf{K}$ such that for all integer polynomial $P(X)$ of degree at most $d$ that satisfies*

$$|P(\alpha)| \le \lambda' H(P)^{-d}, \tag{3.7}$$

*there exist a positive integer $m$ and $\gamma$ in $\Gamma$ for which*

$$P(\alpha) = \gamma \eta_m .$$

*Proof.* Below, all the constants implicit in $\ll$ depend on $\mathbf{K}$ and on $\lambda'$. Let $m$ be a positive integer such that

$$H(P) \asymp e^m ,$$

and set

$$\gamma = P(\alpha)\eta_m^{-1}.$$

Since $D\alpha^k$ is an algebraic integer for $k = 0, \ldots, d$, the algebraic number $D\gamma$ is an algebraic integer, and, by (3.7),

$$|\gamma| \ll 1 .$$

Furthermore, for $j = 1, \ldots, d$, we have

$$|\sigma_j(\gamma)| = |P(\sigma_j(\alpha))| \cdot |\sigma_j(\eta_m^{-1})| \ll H(P)e^{-m} \ll 1 .$$

The algebraic integers $D\gamma \in \mathcal{O}$ and all their complex conjugates being bounded, they form a finite set. $\square$

## 4. Proof of Theorem 1

Let $\delta$ be in $(0, 1)$ to be selected later. Apply Lemma 3 with this $\delta$ to get a unit $\eta$ and apply Lemma 1 with this unit and with $\gamma = \alpha^d$. Since $D^2\alpha^d\eta \in \mathbf{Z} + \ldots + \alpha^d\mathbf{Z}$, we get

$$D^2\eta\alpha^d = a_0 + a_1\alpha + \ldots + a_d\alpha^d = P(\alpha),$$

where, by Corollary 1, $P(X)$ is an integer polynomial of degree $d$ and

$$|P(\alpha)| \asymp H(P)^{-d} \asymp H^{-d}.$$

By Lemma 2, each coefficient of $P(X)$ has its $p$-adic absolute value $\ll (\log 3H(P))^{-1/r}$, except the leading coefficient, whose $p$-adic absolute value equals $|D|_p^2$.

We then infer from Lemma 4 that all the roots of $P(X)$ in $\mathbf{C}_p$ have their $p$-adic absolute value $\ll (\log 3H(P))^{-1/(dr)}$. This proves (2.2).

It now remains for us to guarantee that $P(X)$ has a root very close to $\alpha$. To this end, we proceed to check that

$$|P'(\alpha)| \gg H(P) .$$

Since

$$P'(\alpha) = a_1 + \ldots + da_d\alpha^{d-1} ,$$

we get

$$P'(\alpha) = D^2 \left( \mathrm{Tr}(\eta\alpha^d\beta_1) + 2\alpha\mathrm{Tr}(\eta\alpha^d\beta_2) + \ldots + d\alpha^{d-1}\mathrm{Tr}(\eta\alpha^d\beta_d) \right) ,$$

hence,

$$P'(\alpha) = D^2 \sum_{i=0}^{d} \sum_{k=1}^{d} k\alpha^{k-1} \sigma_i(\eta\alpha^d \beta_k)\,.$$

Let us write

$$P'(\alpha) = D^2 \sum_{i=0}^{d} A_i \sigma_i(\eta)$$

with

$$A_i = \sigma_i(\alpha^d) \sum_{k=1}^{d} k\alpha^{k-1} \sigma_i(\beta_k)\,, \quad i = 0, \ldots, d.$$

Observe first that

$$\sum_{i=1}^{d} A_i \neq 0\,.$$

Indeed, if this is not the case, then, working with the unit $\eta = 1$, that is, with $P(X) = D^2 X^d$ and $P'(\alpha) = dD^2\alpha^{d-1}$, we get

$$d\alpha^{d-1} = A_0 = \alpha^d \sum_{k=1}^{d} k\alpha^{k-1}\beta_k,$$

hence,

$$d = \sum_{k=1}^{d} k\alpha^k \beta_k\,.$$

Taking the trace, and recalling that $\mathrm{Tr}(\alpha^k\beta_k) = 1$, we get

$$d(d+1) = \sum_{k=1}^{d} k\,,$$

a contradiction.

Write

$$P'(\alpha) = D^2 \sum_{i=1}^{d} A_i \sigma_i(\eta) + O(H^{-d}) = D^2 \sigma_1(\eta) \sum_{i=1}^{d} A_i + B$$

with

$$|B| \leq D^2 \sum_{2 \leq i \leq d} |A_i| \cdot |\sigma_1(\eta)| \cdot \left| \frac{\sigma_i(\eta)}{\sigma_1(\eta)} - 1 \right| + O(H^{-d})\,.$$

Selecting now $\delta$ such that

$$\delta \sum_{2 \leq i \leq d} |A_i| \leq \frac{1}{3} \left| \sum_{i=1}^{d} A_i \right|,$$

we infer from Lemma 3 that

$$|P'(\alpha)| \geq \frac{1}{2} D^2 \left| \sigma_1(\eta) \sum_{i=1}^{d} A_i \right| ,$$

when $H$ is sufficiently large. This gives

$$|P'(\alpha)| \gg |\sigma_1(\eta)| \gg H \, .$$

Consequently, $P(X)$ has a root $\xi$ such that

$$|\alpha - \xi| \ll H(P)^{-d-1} \ll H(\xi)^{-d-1}.$$

Classical arguments (see at the end of the proof of Theorem 2.11 in [2]) show that $\xi$ must be real and of degree $d$ if $H$ is sufficiently large. This proves (2.1). Inequality (2.3) follows from (2.1) and (2.2) together with the fact that $\xi$ is of degree $d$. This completes the proof of the theorem.

## 5. Proof of Theorem 2

The constants implicit in $\ll$ and $\gg$ below depend on $\mathbf{K}$, $p$ and $\lambda$. There exists a positive real number $\lambda'$, depending on $\lambda$ and on $d$, such that the minimal defining polynomial $P(X)$ of any real number $\xi$ of sufficiently large height and for which (2.4) holds is of degree $d$ and satisfies

$$|P(\alpha)| \leq \lambda' H(P)^{-d}.$$

Let $(\eta_m)_{m \geq 1}$ be as in Lemma 5. By Lemma 6, it is sufficient to prove Theorem 2 for the integer polynomials $P(X)$ as above such that

$$P(\alpha) = \gamma \eta_m = a_0 + a_1 \alpha + \ldots + a_d \alpha^d.$$

Let $\xi_i$ be the roots of $P(X)$ in $\mathbf{C}_p$ and set

$$u := \max_{1 \leq i \leq d} |\xi_i|_p \, .$$

Assume that $u \leq 1$ . It follows from Lemma 4 that

$$|a_k|_p \leq u |a_d|_p \, , \qquad\qquad 0 \leq k < d \, .$$

Dividing $P(X)$ by $p^s = |a_d|_p^{-1}$ if necessary, we can assume that $|a_d|_p = 1$, and we obtain that

$$|a_k|_p \leq u \, , \qquad\qquad 0 \leq k < d \, .$$

For $j = 1, \ldots, d$, we then have

$$\gamma\eta_m\alpha^{-d} - \tau_j(\gamma\eta_m\alpha^{-d}) = \sum_{k=0}^{d-1} a_k(\alpha^{k-d} - \tau_j(\alpha^{k-d})),$$

hence,

$$|\gamma\eta_m\alpha^{-d} - \tau_j(\gamma\eta_m\alpha^{-d})|_p \ll u\,.$$

Since $|\eta_m|_p = 1$, we get that

$$\left|\frac{\tau_j(\eta_m)}{\eta_m}\frac{\tau_j(\gamma)\alpha^d}{\gamma\tau_j(\alpha^d)} - 1\right|_p \ll u\,.$$

Upon writing

$$\eta_m = \varepsilon_1^{\mu_{1,m}} \ldots \varepsilon_r^{\mu_{r,m}}\,,$$

we have thus

$$u \gg \left|\left(\frac{\tau_j(\varepsilon_1)}{\varepsilon_1}\right)^{-\mu_{1,m}} \cdots \left(\frac{\tau_j(\varepsilon_r)}{\varepsilon_r}\right)^{-\mu_{r,m}} \frac{\tau_j(\gamma)\alpha^d}{\gamma\tau_j(\alpha^d)} - 1\right|_p$$

If

$$\frac{\tau_j(\eta_m)}{\eta_m} = \frac{\gamma\tau_j(\alpha^d)}{\tau_j(\gamma)\alpha^d}$$

holds for $j = 1, \ldots, d$, the number

$$\gamma\eta_m\alpha^{-d}$$

is equal to all its conjuguates, hence is rational, and we have

$$P(\alpha) = b\alpha^d$$

with $b \in \mathbf{Q}$, hence $P(X) = bX^d$, a contradiction. For every $m$, there thus exists an index $j$ such that $1 \leq j \leq d$ and

$$\left(\frac{\tau_j(\varepsilon_1)}{\varepsilon_1}\right)^{-\mu_{1,m}} \cdots \left(\frac{\tau_j(\varepsilon_r)}{\varepsilon_r}\right)^{-\mu_{r,m}} \frac{\tau_j(\gamma)\alpha^d}{\gamma\tau_j(\alpha^d)} \neq 1\,.$$

Consequently, by the theory of linear forms in non-Archimedean logarithms (see e.g., Kunrui Yu [9]), there exists a positive constant $\kappa$ such that

$$u \gg (\max_{1\leq i\leq r} |\mu_{i,m}|)^{-\kappa}.$$

Since $\eta_m \asymp H(P)^{-d}$ and

$$|\log \eta_m| \asymp \max_{1\leq i\leq r} |\mu_{i,m}|,$$

the matrix $(\log|\sigma_j(\varepsilon_i)|)_{1\le i\le r, 1\le j\le r}$ being regular, we conclude that

$$u \gg (\log 3H(\xi))^{-\kappa}\,.$$

This completes the proof of the theorem. □

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