'Approximation by algebraic numbers' (Cambridge Tracts 160)

Recent developments

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These informal notes gather recent references who would be quoted in an hypothetical new edition of the monograph 'Approximation by algebraic numbers'. So, we comment the present edition, one chapter after another.

Bibliographic references from CUP 160 are indicated by boldface numbers in brackets. Additional references are listed by alphabetic order of the authors at the end of the present notes.

Chapter 1.

Chapter 2.

Section 2.5. The combinatorial criterion of Ferenczi and Mauduit [**246**] has been considerably improved upon by Adamczewski, Bugeaud and Luca [11]. With this new criterion, Adamczewski and Bugeaud [2] established that irrational real numbers whose expansion in a given integer base has sublinear block complexity are transcendental. In particular, irrational automatic numbers are transcendental.

New examples of transcendental numbers with bounded partial quotients can be found in [3, 6, 7, 10, 40]. The paper [40], who was the first (don't trust the publication date!) to point out the importance of continuants, is superseeded by [6] All the results of A. Baker's paper [**37**] are improved in [6]. An alternative proof of the transcendence of the Thue–Morse continued fraction is given in [8] (see also [7]).

Chapter 3.

Section 3.4. The lower estimate (3.14) has been slightly improved upon by Tsishchanka [79], who got the lower bound $n/2 + \varepsilon_n$, where ε_n tends to 3 as n tends to infinity.

Section 3.6. See [36, ??] for further results and questions on exponents of Diophantine approximation.

For subsequent interesting results on the simultaneous approximation of a number and its squares, see Roy's and Fischler's papers [??, ??, 52, 53, 54].

Section 3.8. A. Baker's criterion [**38**] for proving that, under some condition, a real number cannot be a U-number has been used in [12] and improved in [9], where further applications of the new criterion are given.

W. M. Schmidt [76] has defined Mahler and Koksma classification of points in $\mathbf{R}^n$ and $\mathbf{C}^n$.

Chapter 4.

Chapter 5.

Chapter 6.

Chapter 7.

Chapter 8.

Chapter 9.

Section 9.1. Approximation of complex algebraic numbers by algebraic numbers of bounded degree is studied in [34]. Results from [**233**] are improved. Among others, it is proved that, for $n \geq d$, we have $w_n(\xi) = (n-1)/2$ if n is odd and $w_n(\xi) \in \{n/2, (n-1)/2\}$ if n is even, and both cases do happen.

Chapter 10.

Appendix A.

Appendix B.

Addition to the bibliography

[1] B. Adamczewski, *Sur l'exposant de densité des nombres algébriques*, Int. Math. Res. Not. Vol. 2007 (2007), article ID 024, 6 pp.

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