On the Nagell–Ljunggren equation $\frac{x^n-1}{x-1} = y^q$

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Abstract. We establish several new results on the Nagell–Ljunggren equation $(x^n-1)/(x-1) = y^q$. Among others, we prove that, for every solution (x,y,n,q) to this equation, n has at most four prime divisors, counted with their multiplicities.

1. Introduction

The first results on the Diophantine equation

$$\frac{x^n-1}{x-1} = y^q, \quad \text{in integers } x>1, y>1, n>2, q \geq 2, \tag{1}$$

go back to 1920 and Nagell's papers [12, 13]. Some twenty years later, Ljunggren [8] clarified some points in Nagell's arguments and completed the proof of the following statement.

Theorem NL. *Apart from the solutions*

$$\frac{3^5-1}{3-1} = 11^2, \quad \frac{7^4-1}{7-1} = 20^2 \;\; \textit{and} \;\; \frac{18^3-1}{18-1} = 7^3, \tag{$\mathcal{S}$}$$

Equation (1) has no other solution (x,y,n,q) if either one of the following conditions is satisfied:

$$(i) \qquad q = 2,$$

$$(ii) \qquad 3 \;\; \textit{divides} \;\; n,$$

$$(iii) \qquad 4 \;\; \textit{divides} \;\; n,$$

$$(iv) \qquad q = 3 \;\; \textit{and} \;\; n \not\equiv 5 \;\; (\textit{mod} \;\; 6).$$

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Equation (1) asks for pure powers written with only the digit 1 in base x. It has only finitely many solutions when x is fixed, as proved by Shorey and Tijdeman [18]. We refer the reader to [5, 17] for surveys of known results on (1), now called the Nagell–Ljunggren equation. Presumably, the only solutions to (1) are given by $(\mathcal{S})$, however, we are still unable to prove that (1) has only finitely many solutions.

Very recently, the second author [10, 11] established sharp upper bounds for the solutions of the Diophantine equation

$$\frac{x^p - 1}{x - 1} = p^e \cdot y^q, \quad \text{in integers } x > 1, y > 1, e \in \{0, 1\}, \tag{2}$$

where p and q are (not necessarily distinct) odd prime numbers. The main purpose of the present work is to show how these results together with older ones [2, 3, 6], obtained by the first author with collaborators, apply to Equation (1). Among other statements, we establish that, for any solution (x, y, n, q) to (1), the exponent n has at most four prime factors counted with multiplicities.

2. Statement of the results

For any integer $n \geq 2$, we denote by $\omega(n)$ the number of distinct prime factors of n, and by $\Omega(n)$ the total number of prime divisors of n, counted with multiplicities. Observe that we have $1 \leq \omega(n) \leq \Omega(n)$.

Theorem 1. *Let (x, y, n, q) be a solution of Equation (1) not in $(\mathcal{S})$. Then, the least prime divisor of n is at least equal to 29 and $\Omega(n) \leq 4$. Furthermore, n is prime if $q = 3$. Moreover, if q divides n, then $n = q$.*

It is an open problem to prove that (1) has only finitely many solutions (x, y, n, q) with $n = q$. The fact that (1) has no further solution with n even follows from Catalan's Conjecture [9].

Our Theorem 1 considerably improves part (i) of Theorem 2 of Shorey [16], who established that (1) has only finitely many solutions (x, y, n, q) with $\omega(n) > q - 2$ (*). According to Shorey [17], page 477, 'An easier question than the conjecture that (1) has only finitely many solutions is to replace $\omega(n) > q - 2$ by $\omega(n) \geq 2$ in the above result'. Theorem 1 is a step in this direction: presumably, (1) has only one solution with n composite, namely $(7^4 - 1)/(7 - 1) = 20^2$.

Besides the new upper bounds obtained in [10, 11], the main ingredient for the proof of Theorem 1 is a factorisation recalled in Lemma 1 below. It easily follows from Lemma 1 and from Theorem NL that, in order to prove that (1) has no solution outside $(\mathcal{S})$, it is sufficient to solve (2) for any odd prime numbers p and q. We are able to considerably improve this assertion.

(*) Actually, it is explained in [17], page 476, and in [5], Théorème 15, that inserting results from [7] and [1] in the same proof yields that (1) has no solution (x, y, n, q) with $\omega(n) > q - 2$

Theorem 2. *For proving that Equation (1) has no solution outside $(\mathcal{S})$, it is sufficient to establish that, for any odd prime numbers p and q with $p \geq 5$, the Diophantine equation*

$$\frac{x^p - 1}{x - 1} = y^q$$

has no solution in positive integers x, y.

Theorem 2 asserts that for proving that Equation (1) has no fourth solution (x, y, n, q), it is sufficient to establish that it has no fourth solution (x, y, p, q) with p prime. We do not have to deal anymore with Equation (2) with $e = 1$.

3. Auxiliary results

Let φ denote the Euler totient function. For any positive integer n, let $G(n)$ denote the square-free part of n and set $Q_n := \varphi(G(n))$.

We begin by quoting a result of Shorey [15].

Lemma 1. *Let (x, y, n, q) be a solution of (1) with n odd. If the divisor D of n satisfies $(D, n/D) = (D, Q_{n/D}) = 1$, then there exist integers y_1 and y_2 with $y_1 y_2 = y$ and*

$$\frac{(x^D)^{n/D} - 1}{x^D - 1} = y_1^q \quad \textit{and} \quad \frac{x^D - 1}{x - 1} = y_2^q.$$

By successive applications of Lemma 1, we get the first part of the next statement (see [15]). A detailled proof of the second part is given in Ribenboim's book [14].

Lemma 2. *If Equation (1) has a solution (x, y, n, q) where $n = 2^a\, p_1^{u_1} \dots p_\ell^{u_\ell}$, with $a \in \{0, 1\}$, $u_i > 0$, and p_i distinct odd primes, then for each $i = 1, \dots, \ell$, there exists an integer y_i such that*

$$\frac{x^{p_i^{u_i}} - 1}{x - 1} = y_i^q.$$

Furthermore, there exist integers $w_i \geq 2$ and $z_i \geq 2$ such that

$$\frac{w_i^{p_i} - 1}{w_i - 1} = z_i^q \quad \textit{or} \quad p_i \cdot z_i^q,$$

the second possibility occurring only if q divides u_i.

Next Lemmas gather various results useful for our proofs.

Lemma 3. *If Equation (1) has a solution (x, y, n, q) outside $(\mathcal{S})$, then $x \geq 10^6$, $x \geq 2q + 1$ and the least odd prime divisor of n is at least 29.*

Proof. The lower bounds on x are established in [3] and in [6]. The last result of the Lemma follows from Théorème 2 from [2] and [10]. □

Lemma 4. *Let (x, y, p, q) be an integer quadruple satisfying (2) with p and q odd prime numbers. Then, we have $q < (p-1)^2$ and*

$$x < q^{10p^2}, \quad \textit{if} \quad q \le p,$$
$$x < 2q^{10p^2(p-1)}, \quad \textit{if} \quad q \ge p+2.$$

Furthermore, if $p = q$, then $x \le (2p)^p$.

Proof. The first statement is contained in Theorem 1 from [11], and the remaining part of the lemma follows from Theorem 2 from [11]. □

4. Proofs

Proof of Theorem 1.

The first assertion of the theorem is contained in Lemma 3.

Let (x, y, n, q) be a solution of (1) with n even. Write $n = 2^a m$ with m odd. In view of Lemma 1, we may assume that $a = 1$, and thus we get

$$\frac{x^m - 1}{x - 1} \cdot (x^m + 1) = y^q. \tag{3}$$

Clearly, the greatest common divisor of $x^m - 1$ and $x^m + 1$ is at most 2, and is 2 only if x is odd. But in this case $(x^m - 1)/(x - 1)$ is odd, and the two factors in the left-hand side of (3) are coprime. Consequently, $x^m + 1$ is a q-th power in any case. By the proof of Catalan's Conjecture [9], this never happens.

Let (x, y, n, q) be a solution of (1). Write $n = p_1^{u_1} \dots p_\ell^{u_\ell}$ with positive integers $u_1, \dots, u_\ell$ and prime numbers $p_1 > \dots > p_\ell$. Assume that $\ell \ge 2$ and set $D = p_1^{u_1} \dots p_{\ell-1}^{u_{\ell-1}}$. By Lemma 2, the equation

$$\frac{X^{p_\ell^{u_\ell}} - 1}{X - 1} = y^q$$

has the solution $X = x^D$. If $u_\ell = 1$, then we infer from Lemmas 3 and 4 that

$$(2q+1)^D \le x^D < q^{10p_\ell^3}. \tag{4}$$

Since $p_\ell \ge 29$, it follows that $10p_\ell^3 < p_\ell^4 < p_{\ell-1}^4$, and we get $u_1 + \dots + u_{\ell-1} \le 3$. Thus,

$$u_1 + \dots + u_\ell \le 4. \tag{5}$$

If $u_\ell > 1$, then

$$\frac{X^{p_\ell^{u_\ell}} - 1}{X^{p_\ell^{u_\ell - 1}} - 1} \times \frac{X^{p_\ell^{u_\ell - 1}} - 1}{X^{p_\ell^{u_\ell - 2}} - 1} \times \dots \times \frac{X^{p_\ell} - 1}{X - 1} = y^q,$$

and we see that

$$\frac{X^{p_\ell^{u_\ell}} - 1}{X^{p_\ell^{u_\ell - 1}} - 1} = z^q \quad \text{or} \quad p_\ell \cdot z^q,$$

the latter possibility occurring only if p_ℓ divides u_ℓ. Consequently, the equation

$$\frac{X^{p_\ell}-1}{X-1} = p_\ell^e \cdot Y^q$$

has a solution given by $e=0$ or 1 and $X = x^{Dp_\ell^{u_\ell - 1}}$. Arguing as above, we also get (5) in this case, that is $\Omega(n) \le 4$, as claimed.

Assume now that $q=3$. As mentionned after the statement of Theorem 1, we already know that $\omega(n)=1$. Thus, n must be a prime power, say $n=p^a$, with $1 \le a \le 4$ and $p \ge 5$, by Theorem NL and by what has just been proved. Since, again by Theorem NL, Equation (1) has no solution with $n \equiv 1 \pmod 3$, we get that $a=1$ or $a=3$. Assume that there are positive integers x, y and a prime number $p \ge 5$ with

$$\frac{x^{p^3}-1}{x-1} = y^3.$$

Then $X = x^{p^2}$ is a solution of the equation

$$\frac{X^p-1}{X-1} = p^e \cdot y^3, \quad e \in \{0,1\},$$

and from Lemmas 3 and 4 we gather that

$$10^{6p^2} < x^{p^2} < 3^{10p^2},$$

a contradiction. Consequently, $a=1$ and n must be a prime number.

Now, we consider the last assertion of the theorem. Let (x,y,n,q) be a solution to (1) with q divides n. Then, as proved by Shorey [17], n is a q-th power. Consequently, n is either equal to q, q^2, q^3 or q^4. In view of Theorem NL, we may assume that $q \ge 5$, and Lemma 2 implies that if $n \ne q$, then $X = x^q$ satisfies

$$\frac{X^q-1}{X-1} = y^q.$$

The combination of Lemmas 3 and 4 then yields that

$$(2q+1)^q \le x^q \le (2q)^q,$$

a contradiction. Alternatively, we can apply a result of Le [7], asserting that Equation (1) has no solution with x being a q-th power. Consequently, we have proved that if n is a power of q, then $n=q$. □

Proof of Theorem 2.

In view of Lemma 2, we encounter the equation

$$\frac{x^p-1}{x-1} = py^q$$

only if Equation (1) has a solution (x, y, n, q) with $n = p^u$ and q divides u. By Theorem 1, this can only happen when $q = u = 3$. Thus, to establish Theorem 2, it only remains to prove that the Diophantine equation

$$\frac{x^{p^3} - 1}{x - 1} = y^3$$

has no solution, which has already been done in the proof of Theorem 1. □

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