Exponents of Diophantine approximation

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1. Introduction

The well known Dirichlet Theorem asserts that for any irrational real number ξ and *any* real number $Q \geq 1$, there exist integers p and q with $1 \leq q \leq Q$ and

$$|q\xi - p| \leq Q^{-1}. \qquad (1)$$

As observed by Khintchine [34], there is no ξ for which the exponent of Q in (1) can be lowered (see Davenport & Schmidt [17] or Schmidt [49] for a very precise result). However, for any $w > 1$, there clearly exist real numbers ξ for which, for *arbitrarily large* values of Q, the equation

$$|q\xi - p| \leq Q^{-w}$$

has a solution in integers p and q with $1 \leq q \leq Q$. Obviously, the quality of approximation strongly depends on whether we are interested in a uniform statement (i.e., a statement valid for any Q, or for any Q sufficiently large) or in a statement valid for arbitrarily large Q. In the case of rational approximation, these questions are quite well understood, essentially thanks to the continued fraction theory. However, the Dirichlet Theorem extends well to rational simultaneous approximation, and to simultaneous approximation of linear forms. In Section 2, we define exponents of Diophantine approximation related to these questions, and survey known results on them. The two dimensional case is now fully understood and the results are displayed in Section 3. In order to extend possibly these results in higher dimension, we define geometrically in Section 4 more exponents of Diophantine approximation, in connection with the work of Schmidt [50].

It is a notorious fact that questions of Diophantine approximation are in general much more difficult when the quantities we approximate are dependent. Classical examples include the simultaneous rational approximation of the first n powers of a transcendental number, and the approximation of linear forms whose coefficients are precisely the first n powers of a transcendental number. These are considered in Section 5. When $n = 2$, important progress has been made recently by Roy [43, 44, 45, 46, 47, 48], and Section 6 is devoted to his results and some of their extensions.

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The present paper is in great part a survey, however, it contains several new results (e.g., Theorems 8 and 10). General references for the topic investigated here are the seminal paper of Khintchine [34], Cassels' book [14] and the monograph [10].

2. Approximation of independent quantities

We begin with some notations and definitions. If $\underline{\theta}$ is a (column) vector in $\mathbf{R}^n$, we denote by $|\underline{\theta}|$ the maximum of the absolute values of its coordinates and by

$$\|\underline{\theta}\| = \min_{\underline{x} \in \mathbf{Z}^n} |\underline{\theta} - \underline{x}|$$

the maximum of the distances of its coordinates to the rational integers.

Following the convention introduced in [11], we indicate by a 'hat' the exponents of *uniform* Diophantine approximation.

Definition 1. *Let n and m be positive integers and let A be a real matrix with n rows and m columns. We denote by $\omega_{n,m}(A)$ the supremum of the real numbers w for which, for arbitrarily large real numbers X, the inequalities*

$$\|A\underline{x}\| \le X^{-w} \quad \textit{and} \quad |\underline{x}| \le X \tag{2}$$

have a non-zero solution $\underline{x}$ in $\mathbf{Z}^m$. We denote by $\hat{\omega}_{n,m}(A)$ the supremum of the real numbers w for which, for all sufficiently large positive real numbers X, the inequalities (2) have a non-zero integer solution $\underline{x}$ in $\mathbf{Z}^m$.

For a $n \times m$ matrix A, the Dirichlet box principle implies that

$$\omega_{n,m}(A) \ge \hat{\omega}_{n,m}(A) \ge \frac{m}{n}. \tag{3}$$

Furthermore, we have both equalities in (3) for almost all matrices A, with respect to the Lebesgue measure on $\mathbf{R}^{mn}$, as follows from the Borel–Cantelli Lemma.

The left-hand side inequality of (3) has been improved by Jarník [29, 30] as follows.

Theorem 1. *For any $n \ge 2$ and any $n \times 1$ real matrix A with at least two coefficients which are $\mathbf{Q}$-linearly independent modulo $\mathbf{Q}$, we have*

$$\omega_{n,1}(A) \ge \frac{\hat{\omega}_{n,1}^2(A)}{1 - \hat{\omega}_{n,1}(A)}. \tag{4}$$

For any $n \ge 1$ and any $n \times 2$ real matrix A, we have

$$\omega_{n,2}(A) \ge \hat{\omega}_{n,2}(A)(\hat{\omega}_{n,2}(A) - 1). \tag{5}$$

For any $n \geq 1$, any $m \geq 3$ and any $n \times m$ real matrix A with $\hat{\omega}_{n,m}(A) > (5m^2)^{m-1}$, we have

$$\omega_{n,m}(A) \geq \big(\hat{\omega}_{n,m}(A)\big)^{m/(m-1)} - 3\,\hat{\omega}_{n,m}(A).$$

In all what follows, we denote by ${}^tA$ the transpose of the matrix A. It is well-known that $\omega_{n,m}(A)$ and $\omega_{m,n}({}^tA)$ are linked by a transference principle. Dyson [20] established the lower bound

$$\omega_{n,m}(A) \geq \frac{m\,\omega_{m,n}({}^tA) + m - 1}{(n-1)\omega_{m,n}({}^tA) + n}, \tag{6}$$

thus extending earlier results of Khintchine [33, 34] who delt with the case $\min\{n, m\} = 1$. For a proof, the reader is referred to Gruber & Lekkerkerker [24], Section 45.3, Cassels [14], Chapter V, Theorem IV, or Schmidt [49], Chapter IV, Section 5. Inequalities (6) have been shown to be best possible for $\min\{n, m\} = 1$ by Jarník [25, 27], who also got some related results [26]. For general m and n, Jarník [31] proved that (6) is best possible except, possibly, when $1 < n < m$ and $\omega_{n,m}(A) < (m-1)/(n-1)$, in which case his method does not give anything.

Furthermore, extending earlier results of Jarník [28], Apfelbeck [5] established that the uniform exponents $\hat{\omega}_{n,m}(A)$ and $\hat{\omega}_{m,n}({}^tA)$ are linked by the same relation

$$\hat{\omega}_{n,m}(A) \geq \frac{m\,\hat{\omega}_{m,n}({}^tA) + m - 1}{(n-1)\hat{\omega}_{m,n}({}^tA) + n}. \tag{7}$$

Jarník [28] and Apfelbeck [5] succeeded in improving (7) when either $\hat{\omega}_{n,m}(A)$ or $\hat{\omega}_{m,n}({}^tA)$ is large. Before investigating the set of values taken by the functions $\omega_{n,m}$ and $\hat{\omega}_{n,m}$, we introduce the following definition.

Definition 2. *By spectrum of an exponent of Diophantine approximation, we mean the set of values taken by this exponent on the set of n by m real matrices A of maximal rank, i.e. of rank $\min\{m, n\}$.*

Since $\omega_{1,1}((\xi)) = \omega_{2,1}({}^t(\xi, \xi))$ holds for any real number ξ, we give the above definition of spectrum to avoid trivialities.

Except for $m = n = 1$ (in that case, we can use the continued fraction theory), it is in general a difficult problem to construct explicit examples of regular n by m matrices A with prescribed values for $\omega_{n,m}(A)$ and/or for $\hat{\omega}_{n,m}(A)$. However, the spectrum of the function $\omega_{n,m}$ has been completely determined, thanks to a deep result of Dickinson & Velani [19] (see also [7]), who calculated the Hausdorff dimension of the set of matrices A with $\omega_{n,m}(A) = \tau$, for an arbitrary real number τ.

Theorem 2. *For any positive integers n and m, the spectrum of the function $\omega_{n,m}$ is equal to $[m/n, +\infty]$.*

As for the spectra of the exponents $\hat{\omega}_{n,m}$, much less is known. They are contained in $[1/n, 1]$ if $m = 1$ and in $[m/n, +\infty]$ if $m \geq 2$ (this is an immediate consequence of (3)).

In particular, we have $\hat{\omega}_{1,1}((\xi)) = 1$ for any irrational real number ξ. The situation is completely different in the case $(m,n) \neq (1,1)$.

Theorem 3. *For any positive integers m, n with $m \geq 2$ there are continuum many n by m real matrices A whose coefficients are algebraically independent and which satisfy $\hat{\omega}_{n,m}(A) = +\infty$. For any positive integer n, there are continuum many n by 1 real matrices A whose coefficients are algebraically independent and which satisfy $\hat{\omega}_{n,1}(A) = 1$.*

Proof : For $(m,n) = (2,1)$ or $(1,2)$ and the coefficients of the corresponding matrices are linearly independent, this is due to Khintchine [34] (see also Theorem XIV, page 94, of [14]). Further results have been obtained by Chabauty & Lutz [15]. Jarník [32] completed the proof of the theorem, using a quite different approach (see also Lesca [39]). □

We address the following problem, which is likely to be difficult.

Problem 1. *For positive integers m and n, determine the spectrum of the function $\hat{\omega}_{n,m}$.*

Partial results when $\min\{n,m\} = 1$ have been established by Jarník [30].

Theorem 4. *The spectrum of $\hat{\omega}_{1,2}$ (resp. $\hat{\omega}_{2,1}$) is equal to $[2,+\infty]$ (resp. to $[1/2,1]$). For any integers $m \geq 2$ and $n \geq 2$, the spectrum of $\hat{\omega}_{1,m}$ contains $(2^{m-1},+\infty]$ and that of $\hat{\omega}_{n,1}$ contains $((u_n - 2 - u_n^{-n+1})/(u_n - 1), 1]$, where u_n is the largest real root of the polynomial $X^{n-1} - X^{n-2} - \sum_{k=0}^{n-2} X^k$.*

Jarník's proof of Theorem 4 is constructive and rests on the continued fraction theory.

In the above definition of the exponent $\omega_{n,m}$ (resp. $\hat{\omega}_{n,m}$), we do not require that there exists a positive constant c such that, for arbitrarily large real numbers X (resp. for any sufficiently large real number X), the inequalities

$$\|A\underline{x}\| \leq c\, X^{-\omega_{n,m}} \quad \text{and} \quad |\underline{x}| \leq X$$

and

$$\|A\underline{x}\| \leq c\, X^{-\hat{\omega}_{n,m}} \quad \text{and} \quad |\underline{x}| \leq X,$$

respectively, have a non-zero solution $\underline{x}$ in $\mathbf{Z}^m$. Taking this into consideration yields new problems.

Actually, Dirichlet's Theorem implies that, for any $X > 1$, the inequations

$$\|A\underline{x}\| \leq c\, X^{-m/n} \quad \text{and} \quad |\underline{x}| \leq X \tag{8}$$

have a non-zero solution $\underline{x}$ in $\mathbf{Z}^m$, when $c = 1$. This suggests to us to introduce the following definition.

Definition 3. *Let A be a $n \times m$ real matrix. We say that Dirichlet's Theorem can be improved for the matrix A if there exists a positive constant $c < 1$ such that (8) has a solution $\underline{x}$ in $\mathbf{Z}^m$ for any sufficiently large X.*

When $m = n = 1$, that is, when $A = ((\xi))$ for some irrational real number ξ, it is well known that Dirichlet's Theorem can be improved if, and only if, ξ has bounded partial quotients in its continued fraction expansion. A precise statement has been obtained by Davenport & Schmidt [17]. In particular, the set of 1×1 matrices A for which Dirichlet's Theorem can be improved has Lebesgue measure zero and Hausdorff dimension 1. This assertion has been partly extended to linear forms and to simultaneous approximation by Davenport & Schmidt [18].

Theorem 5. *For any positive integer n, the set of $n \times 1$ (resp. of $1 \times n$) matrices for which Dirichlet's Theorem can be improved has n-dimensional Lebesgue measure zero.*

Notice that Khintchine [35] proved that the set of *singular* $n \times m$ real matrices A (meaning that for *all* positive constant c, the inequations (8) have a non-zero solution $\underline{x}$ in $\mathbf{Z}^m$ for all X greater than $X_0(A, c)$), has mn-dimensional Lebesgue measure zero. This weaker result is a consequence of the Borel–Cantelli lemma (see [14], page 92). The proof of Theorem 5 is quite involved. According to Kleinbock and Weiss [36], it can be generalized to $n \times m$ matrices. Actually, a more general result is proved in [36]. Maybe, it is possible to adapt the methods of [18, 36] to solve the following problem, which seems to be rather difficult.

Problem 2. *Let c be a real number with $0 < c < 1$. Determine the Hausdorff dimension of the set of $n \times m$ matrices such that (8) has a solution $\underline{x}$ in $\mathbf{Z}^m$ for any sufficiently large X.*

We end this section by briefly mentionning that we can as well consider inhomogeneous problems in Diophantine approximation (see Chapters III and V from [14]). The corresponding exponents of approximation have been introduced in [12], where it is established that the exponent of approximation to a generic point in $\mathbf{R}^n$ by a system of n linear forms is equal to the inverse of the uniform homogeneous exponent associated to the system of dual linear forms.

3. Approximation in dimension two

We investigate more precisely in this section the above spectra when A is an 1×2 or a 2×1 real matrix. In this case, the uniform exponents $\hat{\omega}(A)$ and $\hat{\omega}({}^tA)$ are linked by an equation due to Jarník [28]. This fact seems to have been completely forgotten since 1938. maintenant

Theorem 6. *For any 1×2 real matrix $A = (\alpha, \beta)$, with α or β irrational, the equality*

$$\hat{\omega}_{2,1}({}^tA) = 1 - \frac{1}{\hat{\omega}_{1,2}(A)}$$

holds.

On the other hand, Khintchine's transference inequalities (6) read here

$$\frac{\omega_{1,2}(A)}{\omega_{1,2}(A)+2} \leq \omega_{2,1}({}^tA) \leq \frac{\omega_{1,2}(A)-1}{2},$$

for any matrix A as in Theorem 6. Next Theorem, which is the main result of [38], refines this latter estimate.

Theorem 7. *For any row vector $A = (\alpha, \beta)$ with $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$, the four exponents*

$$v = \omega_{1,2}(A), \quad v' = \omega_{2,1}({}^tA), \quad w = \hat{\omega}_{1,2}(A), \quad w' = \hat{\omega}_{2,1}({}^tA),$$

satisfy the relations

$$2 \leq w \leq +\infty, \quad w' = \frac{w-1}{w}, \quad \frac{v(w-1)}{v+w} \leq v' \leq \frac{v-w+1}{w}.$$

When $v = +\infty$ we have to understand these relations as $w - 1 \leq v' \leq +\infty$, and $w' = 1, v' = +\infty$ if moreover $w = +\infty$. Conversely, for each quadruple (v, v', w, w') in $(\mathbf{R}_{>0} \cup \{+\infty\})^4$ satisfying the previous conditions, there exists a row vector $A = (\alpha, \beta)$ of real numbers with $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$, such that

$$v = \omega_{1,2}(A), \quad v' = \omega_{2,1}({}^tA), \quad w = \hat{\omega}_{1,2}(A), \quad w' = \hat{\omega}_{2,1}({}^tA).$$

Notice that our estimate

$$\frac{v(w-1)}{v+w} \leq v' \leq \frac{v-w+1}{w} \tag{9}$$

refines Khintchine's inequalities since $w \geq 2$. Theorem 7 implies that the lower bound (5) is optimal when $n = 1$.

Corollary 1. *For any row vector $A = (\alpha, \beta)$ with $1, \alpha, \beta$ linearly independent over $\mathbf{Q}$, the lower bounds*

$$\hat{\omega}_{1,2}(A) \geq 2 \quad \text{and} \quad \omega_{1,2}(A) \geq \hat{\omega}_{1,2}(A)(\hat{\omega}_{1,2}(A) - 1)$$

hold. Moreover, let v and w be positive real numbers with

$$w \geq 2 \quad \text{and} \quad v \geq w(w-1).$$

Then there exists a row vector $A \in \mathbf{R}^2$ such that

$$\omega_{1,2}(A) = v \quad \text{and} \quad \hat{\omega}_{1,2}(A) = w.$$

Similarly, the lower bound (4) of Theorem 1 is best possible for $n = 2$.

Corollary 2. *For any column vector* $A = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ *with* $1, \alpha, \beta$ *linearly independent over* $\mathbf{Q}$, *the inequalities*

$$\frac{1}{2} \le \hat{\omega}_{2,1}(A) \le 1 \quad \text{and} \quad \omega_{2,1}(A) \ge \frac{\hat{\omega}_{2,1}(A)^2}{1 - \hat{\omega}_{2,1}(A)}$$

hold. Moreover, let v' *and* w' *be positive real numbers satisfying*

$$\frac{1}{2} \le w' \le 1 \quad \text{and} \quad v' \ge \frac{w'^2}{1 - w'}.$$

Then there exists a column vector $A \in \mathbf{R}^2$ *such that*

$$\omega_{2,1}(A) = v' \quad \text{and} \quad \hat{\omega}_{2,1}(A) = w'.$$

For the deduction of the corollaries, observe that, for given positive real numbers v and w, the interval

$$\frac{v(w-1)}{v+w} \le v' \le \frac{v-w+1}{w}$$

occurring in Theorem 7, is non-empty exactly when $v \ge w(w-1)$. For the minimal value $v = w(w-1)$, it reduces to the point

$$\frac{(w-1)^2}{w} = \frac{w'^2}{1-w'}.$$

Corollaries 1 and 2 immediately follow.

4. Further problems

Having regard to Section 3, we are led to ask for an extension of Theorem 7 in higher dimension.

Problem 3. *Let* n *and* m *be positive integers with* $n \le m$. *Describe the set of quadruples*

$$(\omega_{n,m}(A), \omega_{m,n}({}^tA), \hat{\omega}_{n,m}(A), \hat{\omega}_{m,n}({}^tA)),$$

when A *ranges over the set of real* $n \times m$ *matrices.*

Even a conjectural answer to Problem 3 is unclear to us, unless when $(m,n) = (1,1)$ or $(m,n) = (2,1)$. A (small) contribution towards the resolution of Problem 3 is the following refinement of Khintchine's transference inequalities for row matrices.

Theorem 8. *Let $m \geq 2$ and A be a $1 \times m$ real matrix. Set*

$$v = \omega_{1,m}(A), \quad v' = \omega_{m,1}({}^tA), \quad w = \hat{\omega}_{1,m}(A), \quad w' = \hat{\omega}_{m,1}({}^tA),$$

Then, we have

$$v' \geq \frac{v}{\frac{(m-1)w}{w-1}(v+1) - v} \quad \text{and} \quad v \geq \frac{(m-1)(1+v')}{1-w'} - 1. \tag{10}$$

When $m = 2$, the combination of Theorems 6 and 8 yields the inequalities (9). With the notation of Theorem 8, inequality (6) reads

$$v' \geq \frac{v}{m(v+1) - v} \quad \text{and} \quad v \geq m(1+v') - 1.$$

This is weaker than (10) since $w \geq m$ and $w' \geq 1/m$.

To establish Theorem 8, we insert in the proof of Theorem II from Chapter V of [14] an upper bound for the second of the successive minima of the convex body involved. This upper bound is obtained by following Section 9 of the same Chapter and making a suitable use of the exponents of uniform approximation. Full details will be given in a subsequent work.

In order to enlighten Problem 3, it might be relevant to introduce more exponents of approximation in intermediate dimensions. In this section, we reformulate and extend geometrically the definitions of the exponents $\omega_{n,m}(A)$ and $\hat{\omega}_{n,m}(A)$.

Let Θ and L be two real linear subvarieties contained in a projective space $\mathbf{P}^N(\mathbf{R})$. Assume that Θ and L are non-empty and distinct from $\mathbf{P}^N(\mathbf{R})$, and set

$$\delta = \dim \Theta \quad \text{and} \quad d = \dim L.$$

Following Schmidt [50] (see also the appendix of [42]) and using Euclidean geometry, we can attach to Θ and L a sequence of sines

$$0 \leq \psi_1(\Theta, L) \leq \ldots \leq \psi_t(\Theta, L) \leq 1, \quad \text{where} \quad t = \min(d+1, N-d, \delta+1, N-\delta),$$

of various acute angles measuring in the ambient space $\mathbf{P}^N(\mathbf{R})$ the proximity of the linear subvarieties Θ and L. When $\delta + d \leq N - 1$, the intersection $\Theta \cap L$ is usually empty, and the smallest sine $\psi_1(\Theta, L)$ can be compared with the minimal projective distance (the normalization does not matter for our purpose) between the points of Θ and L. Notice that we have $t = 1$ exactly when either Θ or L is either a point or an hyperplane of $\mathbf{P}^N(\mathbf{R})$.

When L is rational over $\mathbf{Q}$, we denote by $H(L)$ its height, as defined in [50]. We are now able to extend Definition 1 in the following way.

Definition 4. *Let Θ be a proper real linear subvariety of $\mathbf{P}^N(\mathbf{R})$ of dimension δ. Let d and k be integers with*

$$0 \le d \le N-1 \quad \textit{and} \quad 1 \le k \le \min(\delta+1, N-\delta, d+1, N-d).$$

We denote by $w_{d,k}(\Theta)$ the supremum of the real numbers w such that for arbitrarily large real numbers X there exists a rational linear subvariety $L \subset \mathbf{P}^N(\mathbf{R})$ of dimension d, satisfying the inequations

$$\psi_k(\Theta, L) \le H(L)^{-1}X^{-w} \quad \textit{and} \quad H(L) \le X. \tag{11}$$

Similarly, we denote by $\hat{w}_{d,k}(\Theta)$ the supremum of the real numbers w such that, for all sufficiently large positive real number X, there exists a rational linear subvariety $L \subset \mathbf{P}^N(\mathbf{R})$ of dimension d satisfying (11).

The link with Definition 1 is achieved in the following way. Let A be a real $n \times m$ matrix. Put $N = m+n-1$ and associate to A the $(n-1)$-dimensional linear subvariety

$$\Theta = \mathbf{P}(\mathrm{Span}(A \times I_n)) \subset \mathbf{P}^N(\mathbf{R}) = \mathbf{P}(\mathbf{R}^{m+n}),$$

whose points have homogeneous coordinates belonging to the vectorial subspace of $\mathbf{R}^{m+n}$ spanned by the rows of the $n \times (m+n)$ matrix (A, I_n). Notice that the application $A \mapsto \Theta$ clearly defines a homeomorphism from $\mathbf{R}^{mn}$ to an open set of the Grassmanian of the $(n-1)$-dimensional linear spaces in $\mathbf{P}^N(\mathbf{R})$. When $d = 0$, the single exponent $w_{0,1}(\Theta)$ (resp. $\hat{w}_{0,1}(\Theta)$) measures the approximation (resp. uniform approximation) to Θ by rational points in $\mathbf{P}^N(\mathbf{R})$. It is readily observed that Definitions 1 and 4 are then consistent, in the sense that

$$w_{0,1}(\Theta) = \omega_{m,n}({}^tA) \quad \text{and} \quad \hat{w}_{0,1}(\Theta) = \hat{\omega}_{m,n}({}^tA).$$

Similarly, in maximal dimension $d = N-1$, the exponents $w_{N-1,1}(\Theta)$ and $\hat{w}_{N-1,1}(\Theta)$ measure the approximation to Θ by rational hyperplanes. Then, the equalities

$$w_{N-1,1}(\Theta) = \omega_{n,m}(A) \quad \text{and} \quad \hat{w}_{N-1,1}(\Theta) = \hat{\omega}_{n,m}(A)$$

hold.

We address the following

Problem 4. *Let δ and N be integers with $0 \le \delta \le N-1$. Find the spectrum of the array of exponents*

$$\Big\{ \dots, w_{d,k}(\Theta), \hat{w}_{d,k}(\Theta), \dots ; 0 \le d < N-1, 1 \le k \le \min(\delta+1, N-\delta, d+1, N-d) \Big\},$$

where Θ ranges over the set of δ-dimensional subvarieties of $\mathbf{P}^N(\mathbf{R})$.

For instance, when $\delta = 0, N = 3$ (approximating points in $\mathbf{P}^3$), we have to investigate the possible values of six exponents, while for $\delta = 1, N = 3$ (approximating lines in $\mathbf{P}^3$), we have to look at 8-tuples. Notice that the spectra associated to the pairs of dimensions (δ, N) and $(N-1-\delta, N)$ are permuted by duality.

Theorem 7 provides the answer of Problem 4 when $N = 2$. Informations about the above array of exponents, including lower bounds and transference results, may be deduced from [50]. However, as far as we are aware, the generic value of this array (if it does exist) remains unclear, unless when $\delta = 0$ or when $\delta = N - 1$.

5. Approximation of dependent quantities

In this section, we assume that either m or n is equal to 1. In other words, we let $\xi_1, \ldots, \xi_n$ be n real numbers, and we take for A either the matrix $(\xi_1, \ldots, \xi_n)$ or the matrix ${}^t(\xi_1, \ldots, \xi_n)$. In Section 2, we have assumed that the ξ_j's are independent. We deal now with the more complicated situation of dependent ξ_j's ; typically, we assume that $(\xi_1, \ldots, \xi_n)$ belong to some given manifold. There is a broad literature on this subject, and we direct the reader to the book of Bernik & Dodson [9] for results and many bibliographical references. In the sequel, we restrict our attention to the case where $\xi_j = \xi^j$ for a given real number ξ and any integer j with $1 \le j \le n$.

This situation is the most classical one. Indeed, in order to define in 1932 his classification of the real numbers ξ, Mahler [40] introduced the exponents of Diophantine approximation $w_n(\xi)$, which correspond to the exponents $\omega_{1,n}((\xi, \xi^2, \ldots, \xi^n))$ defined in Section 2, when ξ is not algebraic of degree at most n. In view of the particular questions investigated in the present section, we do not keep the notation of Definition 1, and we rather use the classical notation, recalled in Definition 5 below.

Definition 5. *Let $n \ge 1$ be an integer and let ξ be a real number. We denote by $w_n(\xi)$ (resp. by $\hat{w}_n(\xi)$) the supremum of the real numbers w such that, for arbitrarily large real numbers X (resp. any sufficiently large real number X), the inequalities*

$$0 < |x_n \xi^n + \ldots + x_1 \xi + x_0| \le X^{-w}, \qquad \max_{0 \le m \le n} |x_m| \le X,$$

have a solution in integers $x_0, \ldots, x_n$. We denote by $\lambda_n(\xi)$ (resp. by $\hat{\lambda}_n(\xi)$) the supremum of the real numbers λ such that, for arbitrarily large real numbers X (resp. any sufficiently large real number X), the inequalities

$$0 < |x_0| \le X, \qquad \max_{1 \le m \le n} |x_0 \xi^m - x_m| \le X^{-\lambda},$$

have a solution in integers $x_0, \dots, x_n$.

Observe that $\lambda_n(\xi) = \omega_{n,1}({}^t(\xi, \xi^2, \dots, \xi^n))$ and $\hat{\lambda}_n(\xi) = \hat{\omega}_{n,1}({}^t(\xi, \xi^2, \dots, \xi^n))$ hold for any $n \geq 1$ and any real number ξ not algebraic of degree at most n.

Unfortunately, Theorem 2 and 7 do not imply any information regarding the values of the functions w_n and λ_n. Solving a long-standing conjecture of Mahler, Sprindžuk [51] proved in 1965 that $w_n(\xi) = n$ holds for any $n \geq 1$ for almost all (with respect to the Lebesgue measure) real numbers ξ. By (6), this implies that $\lambda_n(\xi) = 1/n$ holds for any $n \geq 1$, for almost all real numbers ξ.

Furthermore, it follows from the Schmidt Subspace Theorem (see e.g. [49]) that

$$\hat{w}_n(\xi) = w_n(\xi) = 1/\lambda_n(\xi) = 1/\hat{\lambda}_n(\xi) = \min\{d-1, n\} \tag{12}$$

hold for any real algebraic number ξ of degree d. Thus, to investigate the sets of values taken the functions w_n, $\hat{w}_n$, λ_n and $\hat{\lambda}_n$, we need only to consider them on transcendental numbers.

Definition 6. *By spectrum of the function w_n (resp. $\hat{w}_n$, λ_n and $\hat{\lambda}_n$), we mean the set of values taken by w_n (resp. $\hat{w}_n$, λ_n and $\hat{\lambda}_n$) on the set of transcendental real numbers.*

It is possible to construct explicit examples of real numbers ξ with $w_n(\xi) = w$, for any given real number $w > (2n + 1 + \sqrt{4n^2+1})/2$, see Theorem 7.7 [10] for references. As in the proof of Theorem 3, the theory of Hausdorff dimension is a crucial tool for determining the spectrum of w_n, a problem solved in 1983 by Bernik [8].

Theorem 9. *For any positive integer n, the spectrum of w_n is equal to $[n, +\infty]$.*

Proof : It follows from [8] that, for any real number $\tau \geq n$, we have

$$\dim_{\mathrm{H}}\{\xi \in \mathbf{R} : w_n(\xi) = \tau(n+1) - 1\} = \frac{1}{\tau},$$

where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension. We finish the proof of the theorem by observing that any Liouville number ξ satisfies $w_n(\xi) = w_1(\xi) = +\infty$. □

As far as we are aware, the spectra of the functions λ_n have not been studied up to now when $n \geq 2$.

Theorem 10. *For any integer $n \geq 1$, the spectrum of λ_n includes the interval $((3 + \sqrt{4n^2+1})/(2n), +\infty]$.*

Proof : Notice that the upper bound

$$\max_{1 \leq k \leq n} |q^n \xi^k - q^{n-k} p^k| \leq n \max\{1, |\xi|\}^{n-1} \max\{|p|, |q|\}^{n-1} |q\xi - p|,$$

which holds for all integers p and q, implies the lower bound

$$\lambda_n(\xi) \geq \frac{w_1(\xi) - n + 1}{n}.$$

On the other hand, Khintchine's transference principle (6) provides us with the upper bound (see also Theorem 3.9 of [10])

$$\lambda_n(\xi) \leq \frac{w_n(\xi) - n + 1}{n}.$$

Now, Theorem 7.7 in [10] asserts that for any given real number $w > (2n+1+\sqrt{4n^2+1})/2$, there exists a real number ξ (that can be given explicitly) with $w_1(\xi) = w_n(\xi) = w$. Then, the equality

$$\lambda_n(\xi) = \frac{w - n + 1}{n}$$

holds. □

Problem 5. *Let $n \geq 1$ be an integer. Is the spectrum of the function λ_n equal to $[1/n, +\infty]$?*

We now turn our attention to the exponents of uniform approximation $\hat{w}_n$ and $\hat{\lambda}_n$, introduced explicitly for the first time in [11], but already studied by Davenport & Schmidt [16] in 1969.

Theorem 11. *For any integer $n \geq 1$ and any transcendental real number ξ, we have*

$$\hat{\lambda}_n(\xi) \leq 1/\lceil n/2 \rceil \quad \text{and} \quad \hat{w}_n(\xi) \leq 2n - 1. \tag{13}$$

Furthermore, we have

$$\hat{\lambda}_2(\xi) \leq (\sqrt{5} - 1)/2 \quad \text{and} \quad \hat{w}_2(\xi) \leq (3 + \sqrt{5})/2. \tag{14}$$

Proof : The two bounds (13) combine results by Davenport & Schmidt [16] and Laurent [37]. We now sketch a new proof of (14), as a consequence of Theorems 6 and 7. We first establish that $\lambda_2(\xi) \leq 1$ whenever $\hat{\lambda}_2(\xi) > 1/2$, which obviously may be assumed. Let $\epsilon > 0$ and let $\underline{x} = (x_0, x_1, x_2)$ be a non-zero integer triple with large norm $|\underline{x}| = X$ such that

$$\max\{|x_0\xi - x_1|, |x_0\xi^2 - x_2|\} \leq X^{-\lambda_2(\xi)+\epsilon}.$$

It is shown in Lemma 2 of [16] that we may suppose without loss of generality that the Hankel determinant $x_1^2 - x_0x_2$ is non-zero. Arguing as Davenport & Schmidt, we deduce the upper bound $\lambda_2(\xi) \leq 1 + \epsilon$ from the estimate

$$1 \leq |x_1^2 - x_0x_2| \leq 2X^{1-\lambda_2(\xi)+\epsilon},$$

which is valid for arbitrarily large values of X. Using now Corollary 2, we find the inequalities

$$1 \geq \lambda_2(\xi) \geq \frac{\hat{\lambda}_2(\xi)^2}{1 - \hat{\lambda}_2(\xi)},$$

from which follows the first upper bound of (14). Notice finally that the two upper bounds of (14) are equivalent by Theorem 6. An alternative proof of the bound $\hat{\omega}_2(\xi) \leq (3+\sqrt{5})/2$ may also be found in Arbour & Roy [6]. □

For a long time, it was believed that the upper bounds in (13) could possibly be improved to $1/n$ and n, respectively. This is, however, not true for $n = 2$, as recently proved by Roy in a series of remarkable papers: the upper bounds given in (14) are best possible. Section 6 is devoted to Roy's recent works and some of their extensions.

6. Some computations of exponents

In this section, we restrict our attention to the functions $\hat{w}_2$ and $\hat{\lambda}_2$, which, as first shown by Roy [43, 44], take values strictly larger than 2 and $1/2$, respectively, at some transcendental points.

Theorem 12. *There are real transcendental numbers ξ with $\hat{w}_2(\xi) = (3+\sqrt{5})/2$ and $\hat{\lambda}_2(\xi) = (\sqrt{5}-1)/2$.*

Recall that real numbers ξ with either $\hat{w}_n(\xi) > n$ or $\hat{\lambda}_n(\xi) > 1/n$ for some $n \geq 2$ are transcendental, by Schmidt's Subspace Theorem.

Actually, the result of Roy is slightly more precise, since he constructed numbers ξ for which there exists some positive constant c such that the systems of inequalities

$$\begin{aligned} |x_2\xi^2 + x_1\xi + x_0| &\leq c\,X^{-(3+\sqrt{5})/2}, \\ |x_1|, |x_2| &\leq X, \end{aligned} \tag{15}$$

and

$$\begin{aligned} |x_0'\xi + x_1'| &\leq c\,X^{-(\sqrt{5}-1)/2}, \\ |x_0'\xi^2 + x_2'| &\leq c\,X^{-(\sqrt{5}-1)/2}, \\ |x_0'| &\leq X, \end{aligned} \tag{16}$$

have non-zero integer solutions (x_0, x_1, x_2) and (x_0', x_1', x_2'), respectively, for any real number $X > 1$. Such a result is quite surprising, since the volumes of the convex bodies defined by (15) and (16) tend rapidly to zero as X grows to infinity. According to Roy, such a real number ξ is called an extremal number. The set of extremal numbers, which is countable [44], has been further studied in [48].

Let us now give a real number with these extremal properties. Let $\{a,b\}^*$ denote the monoid of words on the alphabet $\{a,b\}$ for the product given by the concatenation. The Fibonacci sequence in $\{a,b\}^*$ is the sequence of words $(f_i)_{i\geq 0}$ defined recursively by

$$f_0 = b, \ \ f_1 = a, \quad \text{and} \quad f_i = f_{i-1}f_{i-2}\,(i \geq 2).$$

Since, for every $i \geq 1$, the word f_i is a prefix of f_{i+1}, this sequence converges to an infinite word $f = abaabab\ldots$ called the Fibonacci word on $\{a,b\}$. For two positive distinct integers a and b, let $\xi_{a,b} = [0; a, b, a, a, b, a, \ldots]$ be the real number whose sequence of partial quotients is given by the letters of the Fibonacci word on $\{a,b\}$. Then, $\xi_{a,b}$ satisfies the properties stated in Theorem 12.

Sketch of the proof of Theorem 12.

Following [43], we show that there exists a suitable constant c for which the system (16) with $\xi = \xi_{a,b}$ has a non-zero integer solution for any real number $X > 1$.

We begin with a property of the Fibonacci word f. Let $(F_m)_{m\geq 0}$ denote the Fibonacci sequence defined by $F_0 = 0$, $F_1 = 1$, and $F_{m+2} = F_{m+1} + F_m$, for $m \geq 0$. It is well known (see e.g. [3]) that the sequence $(\phi_n)_{n\geq 2}$ formed by the prefixes ϕ_n of f of length $F_{n+2} - 2$ has the following property: for any $n \geq 2$, the word ϕ_n is a palindrome. Observe that $\phi_2 = a$, $\phi_3 = aba$, and $\phi_4 = abaaba$. Furthermore, we have

$$\phi_n = \phi_{n-1}ab\phi_{n-2}, \qquad \text{for } n \geq 4 \text{ even}, \tag{17}$$

and

$$\phi_n = \phi_{n-1}ba\phi_{n-2}, \qquad \text{for } n \geq 5 \text{ odd}. \tag{18}$$

Before going on with the proof, we make the following observation, extracted from [1]. Let $\eta = [0; a_1, a_2, \ldots]$ be a positive real irrational number, and denote by p_n/q_n its convergents, that is, $p_n/q_n = [0; a_1, a_2, \ldots, a_n]$. By the theory of continued fraction, we have

$$M_n := \begin{pmatrix} q_n & q_{n-1} \\ p_n & p_{n-1} \end{pmatrix} = \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix},$$

and, since such a decomposition is unique, the matrix M_n is symmetrical if, and only if, the word $a_1a_2\ldots a_n$ is a palindrome, that is, if, and only if, we have $a_j = a_{n+1-j}$ for any integer j with $1 \leq j \leq n$. In this case, we have $p_n = q_{n-1}$ and, by the theory of continued fraction, we get

$$\left|\eta - \frac{p_n}{q_n}\right| < \frac{1}{q_n^2} \quad \text{and} \quad \left|\eta - \frac{p_{n-1}}{q_{n-1}}\right| < \frac{1}{q_{n-1}^2}.$$

We then infer from $0 < \eta < 1$, $a_1 = a_n$ and $q_n \leq (a_n + 1)q_{n-1}$ that

$$\begin{aligned} \left|\eta^2 - \frac{p_{n-1}}{q_n}\right| &\leq \left|\eta^2 - \frac{p_{n-1}}{q_{n-1}} \cdot \frac{p_n}{q_n}\right| \leq \left|\eta + \frac{p_{n-1}}{q_{n-1}}\right| \cdot \left|\eta - \frac{p_n}{q_n}\right| + \frac{1}{q_nq_{n-1}} \\ &\leq 2\left|\eta - \frac{p_n}{q_n}\right| + \frac{1}{q_nq_{n-1}} < \frac{a_1 + 3}{q_n^2}. \end{aligned}$$

Consequently, if the sequence of the partial quotients of η is bounded and begins with infinitely many palindromes, then η and η^2 are simultaneously very well approximable by rational numbers of the same denominator, we have $\lambda_2(\eta) = 1$ and η is either quadratic, or transcendental, by (12). As noted in [1, 2], this observation gives a very short proof of the transcendence of the Thue–Morse continued fraction, first established by Queffélec [41] (see also [4]).

For any $n \geq 2$, denote by Q_n the denominator of the rational number whose partial quotients are given by the letters of ϕ_n. The above observation shows that, for a suitable constant c_1 and any $n \geq 4$, the system

$$\begin{aligned} |x_0 \xi_{a,b} + x_1| &\leq c_1 \, Q_n^{-1}, \\ |x_0 \xi_{a,b}^2 + x_2| &\leq c_1 \, Q_n^{-1}, \\ |x_0| &\leq Q_n, \end{aligned}$$

has a non-zero solution, that we denote by $(x_0^{(n)}, x_1^{(n)}, x_2^{(n)})$. Observe that $Q_n = x_0^{(n)}$.

Furthermore, it follows from (17) and (18) that

$$\begin{pmatrix} x_0^{(n)} & x_1^{(n)} \\ x_1^{(n)} & x_2^{(n)} \end{pmatrix} = \begin{pmatrix} x_0^{(n-1)} & x_1^{(n-1)} \\ x_1^{(n-1)} & x_2^{(n-1)} \end{pmatrix} \times S_n \times \begin{pmatrix} x_0^{(n-2)} & x_1^{(n-2)} \\ x_1^{(n-2)} & x_2^{(n-2)} \end{pmatrix},$$

where

$$S_n = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix},$$

according as n is even or odd. This yields

$$x_0^{(n)} = \begin{pmatrix} x_0^{(n-1)} & x_1^{(n-1)} \end{pmatrix} \times S_n \times \begin{pmatrix} x_0^{(n-2)} \\ x_1^{(n-2)} \end{pmatrix},$$

which implies

$$\begin{aligned} \lim_{n \to +\infty} \frac{Q_n}{Q_{n-1} Q_{n-2}} &= \begin{pmatrix} 1 & \xi_{a,b} \end{pmatrix} \times \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} b & 1 \\ 1 & 0 \end{pmatrix} \times \begin{pmatrix} 1 \\ \xi_{a,b} \end{pmatrix} \\ &= \xi_{a,b}^2 + (a+b)\xi_{a,b} + (ab+1). \end{aligned} \tag{19}$$

Set $\gamma = (1+\sqrt{5})/2$ and $\kappa_n = Q_n \, Q_{n-1}^{-\gamma}$, for any integer $n \geq 2$. Since $\gamma = 1 + 1/\gamma$, we get

$$\kappa_n = \frac{Q_n}{Q_{n-1} Q_{n-2}} \, \kappa_{n-1}^{-1/\gamma},$$

thus, by (19), there exist positive constants $c_3 > c_2$ such that

$$c_2 \kappa_{n-1}^{-1/\gamma} < \kappa_n < c_3 \kappa_{n-1}^{-1/\gamma}, \qquad \text{for any } n \geq 2.$$

By induction, this yields

$$c_4\, Q_n^{(1+\sqrt{5})/2} \le Q_{n+1} \le c_5 Q_n^{(1+\sqrt{5})/2}, \qquad \text{for any } n \ge 4, \tag{20}$$

with $c_4 = \min\{Q_2, c_2^\gamma/c_3\}$ and $c_5 = \max\{Q_2, c_3^\gamma/c_2\}$.

Let then X be a sufficiently large real number. There exists an integer $n \ge 4$ such that $Q_n \le X < Q_{n+1}$. The system

$$\begin{aligned} |x_0\xi_{a,b} + x_1| &\le X^{-1/2}, \\ |x_0\xi_{a,b}^2 + x_2| &\le X^{-1/2}, \\ |x_0| &\le X \end{aligned}$$

has the non-zero integer solution $(x_0^{(n)}, x_1^{(n)}, x_2^{(n)})$, and we have

$$\max\{|x_0^{(n)}\xi_{a,b} + x_1^{(n)}|, |x_0^{(n)}\xi_{a,b}^2 + x_2^{(n)}|\} \le c_1\, Q_n^{-1} \le c_1\, X^{-(\log Q_n)/(\log X)} \le c_6\, X^{-2/(1+\sqrt{5})},$$

for a suitable positive constant c_6, by (20). This shows that $\hat{\lambda}_2(\xi_{a,b}) \ge (\sqrt{5}-1)/2$. By Theorems 6 and 11, this yields Theorem 12. □

A natural question is now the study of the spectra of the functions $\hat{w}_2$ and $\hat{\lambda}_2$. Roy [44] proved that there are only countably many real numbers ξ for which the system (15) (resp. (16)) has a non-zero solution for any $X > 1$. Shortly thereafter, Bugeaud & Laurent [11] extended Roy's construction and found uncountably many values taken by $\hat{w}_2$ and $\hat{\lambda}_2$.

Theorem 13. *Let $(s_j)_{j\ge1}$ be a bounded sequence of integers and set*

$$\sigma = \liminf_k\, [0; s_k, s_{k-1}, \dots, s_1].$$

There exist real numbers ξ with

$$\lambda_2(\xi) = 1, \qquad w_2(\xi) = 1 + \frac{2}{\sigma},$$
$$\hat{\lambda}_2(\xi) = \frac{1+\sigma}{2+\sigma}, \qquad \hat{w}_2(\xi) = 2 + \sigma.$$

In particular, the spectra of $\hat{w}_2$ and $\hat{\lambda}_2$ have Hausdorff dimension 1.

Notice that these four exponents satisfy the relation

$$\lambda_2(\xi) = \frac{w_2(\xi)(\hat{w}_2(\xi) - 1)}{w_2(\xi) + \hat{w}_2(\xi)}.$$

Thus, the numbers ξ constructed in Theorem 13, provide us with examples of extremal matrices $A = (\xi, \xi^2)$ for which the lower bound $v' \ge v(w-1)/(v+w)$ given by Theorem

7 turns out to be an equality. Theorem 13 is proved in [11], and we refer to Cassaigne [13] and to [11] for further results on the function

$$(s_j)_{j\geq 1} \longmapsto \liminf_k \, [0; s_k, s_{k-1}, \ldots, s_1].$$

It is tempting to believe that the spectra of $\hat{w}_2$ and $\hat{\lambda}_2$ enjoy a structure of 'Markoff spectrum', and that $(3+\sqrt{5})/2$ and $(\sqrt{5}-1)/2$ are isolated points of the spectra of $\hat{w}_2$ and $\hat{\lambda}_2$, respectively. This is, however, not true, as recently established by Roy [47].

Theorem 14. *The spectra of $\hat{w}_2$ and $\hat{\lambda}_2$ are dense in $[2, (3+\sqrt{5})/2]$ and $[1/2, (\sqrt{5}-1)/2]$, respectively.*

To prove Theorem 14, Roy produces countably many real numbers ξ of 'Fibonacci type' by suitably modifying the constructions of Section 6 of [44] and Section 5 of [46]. In view of results from [11], one may ask whether there exist transcendental real numbers ξ not of that type which satisfy $\hat{w}_2(\xi) > 1 + \sqrt{2}$.

Problem 6. *Determine the spectra of the functions $\hat{w}_2$ and $\hat{\lambda}_2$.*

Works of Fischler [21, 22, 23] bring some light on Problem 6. Furthermore, he informed us that he established the existence of a (small) explicitly computable positive number ϵ such that the intersection of the spectrum of $\hat{w}_2$ with $[(3+\sqrt{5})/2-\epsilon, (3+\sqrt{5})/2]$ is countable. Consequently, the spectrum of $\hat{w}_2$ is not equal to the whole interval $[2, (3+\sqrt{5})/2]$.

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