

Zero–infinity laws in Diophantine approximation

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Abstract. *It is shown that for any translation invariant outer measure \mathcal{M} , the \mathcal{M} -measure of the intersection of any subset of \mathbf{R}^n that is invariant under rational translations and which does not have full Lebesgue measure with the closure of an open set of positive measure cannot be positive and finite. Analogues for p -adic fields and fields of formal power series over a finite field are established. The results are applied to some problems in metric Diophantine approximation.*

1. Introduction

‘Zero–infinity’ laws for the measure of a set are a natural analogue of the more familiar ‘zero–one’ laws of probability theory. They arise in the setting of the real line or with measures other than Lebesgue. Unlike a ‘zero–one’ law, a ‘zero–infinity’ law does not imply that a set is null or full (or has null complement), so in this respect, ‘zero–infinity’ laws are weaker than the familiar ‘zero–one’ laws. This of course is a consequence of the special properties of the Lebesgue measure.

For clarity, we will say that a set satisfies a *full* ‘zero–infinity’ law with respect to some measure if either the set or its complement is null. It follows straightforwardly from the Lebesgue density theorem (see *e.g.* [25], Lemma 7, p. 21) that such a full ‘zero–infinity’ law holds for Lebesgue measurable real sets invariant under rational translations.

Theorem A. *Let $E \subseteq \mathbf{R}$ be a Lebesgue measurable set such that $\xi + p/q$ is in E for any ξ in E and any rational number p/q . Then either E or its complementary set has Lebesgue measure zero.*

One may ask whether Theorem A can be extended to other measures than Lebesgue, and in particular to Hausdorff measures, for which there is no analogue of the Lebesgue density theorem. Clearly, one cannot hope for a complete analogue. Indeed, one can

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easily construct examples of subsets of an interval where both the set and its complement have infinite Hausdorff measures. In view of this, we can ask that the set E be – if not full – at least ‘thick’, *i.e.*, roughly speaking that for any ‘reasonable’ set F , we have $\mathcal{M}(E \cap F) \geq c\mathcal{M}(F)$ for some $c > 0$ (our notion of thickness of a set should not be confused with the one used in e.g. [16], which says that a set E is thick if given any open ball B , the set $E \cap B$ has maximal Hausdorff dimension). This means that E is sufficiently ‘spread out’ to ensure that the measure of the intersection with F is always comparable to the measure of F (which in most cases will be infinite). In particular, one can hope to show that such a set must have Hausdorff measure either zero or infinity for all dimension functions.

The first contribution to this problem is due to Jarník [13, 14] (see also [8], since these papers are not readily available). He showed that for any dimension function f and any real number $\tau > 2$, the Hausdorff f -measure of the intersection of the unit interval $[0, 1]$ with

$$\mathcal{V}_\tau = \left\{ \xi \in \mathbf{R} : \left| \xi - \frac{p}{q} \right| < \frac{K(\xi)}{q^\tau} \text{ for infinitely many } \frac{p}{q} \in \mathbf{Q}, \text{ for some } K(\xi) > 0 \right\},$$

the set of real numbers approximable by rational numbers to order τ , is either 0 or $+\infty$. This set is invariant under rational translations and is closely related to the set

$$\mathcal{K}_\tau := \left\{ \xi \in \mathbf{R} : \left| \xi - \frac{p}{q} \right| < \frac{1}{q^\tau} \text{ for infinitely many } \frac{p}{q} \in \mathbf{Q} \right\},$$

and indeed for any $\varepsilon > 0$, we have $\mathcal{V}_{\tau+\varepsilon} \subset \mathcal{K}_\tau \subset \mathcal{V}_\tau$. A little earlier Jarník had shown in [12] that $\mathcal{H}^s(\mathcal{K}_\tau)$, the Hausdorff f -measure for $f(x) = x^s$, was infinite when $s < 2/\tau$ and vanished when $s > 2/\tau$, whence the Hausdorff dimension of \mathcal{K}_τ is given by

$$\dim \mathcal{K}_\tau = 2/\tau \text{ when } \tau > 2$$

($\mathcal{K}_\tau = \mathbf{R}$ when $\tau \leq 2$). It is not clear whether \mathcal{K}_τ is invariant under rational translates but in his later more general paper of 1931 [15] on the Hausdorff s -measure analogue of Khintchine’s theorem for simultaneous Diophantine approximation, Jarník proved that $\mathcal{H}^{2/\tau}(\mathcal{K}_\tau \cap [0, 1]) = +\infty$ when $\tau > 2$. (See *e.g.* Falconer [9] or Rogers [22] for background on the theory of Hausdorff measure and dimension.) This shows that $\mathcal{V}_\tau \cap [0, 1]$ and $\mathcal{K}_\tau \cap [0, 1]$ are not s -sets (see Chapters 2–4 of [9] for further details of s -sets), unlike for instance the familiar ternary Cantor set (whose Hausdorff s -measure at the critical exponent $\log 2 / \log 3$ is equal to 1). The same conclusion also holds for sets of real numbers well approximable by algebraic numbers of bounded degree (see [4]), the set \mathfrak{B} of badly approximable numbers (this set is invariant under rational translates, see [8]) and many other sets arising in Diophantine approximation.

When the right hand side of the inequalities defining the sets \mathcal{K}_τ is replaced by some arbitrary non-negative function $\psi(q)$, we obtain the set

$$\mathcal{K}(\psi) := \left\{ \xi \in \mathbf{R} : \left| \xi - \frac{p}{q} \right| < \psi(q) \text{ for infinitely many } \frac{p}{q} \in \mathbf{Q} \right\}.$$

Determining the measure of these sets in general is a difficult unsolved problem (the Duffin–Schaeffer conjecture, see *e.g.*, [11, 25]). However, a full zero–infinity law for these sets is known for Lebesgue measure. It is due to Cassels [6] when the rationals p/q are not required to be in their lowest terms and due to Gallagher [10] when they are. It is natural to ask whether there exists a dimension function f for which the sets mentioned above and others intersected with a non-trivial closed interval have \mathcal{H}^f -measure that is always positive and finite. For certain limsup sets such as \mathcal{K}_τ , very general zero-infinity statements for Hausdorff measures are known due to Beresnevich, Dickinson and Velani [3], providing a negative answer in the case of these sets. Recently, Olsen [21] gave a negative answer to this question for a different family of sets, under the assumption that f is strictly concave. The purpose of the present paper is to remove this assumption by showing that Jarník’s arguments actually imply that for any set E in \mathbf{R}^n invariant under certain rational translations (such as the sets \mathcal{V}_τ , \mathfrak{B} and the set \mathcal{L} of Liouville numbers), for any translation invariant outer measure \mathcal{M} , and for any set $F \subseteq \mathbf{R}$ which is the closure of an open set with positive Lebesgue measure, either $\mathcal{M}(E \cap F) = 0$ or $\mathcal{M}(E \cap F) = +\infty$. In particular, for any dimension function f , we have $\mathcal{H}^f(E \cap F) = 0$ or $+\infty$. Also, our general method provides a simple proof of Theorem A. Unfortunately, a general zero–infinity law for the sets $\mathcal{K}(\psi)$ remains out of our reach.

Similar questions may be asked when the reals are replaced by a p -adic field or a field of formal Laurent series with coefficients from a finite field \mathbf{K} . In the latter case, the rationals must be replaced by ratios of polynomials in $\mathbf{K}[X]$. We will prove that any set E invariant under rational translations in either of these situations must also satisfy $\mathcal{H}^f(E \cap F) = 0$ or $+\infty$ for any dimension function f and any set F , which is the closure of an open set of positive Haar measure.

2. Statement of the results

Our main result provides the appropriate analogue of Theorem A for translation invariant outer measures (in particular Hausdorff measures) as well as for higher dimensions. It rests on a clever idea of Jarník [13,14].

Theorem 1. *Let E be a measurable subset of \mathbf{R}^n which does not have full Lebesgue measure. Assume that there exist integers $\ell_1, \dots, \ell_n \geq 2$ such that E is invariant under any translation by a vector of the form $(a_1/\ell_1^k, \dots, a_n/\ell_n^k)$, with $\mathbf{a} = (a_1, \dots, a_n) \in \mathbf{Z}^n$, $k \in \mathbf{Z}$ and $k \geq 1$. Let \mathcal{M} be a translation invariant outer measure and let $F \subseteq \mathbf{R}^n$ be the closure of an open set of positive Lebesgue measure. Then either $\mathcal{M}(E \cap F) = 0$ or $+\infty$. In particular, if \mathcal{M} is outer Lebesgue measure then $\mathcal{M}(E \cap F) = 0$.*

Clearly, this theorem looks somewhat weaker than Theorem A, as it does not show that such a set E is either null or full with respect to \mathcal{M} . However, by taking \mathcal{M} to be Lebesgue measure in the proof and making the obvious modifications, a simple proof of Theorem A, independent of the Lebesgue density theorem, is obtained.

As an immediate application of Theorem 1, we answer a question posed by Mauldin ([7], p. 231):

Does there exist a gauge function g such that the Hausdorff g measure of the set of the Liouville numbers is positive and finite?

Note that Mauldin speaks of gauge functions and not dimension functions. We will use the latter terminology. Recall that the set \mathcal{L} of Liouville numbers is

$$\mathcal{L} := \left\{ \xi \in \mathbf{R} : \text{for all } w > 0, \text{ there exists } \frac{p}{q} \text{ such that } \left| \xi - \frac{p}{q} \right| < \frac{1}{q^w} \right\}.$$

It is readily verified that given any Liouville number ξ and any rational number a/b , the real number $\xi + a/b$ is a Liouville number. This implies

Corollary 1. *Let $I \subseteq \mathbf{R}$ be a non-empty open interval. There does not exist a dimension function f such that $0 < \mathcal{H}^f(\mathcal{L} \cap I) < +\infty$.*

By Corollary 1, $\mathcal{H}^f(\mathcal{L})$ is either 0 or $+\infty$, but as far as we are aware, given a dimension function f , it is an open problem to determine which of these is the value of $\mathcal{H}^f(\mathcal{L})$. There are some partial results on this question: using a covering argument, it is easy to prove $\mathcal{H}^f(\mathcal{L}) = 0$ if there exists $\delta > 0$ such that $\lim_{x \rightarrow 0} x^{-\delta} f(x) = 0$. Conversely, Baker [2] showed that if f satisfies $\liminf_{x \rightarrow 0} x^{-\delta} f(x) > 0$ for any $\delta > 0$, then we have $\mathcal{H}^f(\mathcal{L}) = +\infty$.

In dimensions strictly greater than one, sets of vectors analogous to the sets \mathcal{K}_τ , \mathfrak{B} and \mathcal{L} also exist (see *e.g.* [25] for examples). As these are also invariant under rational translations, Theorem 1 immediately implies the same conclusion for these sets.

We briefly mention other applications of Theorem 1 to Diophantine approximation. In order to classify the real numbers according to their properties of algebraic approximation, Mahler [20] and Koksma [17] introduced, for any positive integer n , the functions w_n and w_n^* defined as follows. Let ξ be a real number. We denote by $w_n(\xi)$ the supremum of the real numbers w for which there exist infinitely many integer polynomials P of degree at most n satisfying

$$0 < |P(\xi)| \leq H(P)^{-w},$$

where $H(P)$ is the naïve height of P , that is, the maximum of the absolute values of its coefficients. Further, we denote by $w_n^*(\xi)$ the supremum of the real numbers w^* for which there exist infinitely many real algebraic numbers α of degree at most n satisfying

$$0 < |\xi - \alpha| \leq H(\alpha)^{-w^* - 1},$$

where $H(\alpha)$ is the naïve height of α , that is, the height of its minimal polynomial over the integers. For results on the functions w_n and w_n^* , the reader is referred to [5,25]. We observe that these functions are invariant by rational translations, thus, by Theorem A, for any positive real numbers w and w^* , any of the sets

$$\{\xi \in \mathbf{R} : w_n(\xi) = w\}, \quad \{\xi \in \mathbf{R} : w_n^*(\xi) = w^*\}$$

have either Lebesgue measure zero, or have full Lebesgue measure. This result (first observed by Sprindžuk [24]) can be extended to Hausdorff measure (or indeed any translation invariant outer measure) thanks to Theorem 1.

Corollary 2. *Let w and w^* be in $(0, +\infty]$. Let n be a positive integer. There does not exist a dimension function f with \mathcal{H}^f not absolutely continuous with respect to the Lebesgue measure, such that the intersection of any of the sets*

$$\{\xi \in \mathbf{R} : w_n(\xi) = w\}, \quad \{\xi \in \mathbf{R} : w_n^*(\xi) = w^*\}$$

with any closed non-trivial interval has positive and finite Hausdorff \mathcal{H}^f -measure.

Theorem 1 also applies to the second example considered by Olsen [21], namely the Besicovitch–Eggleston set $B(\mathbf{p})$ of non-normal numbers in a given integer basis $N \geq 2$ (see [21] for the definition), since this set is invariant under translation by any rational number a/N^k , with $a, k \in \mathbf{Z}$ and $k \geq 1$. It then follows that we have either $\mathcal{H}^f(B(\mathbf{p}) \cap I) = 0$ or $\mathcal{H}^f(B(\mathbf{p}) \cap I) = +\infty$, for any dimension function f with \mathcal{H}^f not absolutely continuous with respect to the Lebesgue measure and any closed non-trivial interval I . This improves Theorem 3 of [21] and an earlier result of Smorodinsky [23]. For a restricted class of Hausdorff measures, the same conclusion has been shown to hold by Ma and Wen [19]. By Theorem 1, the same conclusion also holds for the Cartesian product of such sets $B_{N_1}(\mathbf{p}_1) \times \cdots \times B_{N_n}(\mathbf{p}_n)$, where the N_i denote the bases and the \mathbf{p}_i denote the required distribution of digits. This requires the full force of the theorem.

For the p -adic fields and the fields of formal power series over a finite field, we prove the following result.

Theorem 2. *Let V be an n -dimensional vector space over either a p -adic field or a field of formal Laurent series with coefficients from a finite field \mathbf{K} . Let $E \subseteq V$ be a Haar measurable set which does not have full Haar measure. Assume that E is invariant under any translation by any vector of rationals $(p_1/q_1, \dots, p_n/q_n)$ in the p -adic case or any vector of ratios $(p_1/q_1, \dots, p_n/q_n)$ where $p_i, q_i \in \mathbf{K}[X]$, the polynomial ring over \mathbf{K} , with the $q_i \neq 0$ in the case of formal power series. Let \mathcal{M} be a translation invariant outer measure on V and let $F \subseteq V$ be the closure of an open set of positive Haar measure. We then have $\mathcal{M}(E \cap F) = 0$ or $\mathcal{M}(E \cap F) = +\infty$.*

Note that Theorem 2 immediately implies that there is no dimension function such that the sets analogous to the set of Liouville numbers have positive and finite measure. It has previously been shown that the ordinary Hausdorff dimension of these sets is zero [1,18]. A full analogue of Theorem 1 is possible in the case of formal power series, which implies the same conclusion for the analogues of the Besicovitch–Eggleston sets. However, the proof of the present Theorem 2 is more elegant, and the reader should have no trouble filling in the details to prove the full analogue of Theorem 1.

3. An important lemma

The fundamental tool in the proofs is the following lemma. In the case of the real numbers, it is implicit in Jarník’s papers [13,14]. A certain weak form of ‘quasi-independence’ with respect to an outer measure \mathcal{M} can be defined; and it implies a ‘0- ∞ ’ law for \mathcal{M} and so for any Hausdorff measure. We prove the lemma in high generality.

Lemma 1. Let $F \subset \mathbf{G}$, where \mathbf{G} is a locally compact group and F has finite Haar measure. Let μ denote the restriction of the Haar measure on \mathbf{G} to F , normalised so that $\mu(F) = 1$. Let $E \subset F$ be a measurable set with $\mu(E) < 1$ and let \mathcal{M} be an outer measure on F . Suppose that for every open ball $B(c, \rho) = \{x \in F : d(x, c) < \rho\} \subset F$,

$$\mathcal{M}(E \cap B(c, \rho)) \leq \mu(B(c, \rho))\mathcal{M}(E).$$

Then the measure $\mathcal{M}(E)$ of E is either 0 or infinity.

Note that dividing both sides of the inequality in Lemma 1 yields

$$\frac{\mathcal{M}(E \cap B(c, \rho))}{\mathcal{M}(B(c, \rho))} \leq \frac{\mu(B(c, \rho))}{\mathcal{M}(B(c, \rho))}\mathcal{M}(E).$$

The left hand side suggests a ‘quasi-density’ (though of course the limit need not exist) or a ‘conditional measure’, while the right hand side might be related to a notion analogous to ‘absolute continuity’ of the measures (although such a notion may not be appropriate in the context of outer measures).

Proof. Assume the contrary, *i.e.*, assume $0 < \mathcal{M}(E) < +\infty$. Since $\mu(E) < 1$, there exists a cover of E by open balls $B(c_j, \rho_j)$ such that

$$\sum_j \mu(B(c_j, \rho_j)) < 1.$$

By assumption,

$$\begin{aligned} 0 < \mathcal{M}(E) &= \mathcal{M}\left(E \cap \left(\bigcup_j B(c_j, \rho_j)\right)\right) = \mathcal{M}\left(\bigcup_j (E \cap B(c_j, \rho_j))\right) \\ &\leq \sum_j \mathcal{M}\left(E \cap B(c_j, \rho_j)\right) \leq \mathcal{M}(E) \sum_j \mu(B(c_j, \rho_j)) < \mathcal{M}(E). \end{aligned}$$

This gives the desired contradiction. □

4. Proof of Theorem 1

It suffices to prove the theorem in the case when $F = [0, h_1/\ell_1^{k'_1}] \times \cdots \times [0, h_n/\ell_n^{k'_n}]$. Indeed, suppose that F' is the closure of an open set. Then there is a vector \mathbf{r} of the form given in Theorem 1 and an F such that $F + \mathbf{r} \subseteq F'$. If $\mathcal{M}(E \cap F) = +\infty$, the translation invariance of E implies that $\mathcal{M}(E \cap F') = +\infty$.

Suppose on the other hand that $\mathcal{M}(E \cap F) = 0$. Then we may cover F' by countably many translates of the form given in Theorem 1. In this case, translation invariance of E implies that $\mathcal{M}(E \cap F) = 0$. We will prove the theorem in the case when $F = [0, 1]^n$, the

closed unit hypercube. This is to avoid the notational complications of additional indices. The reader should have no trouble filling in the details for other hyperboxes of the above form.

Let $\mathbf{j} = (j_1, \dots, j_n) \in \mathbf{Z}^n$ and let

$$I(\mathbf{j}; k) = \left[\frac{j_1}{\ell_1^k}, \frac{j_1 + 1}{\ell_1^k} \right] \times \dots \times \left[\frac{j_n}{\ell_n^k}, \frac{j_n + 1}{\ell_n^k} \right].$$

As E is assumed to be invariant under translations by vectors of the form $(j_1/\ell_1^k, \dots, j_n/\ell_n^k)$, we have

$$E \cap I(\mathbf{j}; k) = \left(E \cap \left[0, \frac{1}{\ell_1^k} \right] \times \dots \times \left[0, \frac{1}{\ell_n^k} \right] \right) + (j_1/\ell_1^k, \dots, j_n/\ell_n^k).$$

Since the outer measure \mathcal{M} is assumed to be translation invariant, we get

$$\begin{aligned} \mathcal{M}(E \cap [0, 1]^n) &= \sum_{j_1=0}^{\ell_1^k-1} \dots \sum_{j_n=0}^{\ell_n^k-1} \mathcal{M}(E \cap I(\mathbf{j}; k)) \\ &= \ell_1^k \dots \ell_n^k \mathcal{M} \left(E \cap \left[0, \frac{1}{\ell_1^k} \right] \times \dots \times \left[0, \frac{1}{\ell_n^k} \right] \right). \end{aligned}$$

Consequently, for any $\mathbf{j} \in \mathbf{Z}^n$ and any k with $k \geq 1$ and $0 \leq j_i \leq \ell_i^k - 1$, we have

$$\mathcal{M}(E \cap I(\mathbf{j}; k)) = \frac{1}{\ell_1^k \dots \ell_n^k} \mathcal{M}(E \cap [0, 1]^n). \quad (1)$$

We endow \mathbf{R}^n with the metric induced by the norm $|x|_\infty = \max\{|x_1|, \dots, |x_n|\}$. In this metric, an open ball $B(c, \rho)$ is the Cartesian product of n open intervals, *i.e.*, $B(c, \rho) = (a_1, b_1) \times \dots \times (a_n, b_n)$. Considering the expansions of its endpoints in some base ℓ , every real open interval (a, b) can be represented as a union of a countable set of intervals $[j/\ell^k, (j+1)/\ell^k]$ in such a way that

$$(a, b) = \bigcup_{j,k} \left[\frac{j}{\ell^k}, \frac{j+1}{\ell^k} \right] \quad \text{and} \quad b - a = \sum_{j,k} \frac{1}{\ell^k}.$$

We do this for each coordinate, where we expand the i 'th coordinate interval in base ℓ_i . If necessary, we subdivide intervals again to obtain a representation for $B(c, \rho)$ such that

$$B(c, \rho) = \bigcup_{\mathbf{j}, k} I(\mathbf{j}; k) \quad \text{and} \quad \mu(B(c, \rho)) = \sum_{\mathbf{j}, k} \frac{1}{\ell_1^k \dots \ell_n^k},$$

where μ denotes the Lebesgue measure on \mathbf{R}^n . Hence, we get

$$\mathcal{M}(B(c, \rho)) = \mathcal{M} \left(\bigcup_{\mathbf{j}, k} I(\mathbf{j}; k) \right),$$

and, since $\mathcal{M}(\cdot)$ is an outer measure, it follows from (1) that

$$\begin{aligned}
\mathcal{M}(B(c, \rho) \cap E \cap [0, 1]^n) &= \mathcal{M}\left(\bigcup_{\mathbf{j}, k} E \cap I(\mathbf{j}; k) \cap [0, 1]^n\right) \\
&\leq \sum_{\mathbf{j}, k} \mathcal{M}(E \cap I(\mathbf{j}; k) \cap [0, 1]^n) \\
&\leq \mathcal{M}(E \cap [0, 1]^n) \sum_{\mathbf{j}, k} \frac{1}{\ell_1^k \dots \ell_n^k} \\
&= \mu(B(c, \rho)) \mathcal{M}(E \cap [0, 1]^n).
\end{aligned} \tag{2}$$

Now by hypothesis, E does not have full measure and by taking $X = [0, 1]^n$ in Lemma 1 (as we can without loss of generality), we can suppose that $\mu(E \cap [0, 1]^n) < 1$. It follows from Lemma 1 that $\mathcal{M}(E \cap [0, 1]^n) = 0$ or $\mathcal{M}(E \cap [0, 1]^n) = +\infty$. This is the statement of Theorem 1 in the case when $F = [0, 1]^n$. For other sets F the theorem follows analogously.

Furthermore, taking \mathcal{M} to be the outer Lebesgue measure in the proof, the set $E \cap [0, 1]^n$ must have outer Lebesgue measure either 0 or ∞ . Clearly, $\mathcal{M}(E \cap [0, 1]^n) \neq \infty$, and so we have given a simple proof of Theorem A. \square

5. Proof of Theorem 2

The proof relies on the same idea as the proof of Theorem 1 and is almost identical for p -adics and formal power series. As before, it suffices to consider the case when $F = B(0, 1)^n$, the unit hypercube in V . We let \mathbf{r} denote some vector with rational coordinates in the p -adic case and ratios of polynomials in the case of formal power series. Let k denote p in the p -adic case and $|\mathbf{K}|$ in the case of formal power series.

Let $B(c, \rho)$ denote a closed ball centred at c with radius ρ in the metric induced by the height $\max\{|x_1|, \dots, |x_n|\}$, where $|\cdot|$ denotes the absolute value on the base field. Note that because of the definition of the metric of the underlying space, for any $\rho > 0$, we have $B(c, \rho) = B(c, k^{-r})$ for some $r \in \mathbf{Z}$. We may therefore restrict ourselves to considering balls with radii of this form.

As in the real case, translation invariance implies that for any \mathbf{r} and any $r \in \mathbf{Z}$,

$$E \cap B(\mathbf{r}, k^{-r}) = E \cap B(0, k^{-r}) + \mathbf{r}.$$

Furthermore, using the ultrametric property of the underlying spaces, it is possible to tile the unit ball $B(0, 1)$ with k^{nr} disjoint balls of radius k^{-r} . Using the translation invariance of \mathcal{M} , we get

$$\mathcal{M}(E \cap B(0, 1)) = k^{nr} \mathcal{M}(E \cap B(0, k^{-r})).$$

Hence, for any ball of radius k^{-r} centred at \mathbf{r} , we have

$$\mathcal{M}(E \cap B(\mathbf{r}, k^{-r})) = k^{-nr} \mathcal{M}(E \cap B(0, 1)). \tag{3}$$

Now, the spaces considered are ultrametric, so any interior point of a ball may be taken as the centre of the ball. Furthermore, the set of elements \mathbf{r} is dense in the spaces by construction, so any ball of positive radius has such a point as an interior point. Hence, for any ball $B(c, k^{-r})$, there is an \mathbf{r} such that $B(c, k^{-r}) = B(\mathbf{r}, k^{-r})$. Using (3), for any ball $B(c, k^{-r})$, we obtain

$$\begin{aligned} \mathcal{M}(E \cap B(c, k^{-r}) \cap B(0, 1)) &= \mathcal{M}(E \cap B(\mathbf{r}, k^{-r}) \cap B(0, 1)) \\ &\leq k^{-nr} \mathcal{M}(E \cap B(0, 1)) = \mu(B(c, k^{-r})) \mathcal{M}(E \cap B(0, 1)), \end{aligned}$$

where μ is the Haar measure, normalised so that the closed unit ball has measure 1. On supposing that $\mu(E \cap B(0, 1)) < 1$, we may invoke Lemma 1 to prove the theorem. In the case when $\mu(E \cap B(0, 1)) = 1$, considering a union of translates of this set gives an analogue of Theorem A. \square

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