TRANSCENDENCE WITH ROSEN CONTINUED FRACTIONS

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Abstract. We give the first transcendence results for the Rosen continued fractions. Introduced over half a century ago, these fractions expand real numbers in terms of certain algebraic numbers.

1. Introduction

In 1954, D. Rosen defined an infinite family of continued fraction algorithms [21]. Introduced to aid in the study of certain Fuchsian groups, these continued fractions were applied some thirty years later by J. Lehner [15] in the study of Diophantine approximation by orbits of these groups.

The Rosen continued fractions and variants have been of recent interest. For studies of their dynamical and arithmetical properties, see [8], [19], [12]. For their applications to the study of geodesics on related hyperbolic surfaces, see [24], [7], [18]. For applications to Teichmüller geodesics arising from (Veech) translation surfaces, see [28], [5], [27] and [6]. Several basic questions remain open, including that of arithmetically characterizing the real numbers having a finite Rosen continued fraction expansion; see [16], [14] and [6]. Background on Rosen continued fractions is given in the next section.

The first transcendence criteria for regular continued fractions were proved by E. Maillet, H. Davenport and K. F. Roth, A. Baker, and recently improved by B. Adamczewski and Y. Bugeaud; see [1, 2, 10] and the references

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given there. In particular, Theorem 4.1 of [2] asserts that if ξ is an algebraic irrational number with sequence of convergents $(p_n/q_n)_{n\geq 1}$, then the sequence $(q_n)_{n\geq 1}$ cannot increase too rapidly. It is natural to ask whether similar transcendence results can be proven using Rosen continued fractions. We give the first such results.

Theorem 1.1. *Fix $\lambda = 2\cos\pi/m$ for an integer $m > 3$, and denote the field extension degree $[\mathbb{Q}(\lambda) : \mathbb{Q}]$ by D. If a real number $\xi \notin \mathbb{Q}(\lambda)$ has an infinite expansion in Rosen continued fraction over $\mathbb{Q}(\lambda)$ of convergents p_n/q_n satisfying*

$$\limsup_{n\to\infty} \frac{\log\log q_n}{n} > \log(2D-1)\,,$$

then ξ is transcendental.

For stating our second result, we associate to the Rosen continued fraction expansion

$$[\,\varepsilon_1(x) : r_1(x),\, \varepsilon_2(x) : r_2(x), \ldots,\, \varepsilon_n(x) : r_n(x), \ldots] := \cfrac{\varepsilon_1}{r_1\lambda + \cfrac{\varepsilon_2}{r_2\lambda + \cdots}}$$

of a real number x in $[-\lambda/2, \lambda/2)$ the sequence of pairs of integers $(\varepsilon_i, r_i)_{i\geq 1}$, which we call the *partial quotients*, and thus consider $\mathcal{A} = \{\pm 1\} \times \mathbb{N}$ as the alphabet of the Rosen continued fraction expansions.

As usual, we denote the length of a finite word $U = u_1 \cdots u_k$ as $|U| = k$. For any positive integer s, we write U^s for the word $U \cdots U$ (s times repeated concatenation of the word U). More generally, for any positive real number s, we denote by U^s the word $U^{\lfloor s\rfloor}U'$, where U' is the prefix of U of length $\lceil (s - \lfloor s\rfloor)|U|\rceil$.

Just as Adamczewski and Bugeaud [1, 10] showed for regular continued fraction expansions, a real number whose Rosen continued fraction expansion is appropriately "stammering" must be transcendental.

Theorem 1.2. *Fix $\lambda = 2\cos\pi/m$ for an integer $m > 3$, and denote the field extension degree $[\mathbb{Q}(\lambda) : \mathbb{Q}]$ by D. Let ξ be an infinite Rosen continued fraction with convergents $(p_n/q_n)_{n\geq 1}$ such that*

$$B := \limsup_{n} q_n^{1/n} < +\infty\,.$$

Write

$$b := \liminf_{n} q_n^{1/n} .$$

Assume that there are two infinite sequences $(U_n)_{n\geq 1}$ *and* $(V_n)_{n\geq 1}$ *of finite words over the alphabet* $\mathcal{A}$ *and an infinite sequence* $(w_n)_{n\geq 1}$ *of real numbers greater than* 1 *such that, for* $n \geq 1$*, the word* $U_n V_n^{w_n}$ *is a prefix of the infinite word composed of the partial quotients of* ξ. *If*

$$\limsup_{n\to+\infty} \frac{|U_n| + w_n|V_n|}{2|U_n| + |V_n|} > \frac{3D}{2} \cdot \frac{\log B}{\log b}, \tag{1}$$

then ξ *is either (at most) quadratic over* $\mathbb{Q}(\lambda)$ *or is transcendental.*

Lemma 2.1 gives that $\log b$ is positive.

The key to our proofs is that both numerator and denominator of a Rosen convergent dominate their respective conjugates in an appropriate fashion; see Lemma 3.1. From this one can bound the height of a Rosen convergent in terms of its denominator; see Lemma 3.2. Then, exactly as in the case of regular continued fractions, we apply tools from Diophantine approximation, namely an extension to number fields of the Roth theorem, for the proof of Theorem 1.1, and the Schmidt Subspace Theorem for the proof of Theorem 1.2.

Both theorems are weaker than their analogues for regular continued fractions, since we must work in a number field of degree D rather than in the field $\mathbb{Q}$. However, for $m = 4$ and $m = 6$, that is, for $\lambda_4 = \sqrt{2}$ and $\lambda_6 = \sqrt{3}$, our results can be considerably strengthened and we can get, essentially, the exact analogues of the results established for the regular continued fractions. The key point is that, in both cases, for every convergent p_n/q_n, exactly one of p_n, q_n is in $\mathbb{Z}$, the other being in $\lambda\mathbb{Z}$; see Remark 2 below.

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2. Background

2.1. **Rosen fractions.** We set $\lambda = \lambda_m = 2\cos\frac{\pi}{m}$ and $\mathbb{I}_m = [-\lambda/2, \lambda/2\,)$ for $m \geq 3$. For a fixed integer $m \geq 3$, the Rosen continued fraction map is defined by

$$T(x) = \begin{cases} \left|\frac{1}{x}\right| - \lambda \lfloor \left|\frac{1}{\lambda x}\right| + \frac{1}{2} \rfloor & x \neq 0; \\ \\ 0 & x = 0 \end{cases}$$

for $x \in \mathbb{I}_m$; here and below, we omit the index "m" whenever it is clear from context. For $n \geq 1$, we define

$$\varepsilon_n(x) = \varepsilon(T^{n-1}x) \qquad \text{and} \qquad r_n(x) = r(T^{n-1}x)$$

with

$$\varepsilon(y) = \operatorname{sgn}(y) \qquad \text{and} \qquad r(y) = \left\lfloor \left|\frac{1}{\lambda y}\right| + \frac{1}{2} \right\rfloor.$$

Then, as Rosen showed in [21], the Rosen continued fraction expansion of x is given by

$$[\, \varepsilon_1(x) : r_1(x),\, \varepsilon_2(x) : r_2(x), \ldots,\, \varepsilon_n(x) : r_n(x), \ldots] := \cfrac{\varepsilon_1}{r_1\lambda + \cfrac{\varepsilon_2}{r_2\lambda + \cdots}}.$$

As usual we define the *convergents* p_n/q_n of $x \in \mathbb{I}_m$ by

$$\begin{pmatrix} p_{-1} & p_0 \\ q_{-1} & q_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} = \begin{pmatrix} 0 & \varepsilon_1 \\ 1 & \lambda r_1 \end{pmatrix} \begin{pmatrix} 0 & \varepsilon_2 \\ 1 & \lambda r_2 \end{pmatrix} \cdots \begin{pmatrix} 0 & \varepsilon_n \\ 1 & \lambda r_n \end{pmatrix}$$

for $n \geq 1$. From this definition it is immediate that $|p_{n-1}q_n - q_{n-1}p_n| = 1$, and that the well-known recurrence relations

$$p_{-1} = 1;\, p_0 = 0; \quad p_n = \lambda r_n p_{n-1} + \varepsilon_n p_{n-2},\, n \geq 1$$
$$q_{-1} = 0;\, q_0 = 1; \quad q_n = \lambda r_n q_{n-1} + \varepsilon_n q_{n-2},\, n \geq 1,$$

hold. It also follows that

$$\frac{p_n}{q_n} = [\, \varepsilon_1 : r_1,\, \varepsilon_2 : r_2, \ldots,\, \varepsilon_n : r_n] \tag{2}$$

and

$$\frac{q_{n-1}}{q_n} = [\, 1 : r_n,\, \varepsilon_n : r_{n-1}, \ldots,\, \varepsilon_2 : r_1] \,. \tag{3}$$

We define

$$M_n = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}, \text{ and find that } x = M_n \cdot T^n(x), \tag{4}$$

where $\cdot$ denotes the usual fractional linear operation, namely

$$x = \frac{p_{n-1}T^n(x) + p_n}{q_{n-1}T^n(x) + q_n}.$$

2.2. **Approximation with Rosen fractions.** We briefly discuss the convergence of the "convergents" to x. One can rephrase some of Rosen's original arguments in terms of the (standard number theoretic) natural extension map $\mathcal{T}(x,y) = (\,T(x), \frac{1}{r\lambda+\varepsilon y}\,)$ where $r = r_1(x)$ and $\varepsilon = \varepsilon_1(x)$. The "mirror formula" Equation (3) shows that $\mathcal{T}^n(x,0) = (\,T^n(x), \frac{q_{n-1}}{q_n}\,)$. Extending earlier work of H. Nakada, it is shown in [8] that $\mathcal{T}(x,y)$ has planar domain Ω with y-coordinates between 0 and $R = R(\lambda)$, where $R = 1$ if the index m is even and, otherwise, R is the positive root of $R^2 + (2-\lambda)R - 1 = 0$, in which case we have $1 > R > \lambda/2$ (see Lemma 3.3 of [8]). Therefore, the sequence $(q_n)_{n\geq 1}$ is strictly increasing. But, as Rosen mentions, if x has infinite expansion, then either $\varepsilon_n = 1$ or $r_n > 1$ occurs infinitely often; from this one has both that $q_n \geq 1$ for all n and that the limit as n tends to infinity of q_n is infinite.

One easily adapts Rosen's arguments so as to find the following.

Lemma 2.1. *For every $x \in \mathbb{I}_m$ of infinite expansion, we have*

$$\liminf_n q_n^{1/n} > 1\,.$$

Proof. We know that the sequence $(q_n)_{n\geq 1}$ increases and that, if either $\varepsilon_n = 1$ or $r_n > 1$, then $q_n > \lambda q_{n-1}$. Furthermore, there are no more than h consecutive indices i with $(\varepsilon_i, r_i) = (-1, 1)$, with $h = m/2, (m-3)/2$ depending on the parity of m; see [21] or [8]. Consequently, for any n, there is some $i = 1, \ldots, h+1$ such that $q_{n+i} > \lambda q_{n+i-1}$, giving

$$q_{n+h+1} \geq q_{n+i} > \lambda q_{n+i-1} \geq \lambda q_n\,.$$

As $q_1 \geq \lambda$, letting $s(n) = 1 + \left\lfloor \dfrac{n-1}{h+1} \right\rfloor$, we have $q_n \geq \lambda^{s(n)}$. Since $\lambda > 1$, this proves the lemma. $\square$

Remark 1. In fact, H. Nakada [20] shows that *for almost all* such x, $\lim_{n\to\infty} \dfrac{1}{n} \log q_n$ exists, being equal to one half of the entropy of T. He also shows that the entropy equals $C \cdot (m-2)\pi^2/(2m)$, where $C = 1/\log(1+R)$ when m is odd, and equals $1/\log[(1+\cos\pi/m)/\sin\pi/m]$ when m is even. This C is the normalizing constant of the invariant measure with density $(1+xy)^{-2}$ on the domain Ω of the planar natural extension $\mathcal{T}$; see [8].

Rosen also gave bounds on $|\, x - p_n/q_n \,|$. Using Equation (4)

$$\left| x - \frac{p_n}{q_n} \right| = \frac{1}{q_n^2} \frac{1}{|\, \frac{q_{n+1}}{q_n} + T^{n+1}x \,|},$$

from which Nakada (see [20] Lemma 4) finds

$$\frac{1}{q_n\,(q_{n+1}+q_n)} \leq \left| x - \frac{p_n}{q_n} \right| \leq \frac{c_1}{q_n^2},$$

with $c_1 = c_1(\lambda) = R/(1 - R\lambda/2)$. (The lower bound is in [21].)

Now, from Equation (4) one also finds

$$\left| x - \frac{p_n}{q_n} \right| = \frac{1}{q_n q_{n+1}} \frac{1}{|\, 1 + \frac{q_n}{q_{n+1}} T^{n+1}x \,|}.$$

Since the closed, compact planar region Ω is of finite measure with respect to the measure with density $(1+xy)^{-2}$, we certainly have that $\mathcal{T}^{n+1}(x,0)$ remains a bounded distance from the curve $y = -1/x$. Thus, there is some c_2 such that

$$\left| x - \frac{p_n}{q_n} \right| < \frac{c_2}{q_n q_{n+1}}. \tag{5}$$

Rosen, arguing differently, gave a c_2 with value $1/(1-\lambda/2)$; in particular, convergence of the approximation sequence follows. Rosen's value is not optimal. To see this, one combines Proposition 4.1 of [8] with the approach of Theorems 4.4 and 4.5 (depending on parity of m) also of [8].

2.3. **Traces in Hecke groups.** Rosen introduced his continued fractions to study the Hecke groups. The Hecke (triangle Fuchsian) group G_m with

$m \in \{3, 4, 5, \dots\}$ is the group generated by

$$\begin{pmatrix} 1 & \lambda_m \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

with λ_m as above. The Rosen expansion of a real number terminates at a finite term if and only if x is a parabolic fixed point of G_m; see [21]. These points are clearly contained in $\mathbb{Q}(\lambda_m)$ but in general there are elements of this field that have infinite Rosen expansion; see [16], [14] and [6].

Remark 2. The values of finite Rosen expansion form the set $G_m \cdot \infty$, which is in fact a subset of $\lambda\,\mathbb{Q}(\lambda^2) \cup \{\infty\}$. To see this, one uses induction on word length in the generators displayed above — an ordered pair (a, c) giving a column of any element of G_m must be such that exactly one element of the pair is in $\mathbb{Z}[\lambda^2]$, and the other is in $\lambda\mathbb{Z}[\lambda^2]$. Note that this also applies to convergents p_n/q_n: exactly one of p_n, q_n is in $\mathbb{Z}[\lambda^2]$, the other being in $\lambda\mathbb{Z}[\lambda^2]$.

When $m = 3$, we have $G_3 = \mathrm{PSL}(2, \mathbb{Z})$. In general each G_m is isomorphic to the free product of a cyclic group of order two and a cyclic group of order m. Recall that a Fuchsian triangle group is generated by *even* words in the reflections about the sides of some hyperbolic triangle. Thus any Fuchsian triangle group is of index two in the group generated by these reflections; for each G_m, we denote this larger group by Δ_m.

Since λ_m is the sum of the root of unity $\zeta_{2m} := \exp 2\pi i/(2m)$ with its complex conjugate, $\mathbb{Q}(\lambda_m)$ is a number field of degree $d := \phi(2m)/2$ over the rationals, where ϕ denotes the Euler totient function.

The following key phenomenon property of Hecke groups can be shown in various manners. The result holds for a larger class of groups, from Corollary 5 of [26], due to [11] (extending the arguments from G_m to Δ_m is straightforward). Independent of this earlier work, Bogomolny-Schmit [7] gave a clever proof of the result specifically for Δ_m. See the next remark for another perspective.

Theorem 2.1. *Fix m as above, and let Δ_m be the full reflection group in which G_m has index two. Then for any $M \in \Delta_m$ whose trace is of absolute* *value greater than* 2*, we have*

$$|\, tr(M)\,| \geq |\, \sigma(\, tr(M)\,)\,|\,,$$

where σ is any field embedding of $\mathbb{Q}(\lambda_m)$.

Remark 3. This result can be proven "geometrically". Up to conjugacy, each of the Hecke groups appears as the Veech group of some translation surface; see [28]. Those elements of trace greater than 2 in absolute value are the "derivatives" of the affine pseudo-Anosov diffeomorphisms of the surface. The dilatation of a pseudo-Anosov ϕ is the dominant eigenvalue λ of the action of ϕ on the integral homology of the underlying surface. (The other eigenvalues are hence conjugates of λ.) The corresponding element of the Veech group has trace of absolute value $\lambda + \lambda^{-1}$ from which it follows that this trace dominates its conjugates.

2.4. **Approximation by algebraic numbers.** The following result was announced by Roth [23] and proven by LeVeque; see Chapter 4 of [17]. (The version below is Theorem 2.5 of [9].) Recall that given an algebraic number α, its *naive height*, denoted by $H(\alpha)$, is the largest absolute value of the coefficients of its minimal polynomial over $\mathbb{Z}$.

Theorem 2.2. *(Roth-LeVeque) Let K be a number field, and ξ a real algebraic number not in K . Then, for any $\epsilon > 0$, there exists a positive constant $c(\xi, K, \epsilon)$ such that*

$$|\, \xi - \alpha\,| > \frac{c(\xi, K, \epsilon)}{H(\alpha)^{2+\epsilon}}$$

holds for every α in K.

The *logarithmic Weil height* of α lying in a number field K of degree D over $\mathbb{Q}$ is $h(\alpha) = \frac{1}{D}\sum_\nu \log^+ \max_{\nu\in M_K}\{||\alpha||_\nu\}$, where $\log^+ t$ equals 0 if $t \leq 1$ and M_K denotes the places (finite and infinite "primes") of the field, and $||\cdot||_\nu$ is the ν-absolute value. This definition is independent of the field K containing α. Recall that the product formula states that the product over $\nu \in M_K$ of the $||\alpha||_\nu$ equals 1. Using this, for $\alpha, \beta \in K$, with $\beta \neq 0$, one has

$$h(\alpha/\beta) \leq \sum_\sigma \frac{1}{D} \log \max\{|\sigma(\alpha)|, |\sigma(\beta)|\}\,, \tag{6}$$

where the sum in σ is taken over the field embeddings of K into the complex numbers, and $|\cdot|$ denotes the usual complex norm.

The two heights are related by

$$\log H(\alpha) \leq \deg(\alpha) h(\alpha) + \log 2, \tag{7}$$

for any non-zero algebraic number α; see Lemma 3.11 from [29].

We recall a consequence of the W. Schmidt Subspace Theorem; see Theorem 9A of [25].

Theorem 2.3. *Let d be a positive integer and ξ be a real algebraic number of degree greater than d. Then, for every positive ε, there exist only finitely many algebraic numbers α of degree at most d such that*

$$|\xi - \alpha| < H(\alpha)^{-d-1-\varepsilon}.$$

Note that the Roth theorem is exactly the case $d = 1$ of Theorem 2.3.

In the proof of Theorem 1.2, we could apply Theorem 2.3, but the algebraic numbers α which we use to approximate ξ are of degree at most 2 over a fixed number field. In this situation, the next theorem, kindly communicated to us by J.-H. Evertse [13], yields a stronger result than the previous one.

Theorem 2.4. *(Evertse) Let K be a real algebraic number field of degree d. Let t be a positive integer and ξ be a real algebraic number of degree greater than t over K. Then, for every positive ε, there exist only finitely many algebraic numbers α of degree t over K and δ over $\mathbb{Q}$ such that*

$$|\xi - \alpha| < H(\alpha)^{-dt(t+1+\varepsilon)/\delta}.$$

Note that Theorem 2.4 extends Theorem 2.2.

2.5. **Sturmian sequences: towards an application of Theorem 1.2.** To give an explicit family of Rosen expansions satisfying the hypotheses of Theorem 1.2, we recall a result of [3] on Sturmian sequences.

Let a and b be letters in some alphabet. The complexity function of a sequence $\mathbf{u} = u_1 u_2 \cdots$ with values in $\{a, b\}$ is given by letting $p(n, \mathbf{u})$ be the number of distinct words of length n that occur in $\mathbf{u}$. A sequence $\mathbf{u}$ is called

Sturmian if its complexity satisfies $p(n, \mathbf{u}) = n + 1$ for all n. As Arnoux [4] writes, one can obtain any such sequence by taking a ray with irrational slope in the real plane and intersecting it with an integral grid, assigning a when the ray intersects a horizontal grid line and b when it meets a vertical grid line. Indeed, the *slope* of a Sturmian sequence is the density of a in the sequence (one shows that the limit as n tends to infinity of the average of the number of occurrences a in $u_1 \cdots u_n$ exists; see [4], Proposition 6.1.10).

Lemma 2.2. *Let* $\mathbf{u}$ *be a Sturmian sequence whose slope has an unbounded regular continued fraction expansion. Then, for every positive integer* n*, there are finite words* U*,* V *and a positive real number* s *such that* UV^s *is a prefix of* $\mathbf{u}$ *and* $|UV^s| \geq n|UV|$.

Proof. This follows from the proof of Proposition 11.1 from [3]. □

Remark 4. We apply the above lemma to Sturmian sequences where both a, b are of the form (ε, r), with $\varepsilon = \pm 1$ and $r \in \mathbb{N}$. In particular, we use this in the context of Rosen expansions to prove Corollary 4.1.

3. Bounding the height of convergents

In what follows, we fix $\lambda = \lambda_m$ for some $m > 3$, and suppose that $\xi \in (0, \lambda/2)$ is a real algebraic number having an infinite Rosen continued fraction expansion over $\mathbb{Q}(\lambda)$. Our goal is to estimate the naive height $H(p_n/q_n)$ of the nth convergent p_n/q_n. In light of Theorem 2.1, we let n_0 be the least value of n such that $q_n > 2$.

Lemma 3.1. *Let* $c_3 = c_3(\lambda)$ *be defined by* $c_3 = \min_\sigma \frac{|\sigma(\lambda)|}{\lambda}$*, where the minimum is taken over all field embeddings of* $\mathbb{Q}(\lambda)$ *into* $\mathbb{R}$*. Then for all* $n \geq n_0$*, and any such* σ*, we have both*

$$q_n \geq c_3\, |\,\sigma(q_n)\,| \quad \text{and} \quad p_n \geq c_3\, |\,\sigma(p_n)\,|\,.$$

Proof. For any $n \geq n_0$, recall that $M_n = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}$; this is clearly an element of Δ_m. By Theorem 2.1 we have $q_n + p_{n-1} \geq |\,\sigma(q_n + p_{n-1})\,|$.

Now let $j \in \mathbb{N}$ and set

$$M_{n,j} = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} \begin{pmatrix} 1 & j\lambda \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} p_{n-1} & p_n + j\lambda p_{n-1} \\ q_{n-1} & q_n + j\lambda q_{n-1} \end{pmatrix}.$$

This is also an element of Δ_m of trace greater than 2, and hence

$$|\, p_{n-1} + q_n + j\lambda q_{n-1}\,| \ge |\, \sigma(p_{n-1} + q_n) + j\sigma(\lambda q_{n-1})\,|\,.$$

Since this holds for *all* positive j, we must have that $\lambda q_{n-1} \ge |\,\sigma(\lambda q_{n-1})\,|$. That is,

$$q_{n-1} \ge \frac{|\,\sigma(\lambda)\,|}{\lambda}\,|\sigma(q_{n-1})| \,\ge \left(\min_{\sigma} \frac{|\,\sigma(\lambda)\,|}{\lambda}\right)|\sigma(q_{n-1})|\,.$$

Similarly, using

$$N_{n,j} = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix}\begin{pmatrix} 1 & 0 \\ j\lambda & 1 \end{pmatrix} = \begin{pmatrix} p_{n-1} + j\lambda p_n & p_n \\ q_{n-1} + j\lambda q_n & q_n \end{pmatrix}\,,$$

we find

$$p_n \ge \frac{|\,\sigma(\lambda)\,|}{\lambda}\,|\sigma(p_n)| \,\ge \left(\min_{\sigma} \frac{|\,\sigma(\lambda)\,|}{\lambda}\right)|\sigma(p_n)|\,.$$

□

Remark 5. We conjecture that in fact q_n is always greater than or equal to its conjugates, thus that in the above one can replace c_3 by 1.

Lemma 3.2. *Let D denote the field extension degree $[\mathbb{Q}(\,\lambda\,) : \mathbb{Q}\,]$. There exists a constant $c_4 = c_4(\lambda)$ such that for all $n \ge n_0$,*

$$H(p_n/q_n) \le c_4 q_n^D.$$

Proof. Since p_n and q_n are algebraic integers of degree at most D, it follows from Lemma 3.1 and Equation (6) that

$$h(p_n/q_n) \le \sum_{\sigma} \frac{1}{D} \log\max\{|\sigma(p_n)|, |\sigma(q_n)|\}$$

where σ runs through the complex embeddings. We thus have

$$h(p_n/q_n) \le c_4' + \log q_n$$

for a suitable positive constant c_4'. Using (7), we get the asserted estimate.

□

Lemma 3.3. *Let α be a real number in $[-\lambda/2, \lambda/2)$ with an ultimately periodic expansion in Rosen continued fraction. Denote by $(p_n/q_n)_{n\ge 1}$ the sequence of its convergents. Denote by μ the length of the preperiod and by ν the length of the period, with the convention that $\mu = 0$ if the expansion is* *purely periodic. Then α is of degree at most 2 over $\mathbb{Q}(\lambda)$, and there exists $c_5 = c_5(\lambda, \alpha)$ such that*

$$H(\alpha) \leq c_5(q_\mu q_{\mu+\nu})^D.$$

Proof. In the notation of Equation (4), α is fixed by $M = M_\mu^{-1} M_{\mu+\nu}$. It thus satisfies a quadratic equation with entries in $\mathbb{Z}[\lambda]$, and hence is of degree at most 2 over $\mathbb{Q}(\lambda)$. Indeed, α is a root of $f(x) = cx + (d-a)x - b$ with a, b, c, d denoting the entries of M. Each entry is a $\mathbb{Z}$-linear combination of monomials of the form rs with r an entry of M_μ and s an entry of $M_{\mu+\nu}$.

Now, α is also a root of $\tilde{f}(x) = \prod_\sigma \sigma(f)(x) \in \mathbb{Z}[x]$, where $\sigma(f)$ denotes the result of applying σ to the coefficients of $f(x)$. By Lemma 3.1, all of the conjugates of each of $p_\mu, p_{\mu-1}, q_{\mu-1}, q_\mu$ can be bounded by the product of q_μ with a constant depending upon α and λ. Similarly for the entries of $M_{\mu+\nu}$. After some computation, we conclude that the height of α is $\ll q_\mu^D q_{\mu+\nu}^D$. (One checks that the case of $\mu = 0$ is subsumed by the above.) $\square$

Remark 6. Whereas a real number whose regular continued fraction expansion is ultimately periodic is exactly of degree two over the field of rational numbers, in the previous lemma the words "at most" are necessary. Indeed, $x = 1$ has an ultimately periodic Rosen expansion with respect to any λ_m with m even; see [21]. Further examples of elements of $\mathbb{Q}(\lambda_m)$ with periodic expansions are easily given when $m \in \{4, 6\}$; see Corollary 1 of [24]. Yet further examples, including cases with $m \in \{7, 9\}$, are given in [22], [14].

4. Transcendence results

As usual, $\ll$ and $\gg$ denote inequality with implied constant.

4.1. Applying Roth–LeVeque: the proof of Theorem 1.1.

We now show that the sequence of denominators of convergents to an algebraic number cannot grow too quickly. Theorem 1.1 then follows.

Proof. Let ε be a positive real number. Let ζ be an algebraic number having an infinite Rosen expansion with convergents r_n/s_n.

By the Roth–LeVeque Theorem 2.2, we have

$$|\zeta - r_n/s_n| \gg H(r_n/s_n)^{-2-\varepsilon}, \quad \text{for } n \geq 1.$$

And, hence by Lemma 3.2, for $n \geq n_0 = n_0(\zeta)$, we have $|\zeta - r_n/s_n| \gg s_n^{-2D-D\varepsilon}$. Inequality (5) then gives that there exists a constant c_6 (independent of $n \geq n_0$) such that

$$s_{n+1} < c_6 s_n^{2D-1+D\varepsilon}\,.$$

Set $a = 2D - 1 + D\varepsilon$. For $j < n_0$, define ℓ_j such that $s_j < \ell_j s_{j-1}^a$. We set $c_7 = \max\{1, c_6, \ell_1, \ldots, \ell_{n_0-1}\}$ and find that for any $n > 1$

$$s_{n+1} < c_7 s_n^a < c_7 (c_7 s_{n-1}^a)^a \leq (c_7 s_{n-1})^{a^2}$$

and continuing in this manner, we have $s_{n+1} < (c_7 s_1)^{a^n}$. Since $s_{n+1} > s_n$ letting $c_8 = c_7 s_1$, gives $\log s_n < a^n \log c_8$. From this follows that

$$\limsup_{n\to+\infty} \frac{\log\log s_n}{n} < \log(D(2+\varepsilon) - 1).$$

Letting ε go to zero, we see that every algebraic number satisfies

$$\limsup_{n\to+\infty} \frac{\log\log s_n}{n} \leq \log(2D - 1),$$

as asserted. $\square$

4.2. **Proof of Theorem 1.2 and an application.**

Proof. With $\lambda = 2\cos\pi/m$ fixed, given ξ of infinite Rosen continued fraction with convergents $(p_n/q_n)_{n\geq 1}$, we let $b = \liminf_n q_n^{1/n}$ and $B = \limsup_n q_n^{1/n}$, and assume that $B < \infty$. Let η be a positive real number with $b-1 < \eta < b$. Since there are only finitely many n with either $q_n^{1/n} < b - \eta$ or $q_n^{1/n} > B + \eta$, we have both that $q_n \gg (b-\eta)^n$ and $q_n \ll (B+\eta)^n$.

Suppose that w is a positive real number and U, V are finite words in $\{\pm 1\} \times \mathbb{N}$ such that UV^w is a prefix of the infinite word composed of the partial quotients of ξ. Denote by α the real number of degree at most two over $\mathbb{Q}(\lambda)$ whose Rosen continued fraction is given by the word UV^∞, where V^∞ means the concatenation of infinitely many copies of V. Set $|U| = u$ and $|V| = v$. Since ξ and α have their first $\lfloor u + vw \rfloor$ partial quotients in common, we have

$$|\xi - \alpha| < c_1\, q_{\lfloor u+vw \rfloor}^{-2} \ll (b-\eta)^{-2(u+vw)}.$$

Furthermore, it follows from Lemma 3.3 that

$$H(\alpha) \ll (q_u\, q_{u+v})^D \ll (B+\eta)^{D(2u+v)}.$$

Combined with the previous inequality, this gives

$$|\xi - \alpha| \ll H(\alpha)^{-2(u+vw)\log(b-\eta)/(D(2u+v)\log(B+\eta))}.$$

Now suppose that ξ is algebraic of degree greater than two over $\mathbb{Q}(\lambda)$. Then, for every $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that every real algebraic number β of degree at most 2 over $\mathbb{Q}(\lambda)$ satisfies

$$|\xi - \beta| > C(\varepsilon) H(\beta)^{-3-\varepsilon}.$$

This follows from Theorem 2.2 if β is in $Q(\lambda)$ and, otherwise, by applying Theorem 2.4 with $t = 2$ and $dt = \delta$ to each subfield K of $\mathbb{Q}(\lambda)$.

This proves that ξ must be transcendental if there are u, v, w such that $u + vw$ is arbitrarily large and

$$\frac{2(u+vw)\log b}{D(2u+v)\log B} > 3,$$

as asserted. □

Corollary 4.1. *A Rosen continued fraction whose sequence of partial quotients is Sturmian with slope of unbounded regular continued fraction partial quotients represents a transcendental number.*

Proof. Combine Lemma 2.2 with Theorem 1.2. □

Remark 7. Using the Subspace Theorem as in [1, 10] does not yield in general an improvement of Theorem 1.2. In case $u = 0, b = B$, inequality (1) reduces to $w > 3D/2$, while, proceeding as in [1, 10], we would get $w > 2D-1$. However, if b is much smaller than B and D is small, then the approach of [1, 10] presumably gives a slightly better result than Theorem 1.2.

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