On integers with identical digits

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1. Introduction

A longstanding conjecture claims that the Diophantine equation

$$\frac{x^n-1}{x-1} = y^q \quad \text{in integers } x>1, y>1, n>2, q\geq 2 \tag{1}$$

has finitely many solutions, and, maybe, only those given by

$$\frac{3^5-1}{3-1} = 11^2, \quad \frac{7^4-1}{7-1} = 20^2 \quad \text{and} \quad \frac{18^3-1}{18-1} = 7^3.$$

Among the known results, let us mention that Ljunggren [13] solved completely (1) when $q=2$ and that Nagell [9] treated the cases $3 \mid n$ and $4 \mid n$. For more information and in particular for finiteness type results under some extra hypotheses, we refer the reader to Shorey & Tijdeman [18], [19] and to the recent survey of Shorey [17].

In the present work, we solve completely (1) when x is any integer in the interval $[2, 10]$ and, under some restriction, when x is a power of a prime number. In these cases, it is already known that the number of solutions is finite and that they are bounded by an effectively computable constant. As in the paper [4], we combine several methods in Diophantine approximation together with some computer calculations. Our main tool is a new lower bound for linear forms in two p-adic logarithms (see [2]), which applies very well to (1) and allows us to considerably reduce the time of computation.

We would like to point out that we solve an old open problem in proving that any number $x = 11\ldots 11$, with all digits equal to 1 in base 10, is not a pure power ... except $x=1$.

2. Statement of the results

First, we warn the reader that the proofs of the results obtained in Le [12] and in Yu & Le [21] are incorrect. Indeed, they all depend on Lemma 3 of [12], which is false, see the comment of Yuan Ping-Zhi [22]. In the present work, we supply a correct proof of some of their claims. However, we remark that our method is essentially different from theirs.

Our main result is the following.

Theorem 1. *Let $t \geq 1$ be a rational integer. Let z be an integer in the interval $[2, 10]$ and let $p \geq 11$ be a prime number. Then equation (1) has only two solutions (x, y, n, q) with $x = z^t$, namely $(x, y, n, q) = (3, 11, 5, 2)$ and $(7, 20, 4, 2)$. Further, equation (1) has no solution (x, y, n, q) with $x = p^t$ and $\gcd(q, p-1) = 1$.*

Remarks. • Our method can be applied to solve (1) with $x = z^t$, where z is a fixed integer satisfying some hypothesis. The details can be found in [2], however, let us mention that this hypothesis is satisfied by every integer between 2 and 20, except 11. Moreover, we are unable to prove the following assertion.

Problem. *The Diophantine equation*

$$\frac{11^{tn}-1}{11^t-1} = y^5 \quad \text{in integers } t \geq 1, y \geq 2 \text{ and } n \geq 3$$

has only finitely many solutions.

• Let p be a given prime number and let t be a given rational integer, such that p^t is not too large, in order that some computation can be done. By Step 3 and Step 4 of the proof of Theorem 1, we are able to show that equation (1) has no solution (p^t, y, n, q) with q dividing $p - 1$.

• The new lower bounds for linear forms in two p-adic logarithms obtained in [2] play here a key role, and it is likely that the results of [3] would have led to excessive computation. Indeed, the numerical constants appearing in Proposition 2 would then be much bigger.

Two immediate corollaries of Theorem 1 deserve to be pointed out.

The first one deals with the Diophantine equation

$$3^m - 2y^q = 1 \quad \text{in integers } m \geq 2, y \geq 2 \text{ and } q \geq 2, \tag{2}$$

which appears in the theory of finite groups. Cresenzo [6] claimed that (2) has no solution, but this is obviously false since $3^5 - 2 \cdot 11^2 = 1$, as observed by Sun [20] and, later, by Le [11] (Notice that both Lemmas 1 and 2 of [11] are incorrect). Further, Yu & Le [21] asserted that (2) has only the above quoted solution, but there is a mistake in their proof. Theorem 1 implies the following statement.

Corollary 1. *Equation (2) has only the solution $(m, y, q) = (5, 11, 2)$.*

The second corollary is related to numbers with all digits equal to a given digit a, in a given base b. Combined with a result of Inkeri [7], extending previous work of Obláth [15], Theorem 1 allows us to complete Theorem 6 of Shorey & Tijdeman [18].

Theorem 2. *Let a and b be rational integers with $2 \leq b \leq 10$ and $1 \leq a \leq b-1$. The integer N with all digits equal to a in base b is not a pure power, except for $N = 1, 4, 8$ or 9, for $N = 11111$ written in base $b = 3$, for $N = 1111$ written in base $b = 7$ and for $N = 4444$ written in base $b = 7$.*

More precisely, when compared with Obláth and Inkeri, the only new case here is the case $a = 1$, which remained open for about fifty years.

3. Auxiliary results

The main tool of our proof is a sharp lower bound for the p-adic distance between two rational numbers, obtained in [2]. We normalize the p-adic valuation by $v_p(p) = 1$.

Proposition 1. *Let p be a prime number and let α_1 and α_2 be two rational numbers satisfying $v_p(\alpha_1 - 1) > 0$ and $v_p(\alpha_2 - 1) > 0$. We assume that there exists a real number $E > 0$ such that*

$$v_p(\alpha_1 - 1) \geq E > \frac{1}{p-1},$$

and that, in the case $p = 2$, we have $v_p(\alpha_2 - 1) \geq 2$. Let b_1 and b_2 be two strictly positive integers, denote by p^u the greatest power of p which divides simultaneously b_1 and b_2 and assume that p does not divide b_2/p^u. Let $K \geq 3$, $L \geq 2$, R_1, R_2, S_1, S_2 be positive integers and set

$$R = R_1 + R_2 - 1, \quad S = S_1 + S_2 - 1, \quad N = KL \text{ and } \gamma = \frac{1}{2} - \frac{N}{6RS}.$$

Put

$$b = \frac{(R-1)b_2 + (S-1)b_1}{2} \left(\prod_{k=1}^{K-1} k! \right)^{-2/(K^2-K)}.$$

and assume that

$$\begin{aligned} &\mathrm{Card}\,\{\alpha_1^r \alpha_2^s\,;\, 0 \leq r < R_1,\, 0 \leq s < S_1\} \geq L, \\ &\mathrm{Card}\Big\{rb_2 + sb_1;\, 0 \leq r < R_2,\, 0 \leq s < S_2\Big\} > (K-1)L. \end{aligned} \tag{3}$$

Under the condition

$$\begin{aligned} K(L-1)E\log p - 3\log N - &(K-1)\log b \\ &- \gamma\, L\big(R\,\mathrm{h}(\alpha_1) + S\,\mathrm{h}(\alpha_2)\big) \;>\; 0, \end{aligned} \tag{4}$$

we have

$$v_p(\alpha_1^{b_1} - \alpha_2^{b_2}) \leq E\,(KL - 1/2) + u.$$

Proof : This is Theorem 1 of [2] in the rational case. $\square$

Remark. This result slightly sharpens the estimates of [3] thanks to the new parameter E, which allows us, in the applications, to improve considerably the numerical bounds. Its full strength is used in *Step 2* of the proof and also in Proposition 2 below.

Proposition 2. *Let (p^t, y, n, q) be a solution to (1) with $q \geq 3$, p prime, $t \geq 1$ and $\gcd(q, p-1) = 1$. If $p \neq q$, then we have $p^t \leq 4063$, and if $p = q$ we have $p \leq 1900$ and $p^{t-2} \leq 56000$. Moreover, if $(6^t, y, n, q)$ (resp. $(10^t, y, n, q)$) is a solution of (1) with $t \geq 1$ and $q \geq 3$, then we have $t \leq 7$ (resp. $t \leq 5$).*

Proof : This is Theorem 4 of [2] for the first part. The last statement is proved exactly like Theorem 4 of [2] in the case of composite numbers. Let us briefly recall the arguments. Assume there exists $t \geq 1$, $y \geq 2$, $n \geq 3$ and $q \geq 3$ such that $(6^t, y, n, q)$ is a solution to (1). We apply Proposition 1 to bound from above the 3-adic valuation of $(1 - 6^t) - y^{-q}$, which is, in view of (1), equal to tn. Hence, we get $q \leq 7000$, independently of t. To derive an upper bound for t, we apply Lemma 18 of Saradha & Shorey [16] and we have to distinguish the cases $q = 3$ and $q > 3$. $\square$

A third important tool of our proof is the following result of Bennett & de Weger [1].

Proposition 3. *If a, b and n are integers with $b > a \geq 1$ and $n \geq 3$, then the equation*

$$|ax^n - by^n| = 1$$

has at most one solution in positive integers (x, y), except possibly for the cases where $b = a + 1$, $2 \leq a \leq \min\{0.3n, 83\}$ and $17 \leq n \leq 347$.

Proof : This is Theorem 1.1 of [1]. $\square$

We also need a result of Le [10].

Proposition 4. *Equation (1) has no solution (x, y, n, q) with x a perfect q-th power.*

Proof : This is Theorem 1 of [10]. However, using the work of Bennett & de Weger [1], we can give the following very short proof. Assume that $(x^n-1)/(x-1) = y^q$ with $x = z^q$. Then the equation $X^q-(z^q-1)Y^q = 1$ has the two solutions $(X, Y) = (z^n, y)$ and $(X, Y) = (z, 1)$, contradicting Proposition 3. $\square$

Finally, we recall the main result of [4].

Proposition 5. *Equation (1) has no solution (x, y, n, q) with x being a square.*

Proof : This is Theorem 1 of [4]. $\square$

4. Proof of Theorem 1

Let z and $t \geq 1$ be two rational integers with $2 \leq z \leq 10$ or z prime. Set $x = z^t$ and assume that there exist integers $y \geq 2$, $n \geq 3$ and $q \geq 3$ such that (x, y, n, q) is a solution of equation (1). If z is a prime number with $z \geq 11$, we also assume that $\gcd(q, z-1) = 1$. Proposition 5 implies that t is odd. Further, we can suppose that $x \neq 2$, since it is known that the equation $2^N - 1 = Y^Q$ has no solution in integers $N \geq 2$, $Y \geq 2$ and $Q \geq 2$; see for instance Cassels [5].

We divide the proof into five steps.

• *Step 1.* Application of Proposition 2.

We first apply Proposition 2 to obtain an upper bound for t, which appears to be relatively small. Namely, for prime values of z, we get

z	2	3	5	7	11	13	≥ 17
$t \leq$	13	7	5	5	3	3	1

in the case when $z \neq q$ and

z	3	5	7	≥ 11	≥ 41
$t \leq$	11	7	7	5	3

if $z = q$. Further, we have $t \leq 7$ for $z = 6$ and $t \leq 5$ for $z = 10$. Recall also that $\gcd(q, t) = 1$, by Proposition 4.

• *Step 2.* Sharp upper bound for q.

For each of the remaining equations, we apply Proposition 1 to obtain a very precise upper bound for q. Indeed, we bound from above the p-adic valuation of

$$\Lambda = (1 - z^t) - \left(\frac{1}{y}\right)^q = -z^{tn} y^{-q},$$

where $p = z$ if z is prime and $p = 3$ (*resp.* $p = 5$) if $z = 6$ (*resp.* $z = 10$). With an extensive computer search, we choose the optimal values for the parameters K, L, R_1, R_2, S_1 and S_2 in order that condition (4) is satisfied and the product KL is as small as possible. Surprisingly, we remark that these parameters are rather small. Indeed, L is always less than 10, R_1 and S_1 are both less than 5, R_2 and S_2 are not bigger than 60, and K is nearly equal to $R_2 S_2 / L$. Observe also that $1 - z^t$ and y are multiplicatively independent and that $S_1 + S_2 - 1 \leq q$, hence condition (3) is fulfilled. At this stage, we can summarize our result in the following table.

z	2	2	3	3	5	5	6	6
t	3	≥ 5	1	≥ 3	1	≥ 3	1	≥ 3
$q \leq$	137	37	1900	73	947	47	6172	188

z	7	7	10	10	≥ 11	≥ 11
t	1	≥ 3	1	≥ 3	1	≥ 3
$q \leq$	682	36	2063	92	452	31

Notice that the upper bounds for q obtained here do not depend on whether or not q divides z. In particular, we have already slightly improved Proposition 2 in showing that, for any prime $p \geq 457$, equation (1) has no solution (p^t, y, n, p) with $t \geq 1$, $y \geq 2$ and $n \geq 3$.

- *Step 3.* Use of congruences.

In view of Proposition 4, we can assume that x is not a q-th perfect power. Further, we use congruences to prove that, in *all* remaining cases, we have $n \equiv 1 \pmod q$. The method is competely elementary: we consider congruences modulo a suitable power of q and modulo primes p congruent to 1 mod q. Generally only a few tries are enough. The whole verification took less than a day (we used PARI).

- *Step 4.* Use of Bennett - de Weger's result.

In view of the last step, there is an integer $\nu \geq 1$ such that $n = \nu q + 1$. Plugging this into equation (1), we obtain

$$x\left(x^{\nu}\right)^{q} - (x-1)\, y^{q} = 1.$$

Combined with the computations of Step 3 this solves all the cases treated in Bennett - de Weger's paper and listed in Proposition 3.

- *Step 5.* The last cases.

The remaining cases have been solved very recently in two different and independent ways which were presented in the conference on Diophantine Approximation in Luminy, May 1998. Using Padé approximation, M. Bennett solved positively the (about) three thousand cases which remained after Proposition 3. Using the general estimate for linear forms in two logarithms obtained in [8], M. Mignotte could decrease the number of cases in Proposition 3 to 46. Then, with the help of our colleague Yves Roy, the remaining corresponding cases of equation (1) were solved completely. This verification took about one month of computation, the upper bounds on n were provided by estimates on Thue equations due to Y. Bugeaud; the details are presented in [4].

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