A KIT ON LINEAR FORMS IN THREE LOGARITHMS

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Abstract. In this paper we give a general presentation of the results to be used to get a 'good' lower bound for a linear form in three logarithms of algebraic numbers in the so-called rational case. We recall the best existing general result — Matveev's theorem — and we add a powerful new lower bound for linear forms in three logarithms. We treat in detail the 'degenerate' case, *i.e.* the case when the conditions of the zero-lemma are not satisfied.

1. Introduction

In this paper we give a general presentation of the results to be used to get a 'good' lower bound for a linear form in three logarithms of algebraic numbers. We recall the best existing general result — Matveev's theorem — and we add a powerful new lower bound for linear forms in three logarithms in the so-called rational case, *i.e.* when the coefficients of the linear form are rational integers. We use this result as a first step in our computation of a lower bound. Even if this is not necessary from the logical point of view, this helps a lot for the study of the 'degenerate' case, *i.e.* the case when the conditions of the zero-lemma are not satisfied. We treat in detail the degenerate case, using linear forms in two logarithms. In the degenerate case, another approach (see [12]) is to use determinants of interpolation especially built according to the conditions of the zero-lemma; maybe this approach gives better results but this is not clear in our case. It seems that the published results in this case give weaker results than ours.

Essentially, the present paper is extracted from [2] and [3], but we give much more details in order that this presentation is almost self-contained. Our method is the method of interpolation determinants introduced by Michel Laurent in [5], [6] and [7]. In the case of three logarithms, this method was used by C.D. Bennett *et al.* [1]. But the present paper brings some progress when compared to [1]: we treat the general case of algebraic numbers (not only multiplicatively independent rational integers), many technical details have been improved and, more importantly, a new zero-lemma of Michel Laurent leads to much better estimates.

Our aim, suggested by the title *a kit...*, is to explain how to deal with concrete cases to get a lower bound of a linear form Λ in three logarithms of algebraic numbers. The process contains three steps. First, using a general estimate of Matveev, we obtain some lower bound, say B_1. Then, this first result is used in our estimate for which there are two cases, the *non-degenerate* case and the *degenerate* case. In the non-degenerate case we get a second lower bound B_2, and if B_2 is smaller than B_1 we study the degenerate case. In this case, we consider our linear form in three logarithms as a linear form in two logarithms and we apply the results of Laurent-Mignotte-Nesterenko [9] to this linear form and get a third lower bound B_3. Of course, the conclusion is $|\Lambda| \geq \min\{B_2, B_3\}$. In the degenerate case, there are other ways to proceed in the literature, see the comments in Section 5.

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2. Matveev's theorem for three logarithms

First, we need the special case of three logarithms of the Theorem of E. M. Matveev, thus we quote his result. This theorem enables us to get a first bound in our studies and this bound can be used as the departure for further improvements. The reason for this should appear later.

Theorem 1 (Matveev)**.** *Let $\lambda_1,\ \lambda_2,\ \lambda_3$ be $\mathbb{Q}$–linearly independent logarithms of non-zero algebraic numbers and let $b_1,\ b_2,\ b_3$ be rational integers with $b_1 \neq 0$. Define $\alpha_j = \exp(\lambda_j)$ for $j = 1,\ 2,\ 3$ and*

$$\Lambda = b_1\lambda_1 + b_2\lambda_2 + b_3\lambda_3.$$

Let D be the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ over $\mathbb{Q}$. Put

$$\chi = [\mathbb{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{R}].$$

Let $A_1,\ A_2,\ A_3$ be positive real numbers, which satisfy

$$A_j \geq \max\bigl\{D\mathrm{h}(\alpha_j), |\lambda_j|, 0.16\bigr\} \quad (1 \leq j \leq 3).$$

Assume that

$$B \geq \max\Bigl\{1, \max\bigl\{|b_j|A_j/A_1;\ 1 \leq j \leq 3\bigr\}\Bigr\}.$$

Define also

$$C_1 = \frac{5 \times 16^5}{6\chi}\, e^3\, (7 + 2\chi) \left(\frac{3e}{2}\right)^{\chi} \Bigl(20.2 + \log\bigl(3^{5.5} D^2 \log(eD)\bigr)\Bigr).$$

Then

$$\log|\Lambda| > -C_1\, D^2\, A_1\, A_2 A_3\, \log\bigl(1.5\, eDB \log(eD)\bigr).$$

Proof. See [10]. □

3. A new estimate on linear forms in three logarithms

We present the type of linear forms in three logarithms that we shall study. We consider three non-zero algebraic numbers α_1, α_2 and α_3 and positive rational integers $b_1,\ b_2,\ b_3$ with $\gcd(b_1, b_2, b_3) = 1$, and the linear form

$$\Lambda = b_2 \log\alpha_2 - b_1 \log\alpha_1 - b_3 \log\alpha_3 \neq 0.$$

We restrict our study to the following cases:

- **the real case**: α_1, α_2 and α_3 are real numbers > 1, and the logarithms of the α_i are all real (and > 0). Moreover, in concrete cases, α_1, α_2 and α_3 are multiplicatively independent. Of course, then the $\log\alpha_j$'s are $\mathbb{Q}$–linearly independent.
- **the complex case**: α_1, α_2 and α_3 are complex numbers $\neq 1$ of modulus one, and the logarithms of the α_i are arbitrary determinations of the logarithm (then any of these determinations is purely imaginary). In practical examples, two of these α's are multiplicatively independent and the third one is a root of unity. We shall see later that (see Corollary 3.10), in this case, the $\log\alpha_j$'s are again $\mathbb{Q}$–linearly independent.

In practice this restriction does not cause any inconvenience since

$$|\Lambda| \geq \max\bigl\{|\Re(\Lambda)|, |\Im(\Lambda)|\bigr\},$$

and so we can always reduce to the above cases.

Without loss of generality, we may assume that

$$b_2|\log\alpha_2| = b_1|\log\alpha_1| + b_3|\log\alpha_3| \pm |\Lambda|.$$

But notice than this introduces some important dissymmetry between the roles of the coefficients b_1, b_2 and b_3.

Like the authors of [1], we use Laurent's method, and consider a suitable interpolation determinant Δ.

We shall choose rational positive integers K, L, R, S, T, with K, $L \geq 2$, we put $N = K^2L$ and we assume $RST \geq N$. Let i be an index such that (k_i, m_i, ℓ_i) runs trough all triples of integers with $0 \leq k_i \leq K-1$, $0 \leq m_i \leq K-1$ and $0 \leq \ell_i \leq L-1$. So each number $0, \ldots, K-1$ occurs KL times as a k_i, and similarly as an m_i, and each number $0, \ldots, L-1$ occurs K^2 times as an ℓ_i.

Put

$$b_1 = d_1 b_1',\ b_3 = d_3 b_3'',\ b_2 = d_1 b_2' = d_3 b_2'',\ \beta_1 = b_1/b_2 = b_1'/b_2',\ \beta_3 = b_3/b_2 = b_3''/b_2'',$$

where

$$d_1 = \gcd(b_1, b_2) \quad \text{and} \quad d_3 = \gcd(b_3, b_2).$$

With the above definitions, let

$$\Delta = \det\left\{ \binom{r_j b_2' + s_j b_1'}{k_i} \binom{t_j b_2'' + s_j b_3''}{m_i} \alpha_1^{\ell_i r_j} \alpha_2^{\ell_i s_j} \alpha_3^{\ell_i t_j} \right\},$$

where r_j, s_j, t_j are non-negative integers less than R, S, T, respectively, such that (r_j, s_j, t_j) runs over N distinct triples.

Let

$$\lambda_i = \ell_i - \frac{L-1}{2}, \quad \eta_0 = \frac{R-1}{2} + \beta_1 \frac{S-1}{2}, \quad \zeta_0 = \frac{T-1}{2} + \beta_3 \frac{S-1}{2},$$

and

$$b = (b_2'\eta_0)(b_2''\zeta_0) \left(\prod_{k=1}^{K-1} k! \right)^{-\frac{4}{K(K-1)}}.$$

Following [9], Lemme 8, we can prove that

$$\begin{aligned} \log b \leq \log \frac{(R-1)b_2 + (S-1)b_1}{2d_1} &+ \log \frac{(T-1)b_2 + (S-1)b_3}{2d_3} \\ - 2\log K + 3 - \frac{2\log(2\pi K/e^{3/2})}{K-1} &+ \frac{2 + 6\pi^{-2} + \log K}{3K(K-1)}. \end{aligned}$$

Then, we have $\sum_{i=0}^{N-1} \lambda_i = 0$ and ([1], formula (2.1))

$$\alpha_1^{\lambda_i r_j} \alpha_2^{\lambda_i s_j} \alpha_3^{\lambda_i t_j} = \alpha_1^{\lambda_i (r_j + s_j \beta_1)} \alpha_3^{\lambda_i (t_j + s_j \beta_3)} e^{\lambda_i s_j \Lambda / b_2} = \alpha_1^{\lambda_i (r_j + s_j \beta_1)} \alpha_3^{\lambda_i (t_j + s_j \beta_3)} (1 + \theta_{ij} \Lambda'),$$

where

$$\theta_{ij} = \frac{e^{\lambda_i s_j \Lambda / b_2} - 1}{\Lambda'}$$

and

$$\Lambda' = |\Lambda| \cdot \frac{LSe^{LS|\Lambda|/(2b_2)}}{2|b_2|},$$

where all $|\theta_{ij}|$ are ≤ 1. Proof: since s_j, b_2, L and $|\Lambda|$ are all positive, $|\lambda_j| \leq L/2$ and also $s_j \leq S$ we have

$$|\theta_{ij}| \leq \frac{e^x - 1}{xe^x} \leq 1, \quad \text{where} \ \ x = \frac{LS|\Lambda|}{2b_2} > 0.$$

3.1. **Preliminaries.** This subsection contains some technical results used in the estimates of the interpolation determinant.

Lemma 3.1. *Let K, L, R, S, T be positive integers, put $N = K^2L$ and assume $N \leq RST$, put also*

$$\ell_n = \left\lfloor \frac{n-1}{K^2} \right\rfloor, \quad 1 \leq n \leq N,$$

and $(r_1, \ldots, r_N) \in \{0, 1, \ldots, R-1\}^N$. Suppose that for each $r \in \{0, 1, \ldots, R-1\}$ there are at most ST indices such that $r_j = r$. Then

$$\left| \sum_{n=1}^{N} \ell_n r_n - M_R \right| \leq G_R,$$

where

$$M_R = \left(\frac{L-1}{2} \right) \sum_{n=1}^{N} r_n \qquad \text{and} \qquad G_R = \frac{NLR}{2} \left(\frac{1}{4} - \frac{N}{12RST} \right).$$

Proof. Apply [9], Lemme 4. □

As in [1] or [12] p. 192, for $(k, m) \in \mathbb{N}^2$, we put $\|(k, m)\| = k + m$. And we put

$$\Theta(K_0, I) = \min\big\{ \|(k_1, m_1)\| + \cdots + \|(k_I, m_I)\| \big\},$$

where the minimum is taken over if the I couples (k_1, m_1), $\ldots$, $(k_I, m_I) \in \mathbb{N}^2$ which are pairwise distinct and satisfy m_1, $\ldots$, $m_I \leq K_0$. Then, we have:

Lemma 3.2. *Let K_0, L and I be positive integers with $K_0 \geq 3$, $L \geq 2$ and $I \geq K_0(K_0+1)/2$. Then*

$$\Theta(K_0, I) \geq \left(\frac{I^2}{2(K_0+1)} \right) \left(1 + \frac{(K_0-1)(K_0+1)}{I} - \frac{K_0(K_0+2)(K_0+1)^2}{12I^2} \right).$$

Proof. This is an improvement of the Lemma 1.4 of [1]. We follow more or less the proof of this result.

The argument is elementary: the smallest value for the sum $\|(k_1, m_1)\| + \cdots + \|(k_I, m_I)\|$ is reached when we choose successively, for each integer $n = 0$, 1, $\ldots$ all the points in the domain

$$D_n = \{(k, m) \in \mathbb{N}^2\,;\; m \leq K_0,\; k + m = n\},$$

and stop when the total number of points is I. Moreover,

$$\mathrm{Card}(D_n) = \begin{cases} n+1, & \text{if } \; n \leq K_0, \\ K_0 + 1, & \text{if } \; n \geq K_0. \end{cases}$$

Hence the number of points obtained when n varies between 0 and, say, $A-1$ (with $A \geq K_0$) is

$$\sum_{n=0}^{K_0-1} (n+1) + \sum_{n=K_0}^{A-1} (K_0+1) = \left(A - K_0 + \frac{K_0}{2} \right) (K_0+1) = \left(A - \frac{K_0}{2} \right) (K_0+1).$$

With this notation, the number I of points can be written as

$$I = \left(A - \frac{K_0}{2} \right) (K_0+1) + r, \quad \text{with } \; 0 \leq r \leq K_0,$$

provided that $I \geq K_0(K_0+1)/2$, which is one hypothesis of the Lemma.

Then, the computation of [1] shows that

$$\Theta(K_0, I) \geq \tilde{\Theta}(K_0, I) := \sum_{n=0}^{K_0-1} n(n+1) + \sum_{n=K_0}^{A-1} n(K_0+1) + rA,$$

where

$$\begin{aligned}
\sum_{n=0}^{K_0-1} & n(n+1) + \sum_{n=K_0}^{A-1} n(K_0+1) \\
&= \frac{(K_0-1)K_0(2K_0-1)}{6} + \frac{(K_0-1)K_0}{2} + \frac{K_0+1}{2}\big(A(A-1) - K_0(K_0-1)\big) \\
&= \frac{(K_0-1)K_0(2K_0+2)}{6} + \frac{K_0+1}{2} A(A-1) - \frac{(K_0-1)K_0(K_0+1)}{2} \\
&= \frac{K_0+1}{2}\left(A(A-1) - \frac{1}{3}K_0(K_0-1)\right).
\end{aligned}$$

And we get

$$\Theta(K_0, I) \geq \frac{K_0+1}{2}\left(A(A-1) - \frac{1}{3}K_0(K_0-1)\right) + rA.$$

In terms of I,

$$A = \frac{K_0}{2} + \frac{I-r}{K_0+1}.$$

We have,

$$\frac{\partial\tilde{\Theta}}{\partial r} = \frac{K_0+1}{2}(2A-1)\frac{\partial A}{\partial r} + A + r\frac{\partial A}{\partial r} = -\frac{2A-1}{2} + A - \frac{r}{K_0+1} = \frac{1}{2} - \frac{r}{K_0+1},$$

which shows that the minimum of $\tilde{\Theta}$ is reached either for $r = 0$ or $r = K_0$. It is easy to verify that $\tilde{\Theta}$ takes the same value for $r = 0$ and $r = K_0 + 1$ (which is indeed out of the range of r), this implies that the minimum is reached for $r = 0$. It follows that

$$\begin{aligned}
\frac{2\Theta(K_0, I)}{K_0+1} &\geq \left(\frac{K_0}{2} + \frac{I}{K_0+1}\right)\left(\frac{K_0}{2} + \frac{I}{K_0+1} - 1\right) - \frac{K_0(K_0-1)}{3} \\
&= \frac{K_0^2}{4} + \frac{I^2}{(K_0+1)^2} + \frac{K_0 I}{K_0+1} - \frac{K_0}{2} - \frac{I}{K_0+1} - \frac{K_0^2}{3} + \frac{K_0}{3} \\
&= \frac{I^2}{(K_0+1)^2} + \frac{(K_0-1)I}{K_0+1} - \frac{K_0^2}{12} - \frac{K_0}{6} \\
&= \left(\frac{I}{K_0+1}\right)^2\left(1 + \frac{(K_0-1)(K_0+1)}{I} - \frac{K_0(K_0+2)(K_0+1)^2}{12I^2}\right),
\end{aligned}$$

which proves the lemma. □

The version of Liouville inequality that we use is the same as in [9] (p. 298–99) or in [12] Ex. 3.2, p. 106:

Lemma 3.3. *Let α_1, α_2 and α_3 be non-zero algebraic numbers and a polynomial $f \in \mathbb{Z}[X_1, X_2, X_3]$ such that $f(\alpha_1, \alpha_2, \alpha_3) \neq 0$, then*

$$|f(\alpha_1,\alpha_2,\alpha_3)| \geq |f|^{-D+1}(\alpha_1^*)^{d_1}(\alpha_2^*)^{d_2}(\alpha_3^*)^{d_3} \times \exp\big\{-\mathcal{D}\big(d_1\mathrm{h}(\alpha_1) + d_2\mathrm{h}(\alpha_2) + d_3\mathrm{h}(\alpha_3)\big)\big\},$$

where $\mathcal{D} = [\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3) : \mathbb{Q}] \,\big/\, [\mathbb{R}(\alpha_1,\alpha_2,\alpha_3) : \mathbb{R}]$,

$$d_i = \deg_{X_i} f, \quad i = 1,\ 2,\ 3, \qquad |f| = \max\big\{|f(z_1,z_2,z_3)|\ ;\ |z_i| \leq 1,\ i = 1,2,3\big\},$$

and $\mathrm{h}(\alpha)$ *is the absolute logarithmic height of the algebraic number* α, *and* $\alpha^* = \max\{1, |\alpha|\}$.

Remark. See also [12] Ex. 3.5, p. 108, for a stronger version using projective height.

Lemma 3.4. *Let* $K > 1$ *be an integer, then*

$$\log\left(\prod_{k=1}^{K-1} k!\right)^{\frac{4}{K(K-1)}} \geq 2\log K - 3 + \frac{2\log(2\pi K/e^{3/2})}{K-1} - \frac{2+6\pi^{-2}+\log K}{3K(K-1)}.$$

Proof. This is a consequence of a variant of the proof of Lemme 8 of [9]. □

3.2. **An upper bound for $|\Delta|$.** Let

$$z_j = r_j + s_j\beta_1 - \eta_0 \qquad \text{and} \qquad \zeta_j = t_j + s_j\beta_3 - \zeta_0,$$

so $|z_j| \leq \eta_0$ and $|\zeta_j| \leq \zeta_0$. Since,

$$\binom{r_j b_2' + s_j b_1'}{k_i} = \frac{{b_2'}^{k_i}}{k_i!}{z_j}^{k_i} + \text{terms in } z_j \text{ of degree less than } k_i,$$

and similary for $\binom{t_j b_2'' + s_j b_1''}{m_i}$, using the multilinearity of determinants we obtain the formula

$$\Delta = \det\left(\frac{{b_2'}^{k_i}{b_2''}^{m_i}}{k_i!\, m_i!} z_j^{k_i}\zeta_j^{m_i}{\alpha_1}^{\ell_1 r_j}{\alpha_2}^{\ell_1 s_j}{\alpha_3}^{\ell_1 t_j}\right).$$

Let

$$M_1 = \frac{L-1}{2}\sum_{j=1}^{N} r_j, \qquad M_2 = \frac{L-1}{2}\sum_{j=1}^{N} s_j, \qquad M_3 = \frac{L-1}{2}\sum_{j=1}^{N} t_j.$$

From the two above relations, and the definition of λ_i, it follows that

$$\Delta = {\alpha_1}^{M_1}{\alpha_2}^{M_2}{\alpha_3}^{M_3}\det\left(\frac{{b_2'}^{k_i}{b_2''}^{m_i}}{k_i!\, m_i!} z_j^{k_i}\zeta_j^{m_i}{\alpha_1}^{\lambda_i(r_j+s_j\beta_1)}{\alpha_3}^{\lambda_i(t_j+s_j\beta_3)}\left(1+\Lambda'\theta_{ij}\right)\right).$$

Since $\sum_i \lambda_i = 0$, we deduce that

$$\Delta = {\alpha_1}^{M_1}{\alpha_2}^{M_2}{\alpha_3}^{M_3}\det\left(\frac{{b_2'}^{k_i}{b_2''}^{m_i}}{k_i!\, m_i!} z_j^{k_i}\zeta_j^{m_i}{\alpha_1}^{\lambda_i z_j}{\alpha_3}^{\lambda_i\zeta_j}\left(1+\Lambda'\theta_{ij}\right)\right).$$

Expanding this determinant, we obtain

$$\Delta = {\alpha_1}^{M_1}{\alpha_2}^{M_2}{\alpha_3}^{M_3}\sum_{\mathcal{I}\subseteq\mathcal{N}}(\Lambda')^{N-|\mathcal{I}|}\Delta_{\mathcal{I}},$$

where $\mathcal{N} = \{0, 1, \ldots, N-1\}$ and $\Delta_{\mathcal{I}}$ is the determinant of a certain matrix $\mathcal{M}_{\mathcal{I}}$ defined below. Let

$$\phi_j(z,\zeta) = \frac{{b_2'}^{k_i}{b_2''}^{m_i}}{k_i!\, m_i!} z^{k_i}\zeta^{m_i}\alpha_1^{\lambda_i z}\alpha_3^{\lambda_i\zeta},$$

[where $\alpha_1^{\lambda_i z} = \exp(\lambda_i z\log\alpha_1)$ and similarly for $\alpha_3^{\lambda_i\zeta}$] and

$$\Phi_{\mathcal{I}}(x)_{ij} = \begin{cases}\phi_j(xz_j, x\zeta_j), & \text{if } i\in\mathcal{I},\\ \theta_{ij}\phi_j(xz_j, x\zeta_j), & \text{if } i\notin\mathcal{I}.\end{cases}$$

Then, $\mathcal{M}_{\mathcal{I}} = \big(\Phi_{\mathcal{I}}(1)_{ij}\big)$ and letting $\Psi_{\mathcal{I}}(x) = \det\big(\Phi_{\mathcal{I}}(x)\big)$, gives

$$|\Delta_{\mathcal{I}}| = \big|\det\big(\Phi_{\mathcal{I}}(1)\big)\big| = |\Psi_{\mathcal{I}}(1)|.$$

Now, let

$$J_{\mathcal{I}} = \mathrm{order}(\Psi, 0),$$

the maximum modulus principle implies

$$|\Psi_{\mathcal{I}}(1)| \le \rho^{-J_{\mathcal{I}}} \cdot \max_{|x|=\rho} |\Psi_{\mathcal{I}}(x)|.$$

Since $|z_j| \le \eta_0$ and $|\zeta_j| \le \zeta_0$,

$$\max_{|x|=\rho} \big|\Psi_{\mathcal{I}}(x)\big| \le N! \, \frac{{b'_2}^{\sum k_i} {b''_2}^{\sum m_i}}{\prod k_i! \prod m_i!} \, (\rho\eta_0)^{\sum k_i} (\rho\zeta_0)^{\sum m_i} \times \max_{\sigma \in \mathfrak{S}(\mathcal{N})} \exp\Big\{\rho\Big(\big(\sum \lambda_i z_{\sigma(i)}\big) \log\alpha_1 + \big(\sum \lambda_i \zeta_{\sigma(i)}\big) \log\alpha_2\Big)\Big\}.$$

Put

$$g = \frac{1}{4} - \frac{N}{12RST}, \quad G_1 = \frac{NLR}{2}\, g, \quad G_2 = \frac{NLS}{2}\, g, \quad G_3 = \frac{NLT}{2}\, g.$$

Then, using Lemma 1 and the relation $\sum_{i=0}^{N-1} \lambda_i = 0$, we get

$$\begin{aligned}\sum_{i=0}^{N-1} \lambda_i z_{\sigma(i)} &= \sum_{i=0}^{N-1} \lambda_i (r_i + s_i\beta_1 - \eta_0) = \sum_{i=0}^{N-1} \lambda_i (r_i + s_i\beta_1) \\ &= \sum_{i=0}^{N-1} \left(\ell_i - \frac{L-1}{2}\right) r_{\sigma(i)} + \beta_1 \sum_{i=0}^{N-1} \left(\ell_i - \frac{L-1}{2}\right) s_{\sigma(i)},\end{aligned}$$

and thus

$$\sum_{i=0}^{N-1} \lambda_i z_{\sigma(i)} \le G_1 + \beta_1 G_2.$$

In a similar way,

$$\sum_{i=0}^{N-1} \lambda_i \zeta_{\sigma(i)} \le G_3 + \beta_3 G_2.$$

It follows that (recall that $b_2|\log\alpha_2| = b_1|\log\alpha_1| + b_3|\log\alpha_3| \pm |\Lambda|$)

$$\begin{aligned}\exp\Big\{\rho\Big(\big(\sum \lambda_i z_{\sigma(i)}\big)|\log\alpha_1| + \big(\sum \lambda_i \zeta_{\sigma(i)}\big)|\log\alpha_3|\Big)\Big\} \\ \le \exp\Big\{\rho\Big((G_1 + \beta_1 G_2)|\log\alpha_1| + (G_3 + \beta_3 G_2)|\log\alpha_3|\Big)\Big\} \\ \le \exp\left\{\rho\left(G_1|\log\alpha_1| + G_2\Big(|\log\alpha_2| + \frac{|\Lambda|}{b_2}\Big) + G_3|\log\alpha_3|\right)\right\}.\end{aligned}$$

As in [1], we see that if

$$(*) \qquad \Lambda' < \rho^{-KL}$$

then, for $\rho \ge 2$,

$$\rho G_2 \frac{|\Lambda|}{b_2} \le \rho g \frac{NLS}{b_2} \frac{|\Lambda|}{2} \le \rho K^2 L \frac{\Lambda'}{4} \le \frac{\rho K^2 L}{2\rho^{KL}} \le \frac{K^2 L^2}{10 \cdot 2^{KL}} < 10^{-3},$$

for $K \ge 3$ and $L \ge 5$. Putting these estimates together, we get that condition (*) implies the upper bound

$$|\Delta| \le 1.001\, {\alpha_1}^{M_1 + \rho G_1}\, {\alpha_2}^{M_2 + \rho G_2}\, {\alpha_3}^{M_3 + \rho G_3}\, N! \times 2^N \rho^{\sum(k_i + m_i)} \times \frac{(b'_2\eta_0)^{\sum k_i}}{\prod k_i!} \frac{(b''_2\zeta_0)^{\sum m_i}}{\prod m_i!} \max_{\sigma\in\mathfrak{S}(\mathcal{N})} \frac{|\Lambda'|^{N-|\mathcal{I}|}}{\rho^{J_{\mathcal{I}}}},$$

where
$$J_{\mathcal{I}} = \mathrm{order}(\Psi_{\mathcal{I}}, 0).$$
Under condition (*), we have
$$\frac{|\Lambda'|^{N-|\mathcal{I}|}}{\rho^{J_{\mathcal{I}}}} \leq \rho^{-KL(N-|\mathcal{I}|)-J_{\mathcal{I}}}.$$

If $|\mathcal{I}| \leq 0.5\,N$ then
$$KL(N-|\mathcal{I}|) \geq 0.5\,KLN \geq \frac{NKL}{4}\left(1+\frac{4}{L}+\frac{1}{2K-1}\right)$$
as soon as $K \geq 3$ and $L \geq 5$, conditions that we assume from now on.

If $|\mathcal{I}| \geq 0.5\,N$, then using Lemma 1.3 of [1] or Lemma 6.4 of [12][1], we obtain
$$J_{\mathcal{I}} \geq \Theta(K_0, |\mathcal{I}|), \qquad \text{for} \quad K_0 = 2(K-1)\,.$$
Now, $|\mathcal{I}| \geq 0.5\,K^2L$ implies $|\mathcal{I}| \geq 2.5\,K^2$ and using Lemma 2 we get (with the notation $I = |\mathcal{I}|$)
$$KL(N-I) + J_{\mathcal{I}} \geq KL(N-I) + \frac{I^2}{2(K_0+1)}\left(1+\frac{(K_0-1)(K_0+1)}{I} - \frac{K_0(K_0+2)(K_0+1)^2}{12I^2}\right).$$
It is easy to verify that the right handside is a decreasing function of I in the range $[N/2, N]$, since $L \geq 5$, and we get (recall that $N = K^2L$ and $K_0 = 2K-2$)
$$\begin{aligned}
KL(N-|\mathcal{I}|) + J_{\mathcal{I}} &\geq \frac{N^2}{2(K_0+1)}\left(1+\frac{K_0^2-1}{N} - \frac{K_0(K_0+2)(K_0+1)^2}{12N^2}\right)\\
&= \frac{N^2}{4K}\left(\frac{2K}{K_0+1} + \frac{2K(K_0-1)}{N} - \frac{KK_0(K_0+1)(K_0+2)}{6N^2}\right)\\
&= \frac{N^2}{4K}\left(1+\frac{1}{2K-1}+\frac{2(2K-3)}{KL} - \frac{2(K-1)(2K-1)}{3K^2L^2}\right)\\
&= \frac{N^2}{4K}\left(1+\frac{4}{L}+\frac{1}{2K-1}-\frac{4}{3L^2}-\frac{6}{KL}+\frac{2}{KL^2}-\frac{2}{3K^2L^2}\right)\\
&\geq \frac{N^2}{4K}\left(1+\frac{4}{L}+\frac{1}{2K-1}-\frac{4}{3L^2}-\frac{6}{KL}\right),
\end{aligned}$$
because $L \geq 5$, and this implies, in all cases,
$$KL(N-|\mathcal{I}|) + J_{\mathcal{I}} \geq \frac{N^2}{4K}\left(1+\frac{4}{L}+\frac{1}{2K-1}-\frac{6}{KL}-\frac{4}{3L^2}\right).$$

Thus, gathering all the previous estimates and using the relations
$$\sum_{i=0}^{N-1} k_i = \sum_{i=0}^{N-1} m_i = \frac{(K-1)K}{2}\,KL = \frac{N}{2}\,(K-1),$$
and the definition of b, we obtain the following result (see [2]).

[1] That is: the function of a complex variable x given by
$$\psi(x) = \det\bigl(f_i(xz_j, x\zeta_j)\bigr)_{1\leq i,\, j\leq I}$$
has a zero at $x=0$ of multiplicity at least $\Theta(K_0, I)$, when $f_i(z,\zeta) = z^{k_i}\zeta^{m_i}\phi_i(l_1z + l_2\zeta)$, where ϕ_i is an analytic function in $\mathbb{C}$.

Proposition 3.5. *With the previous notation, if $K \geq 3$, $L \geq 5$ and $\Lambda' \leq \rho^{-KL}$, for some real number $\rho \geq 2$, then*

$$\log|\Delta| \leq \sum_{i=1}^{3} M_i \log|\alpha_i| + \rho \sum_{i=1}^{3} G_i |\log \alpha_i| + \log(N!) + N\log 2 + \frac{N}{2}(K-1)\log b$$
$$- \left(\frac{NKL}{4} + \frac{NKL}{4(2K-1)} - \frac{NK}{3L} - \frac{N}{2}\right)\log\rho + 0.001.$$

3.3. **A lower bound for $|\Delta|$.** Using a Liouville estimate as in Lemma 3.3 above, we get (as in [2]):

Proposition 3.6. *If $\Delta \neq 0$ then*

$$\log|\Delta| \geq -\frac{\mathcal{D}-1}{2} N\log N + \sum_{i=1}^{3}(M_i + G_i)\log|\alpha_i|$$
$$- 2\mathcal{D}\sum_{i=1}^{3} G_i \mathrm{h}(\alpha_i) - \frac{\mathcal{D}-1}{2}(K-1)N\log b.$$

Proof. We have $\Delta = P(\alpha_1, \alpha_2, \alpha_3)$ where $P \in \mathbb{Z}[X_1, X_2, X_3]$ is given by

$$P(X_1, X_2, X_3) = \sum_{\sigma\in\mathfrak{S}_N} \mathrm{sg}(\sigma) \cdot \prod_{i=1}^{N} \binom{r_{\sigma(i)}b_2' + s_{\sigma(i)}b_1'}{k_i}\binom{t_{\sigma(i)}b_2'' + s_{\sigma(i)}b_3''}{m_i} X_1^{n_{r,\sigma}} X_2^{n_{s,\sigma}} X_1^{n_{t,\sigma}},$$

and where

$$n_{r,\sigma} = \sum_{i=1}^{N} \ell_i r_{\sigma(i)}, \quad n_{s,\sigma} = \sum_{i=1}^{N} \ell_i s_{\sigma(i)}, \quad n_{t,\sigma} = \sum_{i=1}^{N} \ell_i t_{\sigma(i)}.$$

By Lemma 1,

$$\left|\deg_{X_i} P - M_i\right| \leq G_i, \quad i = 1, 2, 3.$$

Let

$$V_i = \lfloor M_i + G_i \rfloor, \qquad U_i = \lceil M_i - G_i \rceil, \quad i = 1, 2, 3,$$

then

$$\Delta = {\alpha_1}^{V_1}{\alpha_2}^{V_2}{\alpha_3}^{V_3}\, \tilde{P}(\alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1}),$$

where

$$\deg_{X_i} \tilde{P} \leq V_i - U_i, \quad i = 1, 2, 3.$$

By our Liouville estimate

$$\log\left|\tilde{P}(\alpha_1^{-1}, \alpha_2^{-1}, \alpha_3^{-1})\right| \geq -(\mathcal{D}-1)\log|\tilde{P}| - \mathcal{D}\sum_{i=1}^{3}(V_i - U_i)\,\mathrm{h}(\alpha_i).$$

Now we have to find an upper bound for $|\tilde{P}|$ (or for $|P|$, which is equal to $|\tilde{P}|$). By the multilinearity of the determinant, for all η, $\zeta \in \mathbb{C}$,

$$P(z_1, z_2, z_3) = \det\left(\frac{(r_j b_2' + s_j b_1' - \eta)^{k_i}}{k_i!}\,\frac{(t_j b_2'' + s_j b_3'' - \zeta)^{m_i}}{m_i!} \cdot {z_1}^{\ell_i r_j} \cdot {z_2}^{\ell_i s_j} \cdot {z_3}^{\ell_i t_j}\right).$$

Choose

$$\eta = \frac{(R-1)b_2' + (S-1)b_1'}{2}, \quad \zeta = \frac{(T-1)b_2'' + (S-1)b_3''}{2}.$$

Notice that, for $1 \leq j \leq N$,

$$|r_j b_2' + s_j b_1' - \eta|^{k_i} \leq \left(\frac{(R-1)b_2 + (S-1)b_1}{2d_1}\right)^{k_i}, \quad |t_j b_2'' + s_j b_3'' - \zeta|^{k_i} \leq \left(\frac{(T-1)b_2 + (S-1)b_3}{2d_3}\right)^{m_i}$$

and that
$$\sum_{i=0}^{N-1} k_i = \sum_{i=0}^{N-1} m_i = \frac{(K-1)K}{2}\, KL = \frac{N}{2}\,(K-1),$$
then Hadamard's inequality implies
$$|P| \le N^{N/2} \left(\frac{(R-1)b_2+(S-1)b_1}{2d_1}\right)^{(K-1)N/2} \left(\frac{(T-1)b_2+(S-1)b_3}{2d_3}\right)^{(K-1)N/2} \times \left(\prod_{i=0}^{K-1} k_i!\right)^{-1} \left(\prod_{i=0}^{K-1} m_i!\right)^{-1}.$$
Recall that
$$b = (b_2'\eta_0)(b_2''\zeta_0)\left(\prod_{k=1}^{K-1} k!\right)^{-\frac{4}{K(K-1)}}, \quad \text{where } \eta_0 = \frac{R-1}{2} + \beta_1\frac{S-1}{2},\ \zeta_0 = \frac{T-1}{2} + \beta_3\frac{S-1}{2}.$$
Thus we get,
$$|P| \le N^{N/2}\, b^{(K-1)N/2}.$$
Collecting all the above estimates, we find
$$\log|\Delta| \ge -(\mathcal{D}-1)\left(\log\left(N^{N/2}\right) + \frac{(K-1)N}{2}\,\log b\right) - \mathcal{D}\sum_{i=1}^{3}(V_i-U_i)\,\mathrm{h}(\alpha_i) + \sum_{i=1}^{3} V_i\log|\alpha_i|.$$
The inequalities $\mathcal{D}\mathrm{h}(\alpha_i) \ge \log|\alpha_i| \ge 0$ imply
$$V_i\log|\alpha_i| - \mathcal{D}(V_i-U_i)\,\mathrm{h}(\alpha_i) \ge (M_i+G_i)\log|\alpha_i| - 2\mathcal{D}G_i\mathrm{h}(\alpha_i)$$
and the result follows. □

3.4. **Synthesis.** Under the hypotheses of the previous Propositions we get
$$\begin{aligned}-\frac{\mathcal{D}-1}{2}\,N\log N &+ \sum_{i=1}^{3}(M_i+G_i)\log|\alpha_i| - 2\mathcal{D}\sum_{i=1}^{3} G_i\mathrm{h}(\alpha_i) - \frac{\mathcal{D}-1}{2}(K-1)\,N\log b\\ &\le \sum_{i=1}^{3} M_i\log|\alpha_i| + \rho\sum_{i=1}^{3} G_i|\log\alpha_i| + \log(N!) + N\log 2 + \frac{N}{2}\,(K-1)\log b\\ &\quad - \left(\frac{NKL}{4} + \frac{NKL}{4(2K-1)} - \frac{NK}{3L} - \frac{N}{2}\right)\log\rho + 0.001.\end{aligned}$$
Or, after some simplification,
$$\begin{aligned}-\frac{\mathcal{D}-1}{2}\,N\log N &\le \sum_{i=1}^{3} G_i\big(\rho|\log\alpha_i| - \log|\alpha_i| + 2\mathcal{D}\mathrm{h}(\alpha_i)\big) + \log(N!) + N\log 2\\ &\quad + \frac{K-1}{2}\,\mathcal{D}N\log b - \left(\frac{NKL}{4} + \frac{NKL}{4(2K-1)} - \frac{KN}{3L} - \frac{N}{2}\right)\log\rho + 0.001.\end{aligned}$$

This result implies (divide by $N/2$ and use $N! < 0.96\,N(N/e)^N$, true for $N > 7$) the following proposition (already appearing in [2]):

Proposition 3.7. *With the previous notation, if $K \geq 3$, $L \geq 5$, $\rho \geq 2$, and if $\Delta \neq 0$ then*

$$\Lambda' > \rho^{-KL}$$

provided that

$$\Big(\frac{KL}{2} + \frac{L}{4} - 1 - \frac{2K}{3L}\Big)\log\rho \geq (\mathcal{D}+1)\log N + gL(a_1R + a_2S + a_3T) + \mathcal{D}(K-1)\,\log b - 2\log(e/2),$$

where the a_i are positive real numbers which satisfy

$$a_i \geq \rho|\log\alpha_i| - \log|\alpha_i| + 2\mathcal{D}\,\mathrm{h}(\alpha_i), \qquad i = 1,\ 2,\ 3.$$

3.5. **A zero-lemma.** To conclude we need to find conditions under which one of our determinants Δ is non-zero, a so-called *zero-lemma*. We use a zero-lemma due to M. Laurent [8] which is already used in [3] and improves [4] and provides an important improvement on the zero-lemma of [1]:

Proposition 3.8 (M. Laurent)**.** *Suppose that K, L are positive integers and that Σ_1, Σ_2 and Σ_3 are finite subsets of $\mathbb{C}^2 \times \mathbb{C}^*$ containing the origin and such that*

(i)
$$\begin{cases} \mathrm{Card}\{\lambda x_1 + \mu x_2 \,:\, (x_1,x_2,y) \in \Sigma_1\} & > K, \quad \forall(\lambda,\mu) \neq (0,0),\\ \mathrm{Card}\{y \,:\, (x_1,x_2,y) \in \Sigma_1\} & > L, \end{cases}$$

and

(ii)
$$\begin{cases} \mathrm{Card}\{(\lambda x_1 + \mu x_2, y) \,:\, (x_1,x_2,y) \in \Sigma_2\} & > 2KL, \quad \forall(\lambda,\mu) \neq (0,0),\\ \mathrm{Card}\{(x_1,x_2) \,:\, (x_1,x_2,y) \in \Sigma_2\} & > 2K^2, \end{cases}$$

and also that

(iii)
$$\mathrm{Card}\,\Sigma_3 > 6K^2L.$$

Then, the only polynomial $P \in \mathbb{C}[X_1, X_2, Y]$ with $\deg_{X_i} P \leq K$ for $i = 1,\ 2$, and $\deg_Y P \leq L$ which is zero on the set $\Sigma_1 + \Sigma_2 + \Sigma_3$, is the zero polynomial.

We now study the above conditions in detail. For $j = 1,\ 2,\ 3$, we shall consider finite sets Σ_j defined by

$$\Sigma_j = \big\{(r + s\beta_1,\, t + s\beta_3,\, \alpha_1^r\alpha_2^s\alpha_3^t) \,:\, 0 \leq r \leq R_j,\ 0 \leq s \leq S_j,\ 0 \leq t \leq T_j\big\},$$

where R_j, S_j and T_j are positive integers and where

$$\beta_1 = \frac{b_1}{b_2} = \frac{b_1'}{b_2'}, \qquad \beta_3 = \frac{b_3}{b_2} = \frac{b_3''}{b_2''}.$$

Of course, this choice corresponds to the entries of the arithmetical matrices introduced previously.

We have to consider the multiplicative group $\mathcal{G}$ generated by the three algebraic numbers α_1, α_2 and α_3.

Concerning the above group, the following elementary lemma is important.

Lemma 3.9. *Suppose that α_1, α_2 and α_3 are non-zero complex numbers. Let b_1, b_2 and b_3 be non-zero rational integers. Let $\log\alpha_j$ be any determination of the logarithm of α_j for $j = 1$, 2, 3 and put*

$$\Lambda = b_2\log\alpha_2 - b_1\log\alpha_1 - b_3\log\alpha_3.$$

Let

$$\beta_1 = b_1/b_2, \quad \beta_3 = b_3/b_2.$$

Then the following conditions are equivalent:

(a) The map

$$\psi : \mathbb{Z}^3 \to \mathbb{C}^3, \quad (r, s, t) \mapsto (r + \beta_1 s, t + \beta_3 s, \alpha_1^r \alpha_2^s \alpha_3^t)$$

is not one-to-one (not injective).

(b) There exists some positive integer m such that

$$\alpha_2{}^{mb_2} = \alpha_1{}^{mb_1} \alpha_3{}^{mb_3}.$$

(c) The number Λ belongs to the set $i\pi\mathbb{Q}$.

Proof. Clearly, without loss of generality, we may assume that $\gcd(b_1, b_2, b_3) = 1$. Then we put

$$d_1 = \gcd(b_1, b_2),\ b_1 = d_1 b_1',\ b_2 = d_1 b_2', \quad d_3 = \gcd(b_3, b_2),\ b_3 = d_3 b_3'',\ b_2 = d_3 b_2''.$$

Since b_1, b_2, b_3 are coprime, we have $\gcd(d_1, d_3) = 1$, thus

$$b_2 = d_1 d_3 \tilde{b}_2 \text{ (say)}, \quad b_2' = d_3 \tilde{b}_2, \quad b_2'' = d_1 \tilde{b}_2.$$

After these preliminaries, we prove the implication $(a) \Rightarrow (b)$. Suppose that the map ψ is not injective. Then there exist rational integers r, s, t, not all zero, such that

$$\psi(r, s, t) = (0, 0, 1).$$

That is,

$$r + s\beta_1 = 0, \quad t + s\beta_3 = 0, \quad \alpha_1^r \alpha_2^s \alpha_3^t = 1.$$

The first relation implies $r = -kb_1'$ and $s = kb_2' = kd_3\tilde{b}_2$, for some rational integer k. The second relation implies $t = -lb_1'$ and $s = lb_2'' = ld_1\tilde{b}_2$, for some rational integer l. In particular, $kd_3 = ld_1$, hence there exists $m \in \mathbb{Z}$ such that $k = md_1$ and $l = md_3$. Thus

$$r = -mb_1, \quad s = mb_2, \quad t = -mb_3.$$

Clearly $m \neq 0$, and the third relation gives

$$\alpha_2{}^{mb_2} = \alpha_1{}^{mb_1} \alpha_3{}^{mb_3},$$

as wanted.

Clearly, (b) implies (c).

To show that (c) implies (a), we suppose that (c) holds, *i.e.* that $m\Lambda$ belongs to $2i\pi\mathbb{Z}$ for some positive rational integer m. Then it is clear that $\psi(-mb_1, mb_2, -mb_3) = (0, 0, 1)$, proving that the map ψ is not injective. □

Corollary 3.10. *If α_1, α_2 and α_3 are non-zero complex numbers such that (for example) α_1 and α_2 are multiplicatively independent and $\alpha_3 \neq 1$ is a root of unity, and let $\log \alpha_j$ be any determination of the logarithm of α_j for $j = 1, 2, 3$, then the numbers $\log \alpha_1$, $\log \alpha_2$ and $\log \alpha_3$ are linearly independent over the rationals. Indeed, if b_1, b_2 and b_3 are non-zero rational integers then the number $b_1 \log \alpha_1 + b_2 \log \alpha_2 + b_3 \log \alpha_3$ does not belong to the set $i\pi\mathbb{Q}$.*

Proof. Suppose that

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1 - b_3 \log \alpha_3 = 0$$

where b_1, b_2 and b_3 are rational integers not all equal to zero. Then $\alpha_2^{b_2} = \alpha_1^{b_1} \alpha_3^{b_3}$. Assume that $\alpha_3^d = 1$ with $d > 0$, then $\alpha_2{}^{db_2} = \alpha_1{}^{db_1}$, which implies $b_1 = b_2 = 0$ since α_1 and α_2 are multiplicatively independent. Thus $b_3 \neq 0$ and $\Lambda = b_3 \log \alpha_3 \neq 0$, contradiction. This proves the first claim. The second claim is an obvious consequence of the first one. □

We also assume that

$$(\mathbf{I}_1) \qquad \operatorname{Card}\bigl\{(x_1,x_2)\ :\ (x_1,x_2,y)\in\Sigma_1\bigr\} = (R_1+1)(S_1+1)(T_1+1),$$

and

$$(\mathbf{I}_2) \qquad \operatorname{Card}\bigl\{(x_1,x_2)\ :\ (x_1,x_2,y)\in\Sigma_2\bigr\} = (R_2+1)(S_2+1)(T_2+1).$$

Concerning the conditions ($\mathbf{I}_1$) and ($\mathbf{I}_2$), the following very elementary lemma is useful.

Lemma 3.11. *Suppose that α_1, α_2 and α_3 are non-zero complex numbers and that b_1, b_2 and b_3 are positive rational integers which are coprime. Let R, S and T be positive integers and consider the set*

$$\tilde\Sigma = \{(r+sb_1/b_2, t+sb_3/b_2)\ :\ 0\le r\le R,\ 0\le s\le S,\ 0\le t\le T\}.$$

Then

$$\operatorname{Card}\tilde\Sigma = (R+1)(S+1)(T+1)$$

unless

$$b_1\le R \quad\textit{and}\quad b_2\le S \quad\textit{and}\quad b_3\le T.$$

Proof. Let

$$\beta_1 = b_1/b_2,\quad \beta_3 = b_3/b_2.$$

As above, we put

$$d_1=\gcd(b_1,b_2),\ b_1=d_1b_1',\ b_2=d_1b_2',\quad d_3=\gcd(b_3,b_2),\ b_3=d_3b_3'',\ b_2=d_3b_2''.$$

Since b_1, b_2, b_3 are coprime, we have $\gcd(d_1,d_3)=1$, thus

$$b_2 = d_1d_3\tilde b_2 \text{ (say)},\quad b_2' = d_3\tilde b_2,\quad b_2''=d_1\tilde b_2.$$

After these preliminaries, we prove the result. Suppose that the map

$$\psi:\mathbb{Z}^3\to\mathbb{C}^2,\quad (r,s,t)\mapsto(r+\beta_1 s, t+\beta_3 s)$$

is not injective. Then there exist two different triples of rational integers (r,s,t) and (r',s',t'), with $0\le r,\ r'\le R$, $0\le s,\ s'\le S$ and $0\le t,\ t'\le T$ such that

$$\psi(r,s,t)=\psi(r',s',t').$$

That is,

$$(r-r')+(s-s')\beta_1 = 0 \quad\text{and}\quad (t-t')+(s-s')\beta_3=0.$$

The first relation implies $r-r'=-kb_1'$ and $s-s'=kb_2'=kd_3\tilde b_2$, for some rational integer k. The second relation implies $t-t'=-lb_1''$ and $s-s'=lb_2''=ld_1\tilde b_2$, for some rational integer l. In particular, $kd_3=ld_1$, hence there exists $m\in\mathbb{Z}$ such that $k=md_1$ and $l=md_3$. Thus

$$r-r'=-mb_1,\quad s-s'=mb_2,\quad t-t'=-mb_3$$

and the conclusion follows since clearly m is non-zero. □

Because of the Lemma 3.9, we see that

$$\Lambda\notin i\pi\mathbb{Q} \implies \operatorname{Card}\Sigma_j = (R_j+1)(S_j+1)(T_j+1),\quad j=1,\,2,\,3.$$

The conditions of the zero-lemma, Proposition 3.8, are the following:

(i) The first condition is divided into two subconditions, the first subcondition is

$$\text{(i.1)}\qquad \operatorname{Card}\bigl\{\lambda x_1+\mu x_2\ :\ (x_1,x_2,y)\in\Sigma_1\bigr\} > K,\quad \forall(\lambda,\mu)\ne(0,0).$$

This is the most technical of the above conditions, we study it in detail later.

The second subcondition is

$$\text{(i.2)} \qquad \operatorname{Card}\big\{y \,:\, (x_1,x_2,y)\in\Sigma_1\big\} > L.$$

(ii) The second condition of the zero-lemma is also divided into two subconditions, the first being

$$\text{(ii.1)} \qquad \operatorname{Card}\big\{(\lambda x_1+\mu x_2, y) \,:\, (x_1,x_2,y)\in\Sigma_2\big\} > 2KL, \quad \forall(\lambda,\mu)\neq(0,0).$$

We replace it by the stronger condition

$$\operatorname{Card}\big\{y \,:\, (x_1,x_2,y)\in\Sigma_2\big\} > 2KL.$$

The second subcondition of condition (ii) of the zero-lemma is

$$\text{(ii.2)} \qquad \operatorname{Card}\big\{(x_1,x_2) \,:\, (x_1,x_2,y)\in\Sigma_2\big\} > 2K^2.$$

By ($\mathbf{I}_2$) this condition is equivalent to

$$\text{(C.ii.2)} \qquad (R_2+1)(S_2+1)(T_2+1) > 2K^2.$$

(iii) There is just one condition, namely

$$\operatorname{Card}\Sigma_3 > 6KL^2.$$

When Λ does not belong to the set $i\pi\mathbb{Q}$, this is equivalent to

$$\text{(C.iii)} \qquad (R_3+1)(S_3+1)(T_3+1) > 6K^2L.$$

Now we have 'translated' all the conditions of Proposition 3.8, except the subcondition (i.1). We come back to this situation in the following Lemma which brings some extra information to Proposition 3.1.1 of [1], or also [12] Ex 6.4, p. 184.

Lemma 3.12. *Let A, B and C be non-zero rational integers with* $\gcd(A,B,C)=1$ *and let D be an integer. Define*

$$\Pi = \big\{(x,y,z)\in\mathbb{C}^3 \,:\, Ax+By+Cz=D\big\}$$

and consider the set

$$\Sigma = \big\{(x,y,z)\in\mathbb{Z}^3 \,:\, 0\le x\le X,\ 0\le y\le Y,\ 0\le z\le Z\big\},$$

where X, Y and Z are positive integers. Let

$$M = \operatorname{Card}\big\{(x,y,z)\in\Sigma \,:\, Ax+By+Cz=D\big\}.$$

Then

$$M \le \left(1+\left\lfloor\frac{X}{\alpha}\right\rfloor\right)\left(1+\left\lfloor\frac{Y}{|C|/\alpha}\right\rfloor\right) \quad \text{and} \quad M \le \left(1+\left\lfloor\frac{X}{\alpha}\right\rfloor\right)\left(1+\left\lfloor\frac{Z}{|B|/\alpha}\right\rfloor\right),$$

where

$$\alpha = \gcd(B,C).$$

If we suppose that

$$M \ge \max\big\{X+Y+1,\ Y+Z+1,\ Z+X+1\big\}$$

then

$$|A| \le \frac{(Y+1)(Z+1)}{M-\max\{Y,Z\}}, \quad |B| \le \frac{(X+1)(Z+1)}{M-\max\{X,Z\}}, \quad |C| \le \frac{(X+1)(Y+1)}{M-\max\{X,Y\}}.$$

Proof. If the image (by the map $(x,y,z) \mapsto Ax+By+Cz$) of a point $(x,y,z) \in \mathbb{Z}^3$ belongs to the plane Π then

$$Ax \equiv D \pmod{\alpha},$$

where A and α are coprime since $\gcd(A,B,C)=1$. This shows that the number of such x which satisfy $0 \le x \le X$ is

$$\le 1 + \left\lfloor \frac{X}{\alpha} \right\rfloor.$$

To simplify the notation we suppose for a while that A, B and C are positive. Let now x be fixed, with $0 \le x \le X$, and such that the images of two points (x,y,z) and (x,y',z') belong to Π. Then

$$B(y'-y) = C(z-z'),$$

where we suppose (as we may) that y is minimal (then $y' > y$). Hence there exists $k \in \mathbb{N}$ such that

$$y'-y = k(C/\alpha) \quad \text{and} \quad z-z' = k(B/\alpha).$$

It follows that, for x fixed, the number of $(x,y,z) \in \Sigma$ whose image belong to Π is

$$\le 1 + \left\lfloor \frac{Y}{C/\alpha} \right\rfloor.$$

Hence

$$M \le \left(1 + \left\lfloor \frac{X}{\alpha} \right\rfloor\right)\left(1 + \left\lfloor \frac{Y}{C/\alpha} \right\rfloor\right),$$

which proves the first inequality of the Lemma. The proof of the second one is the same (looking at the coordinate z).

For $\xi \ge 1$ put

$$f(\xi) = \left(1 + \frac{X}{\xi}\right)\left(1 + \frac{\xi Y}{C}\right),$$

then

$$M \le f(\alpha).$$

Suppose now

$$M > \max\bigl\{X+1,\, Y+1,\, Z+1\bigr\}.$$

Put

$$\alpha_1 = \max\{1, C/Y\}, \quad \alpha_2 = \min\{C, X\}.$$

- If $C > Y$ and $1 \le \alpha < C/Y$ then we get $M \le X+1$, contradiction, thus
$$C > Y \implies \alpha \ge \alpha_1 \text{ and } f(\alpha_1) = 2\left(1 + \frac{XY}{C}\right).$$
- If $C > X$ and $\alpha > X$ then we get $M \le Y+1$, contradiction, thus
$$C > X \implies \alpha \le \alpha_2 \text{ and } f(\alpha_2) = 2\left(1 + \frac{XY}{C}\right).$$
- If $C \le \min\{X,Y\}$ then $\alpha_1 = 1$ and $\alpha_2 = C$ and
$$f(\alpha_1) = (X+1)\left(1 + \frac{Y}{C}\right), \quad f(\alpha_2) = \left(1 + \frac{X}{C}\right)(Y+1).$$

It is easy to check that f'' is positive and, from the previous study, it follows that

$$M \leq \max\bigl\{f(\alpha_1), f(\alpha_2)\bigr\}.$$

Considering the different cases $C > \max\{X, Y\}$, $X \leq C < Y$, $Y \leq C < X$ and $C \leq \min\{X, Y\}$ we get always

$$M \leq \max\left\{(X+1)\left(1+\tfrac{Y}{C}\right), \left(1+\tfrac{X}{C}\right)(Y+1)\right\} = \begin{cases} (X+1)\left(1+\tfrac{Y}{C}\right), & \text{if } X \geq Y, \\ \left(1+\tfrac{X}{C}\right)(Y+1), & \text{otherwise.} \end{cases}$$

If $X \geq Y$ then

$$M \leq (X+1)\left(1+\frac{Y}{C}\right),$$

which implies

$$M-(X+1) \leq \frac{Y(X+1)}{C}, \qquad \text{hence } \; C \leq \frac{Y(X+1)}{M-(X+1)},$$

and the hypothesis $M \geq X+Y+1$ leads to

$$C \leq \frac{(X+1)(Y+1)}{M-X},$$

otherwise (*i.e.*, if $X < Y$) we get

$$C \leq \frac{(X+1)(Y+1)}{M-Y}.$$

Finally, we always have

$$|C| \leq \frac{(X+1)(Y+1)}{M-\max\{X,Y\}}.$$

In the same way, considering now the z–coordinate, we get

$$|B| \leq \frac{(X+1)(Z+1)}{M-\max\{X,Z\}}.$$

Then, considering y fixed, a similar argument gives

$$|A| \leq \frac{(Y+1)(Z+1)}{M-\max\{Y,Z\}}.$$

□

Corollary 3.13. *Let B and C be non-zero rational integers with* $\gcd(B,C)=1$ *and let D be an integer. Define the plane Π (with $A=0$), i.e.*

$$\Pi = \bigl\{(x,y,z) \in \mathbb{C}^3 \,:\, By + Cz = D\bigr\},$$

and Σ and M as in the above Lemma. Then

$$M \leq (X+1)\left(1+\left\lfloor\frac{Y}{|C|}\right\rfloor\right) \quad \textit{and} \quad M \leq (X+1)\left(1+\left\lfloor\frac{Z}{|B|}\right\rfloor\right).$$

Moreover, if we suppose that

$$M \geq \max\{X+Y+1, X+Z+1\}$$

then

$$|B| \leq \frac{(X+1)(Z+1)}{M-X}, \qquad |C| \leq \frac{(X+1)(Y+1)}{M-X}.$$

Proof. The proof is similar to that of the Lemma, but simpler. We omit the details. □

Lemma 3.14. *Let R_1, S_1 and T_1 be positive integers and consider the set*

$$\tilde{\Sigma}_1 = \bigl\{(x_1, x_2) = (r + s\beta_1, t + s\beta_3) \,:\, 0 \le r \le R_1,\ 0 \le s \le S_1,\ 0 \le t \le T_1\bigr\},$$

where $\beta_1 = b_1/b_2$ and $\beta_3 = b_3/b_2$ with b_1, b_2 and b_3 coprime non-zero rational integers, and assume that

$$\mathrm{Card}\,\tilde{\Sigma}_1 = (R_1+1)(S_1+1)(T_1+1).$$

Put

$$\mathcal{V} = \bigl((R_1+1)(S_1+1)(T_1+1)\bigr)^{1/2}.$$

Let $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$ and let c be a complex number. Let χ be a positive real number. Then, for any c, the number M of elements $(x_1,x_2) \in \tilde{\Sigma}_1$ such that

$$\lambda x_1 + \mu x_2 = c$$

satisfies

$$M \le \max\Bigl\{R_1+S_1+1,\ S_1+T_1+1,\ R_1+T_1+1,\ \chi\mathcal{V}\Bigr\} =: \mathcal{M}, \tag{1}$$

— *except if,* **either** *there exist two non-zero rational integers r_0 and s_0 such that*

$$r_0 b_2 = s_0 b_1$$

with

$$|r_0| \le \frac{(R_1+1)(T_1+1)}{\mathcal{M}-T_1} \qquad \textit{and} \qquad |s_0| \le \frac{(S_1+1)(T_1+1)}{\mathcal{M}-T_1},$$

or *there exist rational integers r_1, s_1, t_1 and t_2, with $r_1 s_1 \ne 0$, such that*

$$(t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2, \qquad \gcd(r_1,t_1) = \gcd(s_1,t_2) = 1,$$

which also satisfy

$$0 < |r_1 s_1| \le \delta \cdot \frac{(R_1+1)(S_1+1)}{\mathcal{M} - \max\{R_1,S_1\}}, \qquad |s_1 t_1| \le \delta \cdot \frac{(S_1+1)(T_1+1)}{\mathcal{M} - \max\{S_1,T_1\}},$$

$$\textit{and} \qquad |r_1 t_2| \le \delta \cdot \frac{(R_1+1)(T_1+1)}{\mathcal{M} - \max\{R_1,T_1\}},$$

where

$$\delta = \gcd(r_1,s_1).$$

Moreover when $t_1 = 0$ we can take $r_1 = 1$, and when $t_2 = 0$ we can take $s_1 = 1$.

If the previous upper bound (1) for M holds then, for all $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, we have

$$\mathrm{Card}\bigl\{\lambda x_1 + \mu x_2 \,:\, (x_1,x_2) \in \tilde{\Sigma}_1\bigr\} \ge \frac{(R_1+1)(S_1+1)(T_1+1)}{\max\Bigl\{R_1+S_1+1,\ S_1+T_1+1,\ R_1+T_1+1,\ \chi\mathcal{V}\Bigr\}}.$$

Proof. Let

$$E_1 = \bigl\{(r,s,t) \in \mathbb{Z}^3 \,:\, 0 \le r \le R_1,\ 0 \le s \le S_1,\ 0 \le t \le T_1\bigr\}.$$

Recall the notation

$$x_1 = r + \beta_1 s, \quad x_2 = t + \beta_3 s, \quad \beta_1 = \frac{b_1}{b_2}, \quad \beta_3 = \frac{b_3}{b_2}.$$

For $(\lambda,\mu) \in \mathbb{C}^2 \setminus \{(0,0)\}$, we consider the cardinality

$$N = \mathrm{Card}\bigl\{\lambda x_1 + \mu x_2 \,:\, (x_1,x_2) \in \tilde{\Sigma}_1\bigr\}.$$

We put

$$M = \max_{c \in \mathbb{C}} \mathrm{Card}\bigl\{(x_1,x_2) \in \tilde{\Sigma}_1 \,:\, \lambda x_1 + \mu x_2 = c\bigr\}$$

and

$$\Pi_c = \{(z_1, z_2) \in \mathbb{C}^2 \,:\, \lambda z_1 + \mu z_2 = c\}.$$

We clearly have

$$N \geq \text{Card}\, \tilde{\Sigma}_1 / M,$$

so that the last claim of the Lemma is proved and we may also suppose that (1) does not hold.

Consider a complex number c such that the number of points $(x_1, x_2) \in \tilde{\Sigma}_1$ for which $\lambda x_1 + \mu x_2$ belongs to Π_c is maximal (and so equal to M). We distinguish the following cases.

- If $\mu = 0$: Apply the previous Corollary with $(x, y, z) \mapsto (r, s, t)$, $(X, Y, Z) \mapsto (R_1, S_1, T_1)$, $(A, B, C) \mapsto (b_2/d_1, b_1/d_1, 0)$, where

$$d_1 = \gcd(b_1, b_2),$$

  and $(b_2/d_1, b_1/d_1) \mapsto (r_0, s_0)$. Then we get the wanted assertion (the 'either' case).

  Now we assume $\mu \neq 0$ and, to simplify the notation we take $\mu = 1$.

- If $\lambda = 0$: Now, we apply the previous Corollary with $(A, B, C) \mapsto (0, b_3/d_3, b_2/d_3)$, where

$$d_3 = \gcd(b_2, b_3),$$

  and $(b_2/d_3, b_3/d_3) \mapsto (s_1, t_2)$. Then we get the asserted relation

$$(t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2$$

  with $r_1 = 1$ and $t_1 = 0$, and the asserted bounds on r_1, s_1, t_1 and t_2.

- If $\lambda b_1 + b_3 = 0$: In this case $(A, B, C) \mapsto (-b_3/d, 0, b_1/d)$, where

$$d = \gcd(b_1, b_3),$$

  and $(b_1/d, -b_3/d) \mapsto (r_1, t_1)$. Then we get the asserted relation

$$(t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2$$

  with $s_1 = 1$ and $t_2 = 0$, and the asserted bounds on r_1, s_1, t_1 and t_2.

- If $\lambda\mu(\lambda b_1 + b_3) \neq 0$: Since $M > S_1 + 1$, there exist two distinct triples (r, s_0, t) and $(r', s_0, t') \in E$ such that

$$\lambda(r + \beta_1 s_0) + (t + \beta_3 s_0) = \lambda(r' + \beta_1 s_0) + (t' + \beta_3 s_0),$$

  which gives $\lambda(r' - r) = t - t'$, where we suppose (as we may) that r is minimal (then $r' > r$) and also that $r' - r > 0$ is minimal. Put $r_1 = r' - r$ and $t_1 = t - t'$, then

$$\lambda = t_1 / r_1.$$

  Since $M > R_1 + 1$, there exist two distinct triples (r_0, s, t) and $(r_0, s', t') \in E$ such that

$$t_1 b_2 r_0 + (t_1 b_1 + r_1 b_3) s + r_1 b_2 t = t_1 b_2 r_0 + (t_1 b_1 + r_1 b_3) s' + r_1 b_2 t',$$

  which gives now a relation of the form

$$(t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2, \quad \text{with } t_1 t_2 \neq 0,$$

  for which we may suppose that

$$\gcd(r_1, t_1) = \gcd(s_1, t_2) = 1.$$

  Now we are ready to apply the above Lemma 3.12 with

$$(A, B, C) \mapsto (t_1 s_1/\delta, r_1 t_2/\delta, r_1 s_1/\delta),$$

  where

$$\delta = \gcd(t_1 s_1, r_1 t_2, r_1 s_1),$$

  and we get the conclusion, except that we have to prove that $\delta = \gcd(r_1, s_1)$.

Suppose that p is a prime divisor of δ, then $p \mid r_1 s_1$. If $p \nmid r_1$ then $p \mid s_1$ and $p \nmid t_1$, thus $p \nmid r_1 t_1$: contradiction. If $p \nmid s_1$ then $p \mid r_1$ and $p \nmid t_1$, thus $p \nmid s_1 t_2$: contradiction. Hence, $p \mid r_1$ and $p \mid s_1$ and $p \nmid t_1 t_2$. And now it is easy to conclude that

$$\delta = \gcd(r_1, s_1).$$

This ends the proof of the Lemma. □

Remark. Before leaving this Subsection, it is important to notice that the conclusion of the zero-lemma, namely '... the only polynomial $P \in \mathbb{C}[X_1, X_2, Y]$ with $\deg_{\mathbf{X_i}} P \le K$ for $i = 1, 2$, and $\deg_Y P \le L$ which is zero on the set $\Sigma_1+\Sigma_2+\Sigma_3$, is the zero polynomial' applied to the interpolation matrix considered above implies that this interpolation matrix is of maximal rank, which means that there exists a determinant Δ as above which is nonzero.

3.6. **Statement of the main result: a lower bound for the linear form.** If we gather the results obtained in the previous subsections, we get the following theorem.

Theorem 2. *We consider three non-zero algebraic numbers α_1, α_2 and α_3, which are either all real and > 1 or all complex of modulus one and all $\neq 1$. We also consider three positive rational integers b_1, b_2, b_3 with $\gcd(b_1, b_2, b_3) = 1$, and the linear form*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1 - b_3 \log \alpha_3,$$

where the logarithms of the α_i are arbitrary determinations of the logarithm, but which are all real or all purely imaginary. We assume that

$$0 < |\Lambda| < 2\pi/w,$$

where w is the maximal order of a root of unity belonging to the number field $\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3)$ [2]. And we assume also that

$$b_2 |\log \alpha_2| = b_1 |\log \alpha_1| + b_3 |\log \alpha_3| \pm |\Lambda|.$$

We put

$$d_1 = \gcd(b_1, b_2), \quad d_3 = \gcd(b_3, b_2), \quad b_2 = d_1 b_2' = d_3 b_2''.$$

Let K, L, R, R_1, R_2, R_3, S, S_1, S_2, S_3, T, T_1, T_2, T_3 be positive rational integers, with

$$K \ge 3, \quad L \ge 5, \quad R > R_1 + R_2 + R_3, \quad S > S_1 + S_2 + S_3, \quad T > T_1 + T_2 + T_3.$$

Let $\rho \ge 2$ be a real number. Assume first that

$$\left(\frac{KL}{2} + \frac{L}{4} - 1 - \frac{2K}{3L}\right) \log \rho \ \ge (\mathcal{D}+1) \log N + gL(a_1 R + a_2 S + a_3 T) + \mathcal{D}(K-1) \log b - 2 \log(e/2), \tag{2}$$

where $N = K^2 L$, $\mathcal{D} = [\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}] \,/\, [\mathbb{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{R}]$, $e = \exp(1)$,

$$g = \frac{1}{4} - \frac{N}{12RST}, \qquad b = (b_2' \eta_0)(b_2'' \zeta_0) \left(\prod_{k=1}^{K-1} k! \right)^{-\frac{4}{K(K-1)}},$$

with

$$\eta_0 = \frac{R-1}{2} + \frac{(S-1)b_1}{2b_2}, \qquad \zeta_0 = \frac{T-1}{2} + \frac{(S-1)b_3}{2b_2},$$

and

$$a_i \ge \rho |\log \alpha_i| - \log |\alpha_i| + 2\mathcal{D}\, \mathrm{h}(\alpha_i), \qquad i = 1,\ 2,\ 3.$$

[2] If D is the degree of this number field then $\varphi(w) \le D$, where φ is the Euler totient function. It is easy to prove that $\varphi(w) \ge (w/2)^{0.63}$, which implies $w < 2D^{1.6}$. Hence the previous condition on Λ is satisfied if $0 < |\Lambda| \le \pi D^{-1.6}$ and then $\Lambda \notin i\pi\mathbb{Q}$. Trivially, this last condition is also satisfied when Λ is real and non-zero.

Put

$$\mathcal{V} = \sqrt{(R_1+1)(S_1+1)(T_1+1)}.$$

If, *for some positive real number* χ,

(i) $(R_1+1)(S_1+1)(T_1+1) > K \cdot \max\Big\{R_1+S_1+1,\, S_1+T_1+1,\, R_1+T_1+1,\, \chi\mathcal{V}\Big\}$,

(ii) $\mathrm{Card}\,\big\{\alpha_1^r\alpha_2^s\alpha_3^t \,:\, 0 \le r \le R_1,\, 0 \le s \le S_1,\, 0 \le t \le T_1\big\} > L$,

(iii) $(R_2+1)(S_2+1)(T_2+1) > 2K^2$,

(iv) $\mathrm{Card}\,\big\{\alpha_1^r\alpha_2^s\alpha_3^t \,:\, 0 \le r \le R_2,\, 0 \le s \le S_2,\, 0 \le t \le T_2\big\} > 2KL$, *and*

(v) $(R_3+1)(S_3+1)(T_3+1) > 6K^2L$,

then either

$$\Lambda' > \rho^{-KL},$$

where

$$\Lambda' = |\Lambda| \cdot \frac{LSe^{LS|\Lambda|/(2b_2)}}{2|b_2|},$$

or *at least one of the following conditions* (**C1**), (**C2**), (**C3**) *hold:*

(**C1**) $$|b_1| \le R_1 \quad \textit{and} \quad |b_2| \le S_1 \quad \textit{and} \quad |b_3| \le T_1,$$

(**C2**) $$|b_1| \le R_2 \quad \textit{and} \quad |b_2| \le S_2 \quad \textit{and} \quad |b_3| \le T_2,$$

(**C3**) **either** *there exist two non-zero rational integers* r_0 *and* s_0 *such that*

$$r_0b_2 = s_0b_1$$

with

$$|r_0| \le \frac{(R_1+1)(T_1+1)}{\mathcal{M}-T_1} \quad \textit{and} \quad |s_0| \le \frac{(S_1+1)(T_1+1)}{\mathcal{M}-T_1},$$

where

$$\mathcal{M} = \max\Big\{R_1+S_1+1,\, S_1+T_1+1,\, R_1+T_1+1,\, \chi\mathcal{V}\Big\},$$

or *there exist rational integers* r_1, s_1, t_1 *and* t_2, *with* $r_1s_1 \ne 0$, *such that*

$$(t_1b_1 + r_1b_3)s_1 = r_1b_2t_2, \qquad \gcd(r_1,t_1) = \gcd(s_1,t_2) = 1,$$

which also satisfy

$$|r_1s_1| \le \delta \cdot \frac{(R_1+1)(S_1+1)}{\mathcal{M}-\max\{R_1,S_1\}}, \quad |s_1t_1| \le \delta \cdot \frac{(S_1+1)(T_1+1)}{\mathcal{M}-\max\{S_1,T_1\}}, \quad |r_1t_2| \le \delta \cdot \frac{(R_1+1)(T_1+1)}{\mathcal{M}-\max\{R_1,T_1\}},$$

where

$$\delta = \gcd(r_1,s_1).$$

Moreover, when $t_1 = 0$ *we can take* $r_1 = 1$, *and when* $t_2 = 0$ *we can take* $s_1 = 1$.

Proof. The assumption $0 < |\Lambda| < 2\pi/w$ implies that $\Lambda \notin i\pi\mathbb{Q}$, by Lemma 3.9, the hypothesis (v) of the Theorem implies condition (iii) of the zero-lemma. By Lemma 3.11, we get the conditions (**C1**) and (**C2**) if, respectively, the condition (i.1) or (i.2) of the zero-lemma are not satisifed. This finishes the proof. □

Warning . — In the above theorem, the roles of (α_1,b_1) and (α_3,b_3) are not completely symmetric. Even if we do not make the hypothesis $a_1 \ge a_3$ (and, of course, do not use it), in practice it is sometimes better to choose the numerotation such that $a_1 \ge a_3$, but one has also to deal with (**C3**) which is also non-symmetrical…

4. An estimate for linear forms in two logarithms

We need to use linear forms in two logarithms in a very special situation (related to condition (**C3**) above) and it is difficult to find an easy-to-use result for such a case. This is the reason why we write a suitable application of [9] in this Section.

Let α_1, α_2 be two non-zero algebraic numbers, and let $\log\alpha_1$ and $\log\alpha_2$ be any determinations of their logarithms. We consider here the linear form

$$\Lambda = b_2\log\alpha_2 - b_1\log\alpha_1,$$

where b_1 and b_2 are positive integers. Without loss of generality, we suppose that the absolute values $|\alpha_1|$ and $|\alpha_2|$ are ≥ 1. Put

$$\mathcal{D} = [\mathbf{Q}(\alpha_1,\alpha_2):\mathbf{Q}]\,/\,[\mathbf{R}(\alpha_1,\alpha_2):\mathbf{R}].$$

4.1. Statement of the main result of [9].

The main result of [9], which we recall for the convenience of the reader, is:

Theorem 3. *Let K be an integer ≥ 3, L an integer ≥ 2, R_1, R_2, S_1, S_2 positive integers. Let ρ be a real number > 1. Put $R = R_1 + R_2 - 1$, $S = S_1 + S_2 - 1$, $N = KL$,*

$$g = \frac{1}{4} - \frac{N}{12RS}, \qquad b = \frac{\big((R-1)b_2 + (S-1)b_1\big)}{2}\left(\prod_{k=1}^{K-1} k!\right)^{-2/(K^2-K)}.$$

Let a_1, a_2 be positive real numbers such that

$$a_i \geq \rho\,|\log\alpha_i| - \log|\alpha_i| + 2\mathcal{D}\,\mathrm{h}(\alpha_i),$$

for $i = 1, 2$. Suppose that:

$$\text{(I)} \qquad \mathrm{Card}\big\{\alpha_1^r\,\alpha_2^s\,;\, 0 \leq r < R_1,\, 0 \leq s < S_1\big\} \geq L,$$

and

$$\text{(II)} \qquad \mathrm{Card}\big\{rb_2 + sb_1\,;\, 0 \leq r < R_2,\, 0 \leq s < S_2\big\} > (K-1)L$$

and also that

$$\text{(III)} \qquad K(L-1)\log\rho - (\mathcal{D}+1)\log N - \mathcal{D}(K-1)\log b - gL\,(Ra_1 + Sa_2) \;>\; 0.$$

Then,

$$|\Lambda'| \geq \rho^{-KL+(1/2)},$$

where

$$\Lambda' = \Lambda\cdot\max\left\{\frac{LSe^{LS|\Lambda|/(2b_2)}}{2b_2},\; \frac{LRe^{LR|\Lambda|/(2b_1)}}{2b_1}\right\}.$$

4.2. **A special estimate for linear forms in two logarithms.** In the case when the number α_1 is not a root of unity we shall deduce the following result from Theorem 3, which is a variant of Théorème 2 of [9], close to Theorem 1.5 of [11].

Proposition 4.1. *Consider the linear form*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

where b_1 *and* b_2 *are positive integers. Suppose that* α_1 *is not a root of unity. Put*

$$\mathcal{D} = [\mathbf{Q}(\alpha_1, \alpha_2) : \mathbf{Q}] \,/\, [\mathbf{R}(\alpha_1, \alpha_2) : \mathbf{R}].$$

Let a_1, a_2, h, k *be real positive numbers, and* ρ *a real number* > 1. *Put* $\lambda = \log \rho$ *and suppose that*

$$h \geq \max\left\{1,\, 1.5\,\lambda,\, \mathcal{D}\left(\log\Bigl(\frac{b_1}{a_2} + \frac{b_2}{a_1}\Bigr) + \log\lambda + f(K)\right) + \varepsilon\right\}, \quad \varepsilon = 0.0262, \tag{4}$$

$$a_i \geq \max\bigl\{4,\, 2.7\,\lambda,\, \rho\,|\log\alpha_i| - \log|\alpha_i| + 2\mathcal{D}\,\mathrm{h}(\alpha_i)\bigr\}, \quad (i = 1, 2), \tag{5}$$

$$a_1 a_2 \geq 20\lambda^2, \tag{6}$$

where

$$f(x) = \log\frac{\bigl(1 + \sqrt{x-1}\bigr)\sqrt{x}}{x-1} + \frac{\log x}{6x(x-1)} + \frac{3}{2} + \log\frac{3}{4} + \frac{\log\frac{x}{x-1}}{x-1}$$

and

$$L = 2 + \lfloor 2h/\lambda \rfloor \geq 5, \quad K = 1 + \lfloor kLa_1a_2 \rfloor.$$

Then we have the lower bound

$$\log|\Lambda| \geq -\lambda k L^2 a_1 a_2 - \max\Bigl\{\lambda(L - 0.5) + \log\Bigl(L^2(1 + \sqrt{k})a_2\Bigr), \mathcal{D}\log 2\Bigr\},$$

provided that k *satisfies* $k \leq 2.2\lambda^{-2}$ *and*

$$kU - V\sqrt{k} - W \geq 0,$$

with

$$U = (L-1)\lambda - h, \quad V = L/3, \quad W = \frac{1}{4}\left(\frac{L}{a_2} + \frac{1}{a_1}\right).$$

4.3. **Estimates for the parameter** k**.** Before proceeding to the proof of the above Proposition, we need to compute upper and lower bounds for the parameter k.

Put $\Delta = V^2 + 4UW$, the condition on k implies $k \geq k_0$, where

$$\sqrt{k_0} = \frac{V + \sqrt{\Delta}}{2U}, \quad k_0 = \frac{V^2 + \Delta + 2V\sqrt{\Delta}}{4U^2} = \frac{V^2}{2U^2} + \frac{W}{U} + \frac{V}{2U}\sqrt{\frac{V^2}{U^2} + \frac{4W}{U}} \geq \frac{V^2}{U^2} + \frac{W}{U},$$

with

$$\frac{8}{9\lambda} \geq \frac{1}{3}\,\frac{\lambda^{-1}(2h+\lambda)}{(2h+\lambda) - (h+\lambda)} \geq \frac{V}{U} = \frac{1}{3}\,\frac{L}{\lambda L - (h+\lambda)} \geq \frac{1}{3}\,\frac{\lambda^{-1}2(h+\lambda)}{2(h+\lambda) - (h+\lambda)} = \frac{2}{3\lambda},$$

since $\partial(V/U)/\partial L < 0$ and $1 + 2h/\lambda \leq L \leq 2(1 + h/\lambda)$, where $h \geq 1.5\lambda$. Moreover W satisfies

$$\frac{W}{U} = \frac{1}{4}\left(\frac{L}{a_2} + \frac{1}{a_1}\right)\frac{1}{\lambda L - \lambda - h} \geq \frac{1}{4a_1(\lambda L - \lambda - h)} + \frac{1}{2a_2\lambda},$$

and also

$$\frac{W}{U} \leq \frac{1}{4}\left(\frac{1 + 2h/\lambda}{a_2} + \frac{1}{a_1}\right)\frac{1}{h} = \frac{2}{a_2\lambda} + \frac{\frac{1}{a_1} + \frac{1}{a_2}}{4h} \leq \begin{cases} \frac{1}{2\lambda} + \frac{1}{8\times 1.5\lambda}, & \text{if } \lambda \leq 1, \\ \frac{2}{2.7\lambda^2} + \frac{2}{2.7\times 6\lambda^2}, & \text{if } \lambda \geq 1, \end{cases}$$

because of our hypotheses on a_1, a_2 and h. Thus we always have
$$\frac{W}{U} \le \frac{7}{8.1\lambda^2}.$$
It is easy to check that the previous inequalities imply
$$\sqrt{k_0} \le \frac{1.48}{\lambda}.$$
Hence $k_0 < 2.2\,\lambda^{-2}$ and we can always choose k satisfying
$$\frac{4}{9\lambda^2} \le k \le \frac{2.2}{\lambda^2},$$
and then
$$kLa_1a_2 \ge \left(\frac{4}{9\lambda^2} + \frac{\frac{L}{a_2} + \frac{1}{a_1}}{4(\lambda L - \lambda - h)} \right) La_1a_2$$
so that
$$kLa_1a_2 \ge \frac{4a_1a_2L}{9\lambda^2} + \frac{a_1L}{2\lambda} + \frac{a_2}{2\lambda} = \psi(L),$$
say.

Clearly ψ increases with L and it is easy to check that $\psi(5) > 54$ (use the fact that $a_1a_2 \ge 20\lambda^2$).

4.4. **Proof of the Proposition.** Now we are ready to prove Proposition 4.1.

We suppose that α_1 is not a root of unity, and we apply Theorem 3 with a suitable choice of the parameters. The proof follows the proof of Théorème 2 of [9]. For the convenience of the reader we keep the numerotation of the formulas of [9], except that formula $(5.i)$ in [9] is here formula $(4.i)$, moreover when there is some change the new formula is denoted by $(4.i)'$.

Put
$$L = 2 + \lfloor 2h/\lambda \rfloor, \quad K = 1 + \lfloor kLa_1a_2 \rfloor,$$
thus $L \ge 5$ and $K \ge 55$,
$$R_1 = L, \quad S_1 = 1, \quad R_2 = 1 + \lfloor \sqrt{k}La_2 \rfloor, \quad S_2 = 1 + \lfloor \sqrt{k}La_1 \rfloor. \tag{4.1$'$}$$
By Liouville inequality,
$$\log|\Lambda| \ge -\mathcal{D}\log 2 - \mathcal{D}\, b_1\mathrm{h}(\alpha_1) - \mathcal{D}\, b_2\mathrm{h}(\alpha_2) \ge -\mathcal{D}\log 2 - \tfrac{1}{2}(b_1a_1 + b_2a_2) = -\mathcal{D}\log 2 - \tfrac{1}{2}b'a_1a_2,$$
where
$$b' = \frac{b_1}{a_2} + \frac{b_2}{a_1}.$$
We consider two cases:
$$b' \le 2\lambda kL^2, \quad \text{or} \quad b' > 2\lambda kL^2.$$
In the first case, Liouville inequality implies
$$\log|\Lambda| \ge -\mathcal{D}\log 2 - \lambda kL^2a_1a_2$$
and Prop. 4.1 holds.

Suppose now that $b' > 2\lambda kL^2$. Then $\max\{b_1/a_2, b_2/a_1\} > \lambda kL^2$, hence
$$b_1 > \lambda\sqrt{k}L \times \sqrt{k}La_2 \quad \text{or} \quad b_2 > \lambda\sqrt{k}L \times \sqrt{k}La_1.$$
Since $k \ge (4/9)\lambda^{-2}$ and $L \ge 2$, we have $\lambda\sqrt{k}L > 1$, which implies
$$b_1 \ge R_2 \quad \text{or} \quad b_2 \ge S_2,$$
hence
$$\mathrm{Card}\{rb_2 + sb_1\,;\, 0 \le r < R_2,\, 0 \le s < S_2\} = R_2S_2 > (K-1)L,$$

by the choice of R_2 and S_2. Moreover, since α_1 is not a root of unity, we have

$$\mathrm{Card}\{\alpha_1^r\alpha_2^s\,;\,0\le r<R_1,\,0\le s<S_1\}=R_1=L.$$

This ends the verification of conditions (I) and (II) of Theorem 3.

Remark. The condition $b'>2k\lambda L^2$ implies

$$\frac{\lambda L}{\mathcal{D}}\ge\frac{2h}{\mathcal{D}}\ge 2\bigl(\log(2k\lambda^2L^2)+f(K)\bigr)\ge 2\left(\log(8L^2/9)+\frac{3}{2}+\log\frac{3}{4}\right)>8.626,$$

using the above estimates on k and $L\ge 5$.

Suppose that (III) holds, then Theorem 3 implies

$$\log|\Lambda'|\ge -KL\lambda+\lambda/2,$$

where

$$\Lambda'=\Lambda\cdot\max\left\{\frac{LSe^{LS|\Lambda|/(2b_2)}}{2b_2},\ \frac{LRe^{LR|\Lambda|/(2b_1)}}{2b_1}\right\}.$$

Notice that

$$R=R_1+R_2-1\le L+\sqrt{k}La_2\quad\text{and}\quad S=S_1+S_2-1\le 1+\sqrt{k}La_1.$$

This shows that

$$\max\{LR,LS\}\le L^2(1+\sqrt{k}a_2)<L^2(1+1.5\lambda^{-1}a_2)=L^2\left(\frac{1}{a_2}+\frac{1.5}{\lambda}\right)a_2<\frac{a_1a_2L^2}{2\lambda}.$$

As we may, suppose that $\log|\Lambda|\le-\lambda kL^2a_1a_2$, then

$$\max\left\{\frac{LR|\Lambda|}{2b_2},\frac{LS|\Lambda|}{2b_1}\right\}\le\frac{(1+\sqrt{k}a_2)L^2\,|\Lambda|}{2}\le\frac{L^2a_1a_2}{4\lambda}e^{-4L^2a_1a_2/(9\lambda)},$$

since $(4/9)\lambda^{-2}\le k\le 2.2\lambda^{-2}$ and $L^2a_1a_2/\lambda>100$ (indeed, we have $L\ge 5$, $a_1\ge 4$ and $a_2\ge 2.7\lambda$, hence $L^2a_1a_2/\lambda\ge 270$), we get

$$\max\left\{\frac{LR|\Lambda|}{2b_2},\frac{LS|\Lambda|}{2b_1}\right\}<10^{-10}.$$

Thus,

$$|\Lambda'|\le|\Lambda|\times L^2(1+\sqrt{k}a_2),$$

which implies

$$\log|\Lambda|\ge-\lambda kL^2a_1a_2-\lambda(L-0.5)-\log\bigl(L^2(1+\sqrt{k}a_2)\bigr)$$

and Prop. 4.1 follows.

Now we have to verify that condition (III) is satisfied: we have to prove that

$$\Phi_0=K(L-1)\log\rho-(\mathcal{D}+1)\log N-\mathcal{D}(K-1)\log b-gL\,(Ra_1+Sa_2)>0,$$

when $b'>2\lambda kL^2$. Notice that the condition $b'>2\lambda kL^2$ implies

$$h\ge\mathcal{D}\bigl(\log(2\lambda^2kL^2)+f(K)\bigr)\ge\mathcal{D}\left(\log\Bigl(\frac{8L^2}{9}\Bigr)+\frac{3}{2}+\log\frac{3}{4}\right)>4.313\,\mathcal{D}.$$

We replace this condition by the two conditions $\Phi>0$, $\Theta>0$, where $\Phi_0\ge\Phi+\Theta$. The term Φ is the main one, Θ is a sum of residual terms. As indicated in [9], the condition $\Phi>0$ leads to the choice of the parameters $(4.1)'$, whereas $\Theta>0$ is a secondary condition, which leads to assume some technical hypotheses on h and a_1, a_2.

As in [9] (Lemme 8) we get

$$\log b \le \log\left(\frac{b_1}{a_2}+\frac{b_2}{a_1}\right)+\log\lambda-\frac{\log\left(2\pi K/\sqrt{e}\right)}{K-1}+f(K)\le\frac{h}{\mathcal{D}}-\frac{\varepsilon}{\mathcal{D}}-\frac{\log\left(2\pi K/\sqrt{e}\right)}{K-1}, \tag{4.17}$$

which follows from the condition

$$h\ge\mathcal{D}\big(\log b'+\log\lambda+f(K)\big)+\varepsilon.$$

Here we have

$$gL(Ra_1+Sa_2)\le\left(\frac{1}{4}-\frac{KL}{12RS}\right)L(Ra_1+Sa_2)=\frac{L(Ra_1+Sa_2)}{4}-\frac{KL^2}{12}\left(\frac{a_1}{S}+\frac{a_2}{R}\right),$$

which implies

$$gL(Ra_1+Sa_2)\le\frac{L}{4}(a_1L+a_2+2L\sqrt{k}a_1a_2)-\frac{KL}{6\sqrt{k}}\le\frac{L}{4}(a_1L+a_2)+\frac{\sqrt{k}L^2a_1a_2}{3}. \tag{4.18}$$

Put

$$\Phi=K(L-1)\lambda-Kh-\frac{\sqrt{k}L^2a_1a_2}{3}-\frac{L(a_1L+a_2)}{4} \tag{4.21}$$

and

$$\Theta=\varepsilon(K-1)+h-\mathcal{D}\log\big(\sqrt{e}L/(2\pi)\big)-\log(KL). \tag{4.22}$$

By (4.17) and (4.18) we see that $\Phi_0\ge\Phi+\Theta$, where $kLa_1a_2<K\le 1+kLa_1a_2$, hence

$$\frac{\Phi}{La_1a_2}>kU-V\sqrt{k}-W,$$

where

$$U=(L-1)\lambda-h,\quad V=\frac{L}{3},\quad W=\frac{1}{4}\left(\frac{L}{a_2}+\frac{1}{a_1}\right).$$

This proves that $\Phi>0$ provided that $kU-V\sqrt{k}-W\ge 0$.

We have

$$\Theta\ge h-\log(kL^2a_1a_2)-\mathcal{D}\log\big(\sqrt{e}L/(2\pi)\big)+\varepsilon(kLa_1a_2-1).$$

To prove that $\Theta\ge 0$, rewrite (4.22) as $\Theta=\Theta_0(D-1)+\Theta_1$, where

$$\Theta_0=\log(\lambda b')+f(K)-\log L+\log\left(\frac{2\pi}{\sqrt{e}}\right),$$

and

$$\Theta_1=\varepsilon K-\log K-2\log L+\log\left(\frac{2\pi}{\sqrt{e}}\right)+\log(\lambda b')+f(K).$$

We conclude by proving that Θ_0 and Θ_1 are both positive.

Since $b'>2k\lambda L^2$ and $k\ge 4/(9\lambda^2)$, we have

$$\log(\lambda b')>\log(2k\lambda^2L^2)>\log(8/9)+2\log L,$$

and this implies that

$$\Theta_0>\log(8L/9)+f(K)+\log(2\pi/\sqrt{e})>\log\frac{8L}{9}+\frac{3}{2}+\log\frac{3}{4}+\log\frac{2\pi}{\sqrt{e}}$$

is positive. This implies also that

$$\Theta_1\ge\varepsilon K-\log K+\log\frac{8}{9}+\log\frac{2\pi}{\sqrt{e}}+f(K).$$

Thus,

$$\Theta_1 \geq 0.0262K - \log K + \log\left(\frac{16\pi}{9\sqrt{e}}\right) + f(K)$$

and an elementary numerical verification shows that Θ_1 is positive for $K \geq 55$, which holds as we saw in the previous Subsection.

Remark. We have proved that, under the hypotheses of our result, we can choose $\varepsilon = 0.0262$, more generally the condition on ε is

$$\varepsilon K - \log K + \log\left(\frac{16\pi}{9\sqrt{e}}\right) + f(K) \geq 0$$

for all $K \geq K_0$, where $K_0 = \lceil k_0 L a_1 a_2 \rceil$.

5. How to use Theorem 2

5.1. **About the multiplicative group $\mathcal{G}$.** In practical examples, generally the following condition holds:

$$(\mathbf{M}) \qquad \begin{cases} \text{either } \alpha_1,\ \alpha_2 \text{ and } \alpha_3 \text{ are multiplicatively independent, or} \\ \text{two multiplicatively independent, the third a root of unity } \neq 1. \end{cases}$$

We use now hypothesis ($\mathbf{M}$), which is clearly stronger than the standard hypothesis 'the multiplicative group $\mathcal{G}$ is of rank at least two', and we also notice that the order in $\mathbb{C}^*$ of a root of unity $\neq 1$ is at least equal to 2, thus the condition (i.2) of Section 3 is satisfied if

$$\text{(C.i.2)} \qquad \frac{2(R_1+1)(S_1+1)(T_1+1)}{W_1+1} > L,$$

where W_1 is defined by

$$W_1 = \begin{cases} R_1, & \text{if } \alpha_1 \text{ is a root of unity,} \\ S_1, & \text{if } \alpha_2 \text{ is a root of unity,} \\ T_1, & \text{if } \alpha_3 \text{ is a root of unity,} \\ 1, & \text{otherwise.} \end{cases}$$

But see also the remark after (C.ii.1) below.

Then, by the study of the case (i.2), we see that, to satisfy the condition (ii.12) of Section 3 it is enough to suppose that (when condition ($\mathbf{M}$) holds)

$$\text{(C.ii.1)} \qquad \frac{(R_2+1)(S_2+1)(T_2+1)}{W_2+1} > KL,$$

where W_2 is defined by

$$W_2 = \begin{cases} R_2, & \text{if } \alpha_1 \text{ is a root of unity,} \\ S_2, & \text{if } \alpha_2 \text{ is a root of unity,} \\ T_2, & \text{if } \alpha_3 \text{ is a root of unity,} \\ 1, & \text{otherwise.} \end{cases}$$

Remark. When (for example) α_3 is a root of unity of order ν, condition (C.ii.1) above can be replaced by

$$\text{(C}'\text{.ii.1)} \qquad \nu\,(R_2+1)(S_2+1) > 2KL,$$

(provided $T_2 \geq \nu - 1$) and condition (C.i.2) can be replaced by

$$\text{(C}'\text{.i.2)} \qquad \nu\,(R_1+1)(S_1+1) > L$$

(provided $T_1 \geq \nu - 1$).

Remark. Under a weaker condition one can obtain similar (but slightly weaker) conclusions, see [12], Ex. 7.5, p. 229.

5.2. **The choice of parameters.** Here we assume that condition (**M**) holds, then by the above Corollary 3.10 we know that $\Lambda \notin i\pi\mathbb{Q}$.

To apply Theorem 2, we consider an integer $L \geq 5$ and real parameters $m > 0$, $\rho \geq 2$ (then one can define the a_i's) and we put

$$K = \lfloor mLa_1a_2a_3 \rfloor.$$

To simplify the presentation, even if we do not really need these conditions, we also assume

$$m \geq 1 \quad \text{and} \qquad \Omega := a_1a_2a_3 \geq 2.$$

We define

$$\begin{aligned} R_1 &= \lfloor c_1a_2a_3 \rfloor, \quad & S_1 &= \lfloor c_1a_1a_3 \rfloor, \quad & T_1 &= \lfloor c_1a_1a_2 \rfloor, \\ R_2 &= \lfloor c_2a_2a_3 \rfloor, \quad & S_2 &= \lfloor c_2a_1a_3 \rfloor, \quad & T_2 &= \lfloor c_2a_1a_2 \rfloor, \\ R_3 &= \lfloor c_3a_2a_3 \rfloor, \quad & S_3 &= \lfloor c_3a_1a_3 \rfloor, \quad & T_3 &= \lfloor c_3a_1a_2 \rfloor, \end{aligned}$$

where the parameters c_1, c_2 and c_3 will be chosen so that the conditions (i) up to (v) of the Theorem are satisfied.

Clearly, condition (i) is satisfied if

$$\left(c_1^3(a_1a_2a_3)^2\right)^{1/2} \geq \chi m a_1a_2a_3L \quad \text{and} \quad c_1^2 \cdot \Omega a \geq 2mL, \quad \text{where} \quad a = \min\{a_1, a_2, a_3\}.$$

Condition (ii) is true when $2c_1^2a_1a_2a_3 \cdot \min\{a_1, a_2, a_3\} \geq L$. Thus, since we suppose $m \geq 1$ and also $\Omega \geq 2$, we can take

$$c_1 = \max\left\{(\chi mL)^{2/3}, \left(\frac{mL}{a}\right)^{1/2}\right\}.$$

To satisfy (iii) and (iv) we can take

$$c_2 = \max\left\{2^{1/3}(mL)^{2/3}, \sqrt{m/a}\, L\right\}.$$

Finally, since $\Lambda \notin i\pi\mathbb{Q}$, by Lemma 3.9 condition (v) holds for

$$c_3 = (6m^2)^{1/3}\, L.$$

Remark. When α_1, α_2, α_3 are multiplicatively independent then it is enough to take c_1 and c_3 as above and

$$c_2 = 2^{1/3}(mL)^{2/3}.$$

Then we have to verify the condition (2) of Theorem 2. When this inequality holds, one obtains

$$|\Lambda'| > \rho^{-KL},$$

and we get

$$\log|\Lambda| > -KL\log\rho - \log(SL),$$

except maybe if at least one of the conditions (**C1**), (**C2**) or (**C3**) holds.

It may be useful to notice that, because of the choice of these parameters, the previous lower bound is essentially of the form

$$\log|\Lambda| \geq -CL^2a_1a_2a_3,$$

where C is some (rather large) constant. One may verify that condition (2) forces to choose L of the order of magnitude of $\mathcal{D}\log b$, so that we have (to simplify)

$$\log|\Lambda| \geq -C\mathcal{D}^2 a_1a_2a_3 \log^2 B, \quad \text{where } B = \max\{b_1, b_2, b_3\},$$

in the *non degenerate case*. We give a more detailed study below.

Remark. In many concrete applications (this is the case for the examples at the end of the paper) one knows only some lower bound, say h_0, for the height h_i of α_i, one of the algebraic numbers α_1, α_2 or α_3. To apply Theorem 2 we have to verify that condition (2) holds, and there is a difficulty. But first notice that this condition is $\Phi = \Phi(K) \geq 0$ where

$$\Phi(K) = AK - B - C\log K - E\,\frac{\log K}{K} - \frac{F}{K},$$

with positive constants A, B, $\ldots$, F, when the factor of g is expessed in terms of K using the definitions of these parameters and of R, S, T (*i.e.*, $Ra_1+Sa_2+Ta_3 \approx 3cm^{2/3}\Omega L$ and $K = \lfloor m\Omega L\rfloor$). Thus the derivative $\Phi'(K)$ satisfies

$$K\Phi'(K) = AK - C - \frac{E}{K} + E\,\frac{\log K}{K} + \frac{F}{K}.$$

This short computation shows that if $\Phi(K_0)$ is positive for some integer $K_0 \geq 3$ then it is positive for any integer $K \geq K_0$, when A, B, $\ldots$, E and F are fixed (this means in particular that m and L are fixed). Notice also that the term b appearing in Theorem 2 is a decreasing function of a_i.[3]

The conjonction of these two remarks shows that we can study condition (2) with the value $a_i = a_0$ (corresponding to h_0) and with L fixed and the other parameters m, c_1, c_2, c_3, R_1, $\ldots$, T_3, $K = K_0$ chosen as above with $a_i = a_0$. When $\Phi(K_0)$ is positive, we also have $\Phi(K) > 0$ for the preceding values of L, m, c_1, c_2, c_3 and any $a_i \geq a_0$.

Now consider the conditions (**C1**), (**C2**) and (**C3**). For conditions (**C1**) and (**C2**) we have in particular

$$(\mathbf{C1}) \text{ or } (\mathbf{C2}) \implies b_2 \leq \max\{S_1, S_2\}.$$

Condition (**C3**) will be studied in detail in the next subsection. Put

$$r_1 = \delta r_1', \qquad s_1 = \delta s_1',$$

where

$$\delta = \gcd(r_1, s_1).$$

We just notice, for the second alternative, namely

$$(t_1b_1 + r_1b_3)s_1 = r_1b_2t_2, \qquad \text{with} \quad \gcd(r_1,t_1) = \gcd(s_1,t_2) = 1,$$

that $r_1' \mid b_1$, say $b_1 = r_1'b_1'$, hence

$$(t_1b_1' + \delta b_3)s_1' = b_2t_2, \quad \text{with } \ b_1 = r_1b_1'.$$

If $t_2 \neq 0$ this shows that $s_1' \mid b_2$, say $b_2 = s_1'b_2'$, so that

$$t_1b_1' + \delta b_3 = b_2't_2, \quad \text{with } \ b_1 = r_1'b_1', \ \text{and } \ b_2 = s_1'b_2'.$$

[3] More precisely, we never use the exact value of this term b but consider its upper bound implied by Lemma 3.4, and the resulting quantity is a decreasing function of a_i.

5.3. **The degenerate case.** We have already seen the (easy) consequences of conditions (**C1**) or (**C2**). We focus our attention on the third condition (**C3**). In this subsection we choose $\chi = 1$.

The first subcase is

$$r_0 b_2 = s_0 b_1,$$

with the above bounds for r_0 and s_0. This implies

$$b_1 = d_1 r_0, \quad b_2 = d_1 s_0$$

and one verifies that essentially (see below)

$$|r_0| \leq \sqrt{c_1} a_2, \qquad |s_0| \leq \sqrt{c_1} a_1, \qquad \text{where } \ c_1 \ll L^{2/3}.$$

We consider the linear form Λ as a linear form in two logarithms:

$$\Lambda = d_1(s_0 \log \alpha_2 - r_0 \log \alpha_1) - b_3 \log \alpha_3$$

and using Theorem 3 we get

$$\log |\Lambda| \gg -(s_0 a_2 + r_0 a_1) a_3 \mathcal{D}^2 \log^2 B \gg -\sqrt{c_1} a_1 a_2 a_3 \mathcal{D}^2 \log^2 B \gg -a_1 a_2 a_3 (\mathcal{D} \log B)^{7/3}.$$

The second subcase is

$$(t_1 b_1 + r_1 b_3) s_1 = r_1 t_2 b_2.$$

If

$$t_2 = 0$$

then we easily get

$$b_1 = d r_1, \quad b_3 = -d t_1, \quad \text{where } \ d = \gcd(b_1, b_3).$$

One verifies that essentially

$$|r_1| \leq \sqrt{c_1} a_3, \qquad |t_1| \leq \sqrt{c_1} a_1, \qquad \text{where } \ c_1 \ll L^{2/3}.$$

We consider the linear form in two logarithms

$$\Lambda = b_2 \log \alpha_2 - d(r_1 \log \alpha_1 - t_1 \log \alpha_3)$$

and using Theorem 3 we get now

$$\log |\Lambda| \gg -(\sqrt{c_1} a_3 a_1) a_2 \mathcal{D}^2 \log^2 B \gg -a_1 a_2 a_3 (\mathcal{D} \log B)^{7/3},$$

just as before.

Similarly, if

$$t_1 = 0$$

then we get

$$b_3 = d_3 t_2, \quad b_2 = d_3 s_1, \quad \text{where } \ d_3 = \gcd(b_2, b_3).$$

And we get once more

$$\log |\Lambda| \gg -(\sqrt{c_1} a_3 a_1) a_2 \mathcal{D}^2 \log^2 B \gg -a_1 a_2 a_3 (\mathcal{D} \log B)^{7/3}.$$

(In this third case, essentially, $s_1 \leq \sqrt{c_1} \cdot \min\{a_1, a_3\}$ and $t_2 \leq \sqrt{c_1}\, a_2$, and then we write Λ as $\Lambda = d_3(s_1 \log \alpha_2 - t_2 \log \alpha_3) - b_1 \log \alpha_1$.)

Thus we may now restrict our attention to the more serious case $t_1 t_2 \neq 0$. Then we have

$$(t_1 b_1 + r_1 b_3) s_1 = r_1 t_2 b_2, \qquad \text{with } \ b_1 = r_1' b_1' \ \text{ and } \ b_2 = s_1' b_2',$$

where

$$r_1 = \delta r_1', \quad s_1 = \delta s_1', \quad \delta := \gcd(r_1, s_1).$$

And we have

$$\gcd(r_1, t_1) = \gcd(s_1, t_2) = \gcd(r_1', s_1') = 1, \quad t_1 b_1' + \delta b_3 = t_2 b_2'.$$

[We have used new notation. The reader should not confuse these new definitions for b_1' and b_2' with the previous ones.] To simplify a little the notation, we put

$$\mathcal{V}_R = \mathcal{V} - \max\{S_1, T_1\}, \quad \mathcal{V}_S = \mathcal{V} - \max\{R_1, T_1\}, \quad \mathcal{V}_T = \mathcal{V} - \max\{R_1, S_1\},$$

where

$$\mathcal{V} = \Big((R_1+1)(S_1+1)(T_1+1)\Big)^{1/2}$$

satisfies

$$\mathcal{V} \geq \max\{R_1 + S_1 + 1,\ S_1 + T_1 + 1,\ R_1 + T_1 + 1\}.$$

Now the previous bounds read

$$0 < |r_1 s_1| < \delta(R_1+1)(S_1+1)/\mathcal{V}_T, \quad |s_1' t_1| < (S_1+1)(T_1+1)/\mathcal{V}_R, \quad |r_1' t_2| < (R_1+1)(T_1+1)/\mathcal{V}_S.$$

Which essentially implies (notice that $R_1 \approx c_1 a_2 a_3$, $S_1 \approx c_1 a_1 a_3$, $T_1 \approx c_1 a_1 a_2$ and $\mathcal{V} \approx c_1^{3/2} a_1 a_2 a_3$)

$$|r_1 s_1| \leq \delta \sqrt{c_1} a_3, \quad |s_1' t_1| \leq \sqrt{c_1} a_1, \quad |r_1' t_2| \leq \sqrt{c_1} a_2.$$

We distinguish three cases according to the size of the terms a_i's.

Case 1: $a_1 = \min\{a_1, a_2, a_3\}$

In this case, we write

$$t_1 \Lambda = t_1 r_1' b_1' \ell_1 + t_1 s_1' b_2' \ell_2 + t_1 b_3 \ell_3 = b_2'(s_1' t_1 \ell_2 + r_1' t_2 \ell_1) + b_3(t_1 \ell_3 - \delta r_1' \ell_1),$$

where $\ell_j = \log \alpha_j$ for $j = 1, 2, 3$. And applying [9] to this linear form in two logs we get

$$-\log|\Lambda| \ll (|s_1' t_1| a_2 + |r_1' t_2| a_2)(|t_1| a_3 + |r_1| a_2)\mathcal{D}^2 \log^2 B,$$

where (being somewhat pessimistic)

$$B = \max\{|b_1|, |b_2|, |b_3|\},$$

and where the implied constant is an absolute constant. And using the upper bounds for the integers $|r_1|, \ldots,$ we get

$$-\log|\Lambda| \ll (\sqrt{c_1} a_1 a_2)(\sqrt{c_1} a_1 a_3)\mathcal{D}^2 \log^2 B \ll a_1^2 a_2 a_3 L^{4/3} \mathcal{D}^2 \log^2 B.$$

Since we have indeed

$$|\delta r_1' s_1'| \ll \sqrt{c_1} a_3, \quad |s_1' t_1| \ll \sqrt{c_1} a_1, \quad |r_1' t_2| \ll \sqrt{c_1} a_2, \quad \text{where } \ c_1 \ll L^{2/3},$$

we get

$$-\log|\Lambda| \ll a_1^2 a_2 a_3 (\mathcal{D} \log B)^{8/3},$$

where the implied constant is absolute.

Case 2: $a_2 = \min\{a_1, a_2, a_3\}$

In this second case, we write

$$t_2 \Lambda = t_2 r_1' b_1' \ell_1 + t_2 s_1' b_2' \ell_2 + t_2 b_3 \ell_3 = b_1'(r_1' t_2 \ell_1 + s_1' t_1 \ell_2) + b_3(t_2 \ell_3 + \delta s_1' \ell_2).$$

Applying [9] to this linear form in two logs we get

$$-\log|\Lambda| \ll (|r_1' t_2| a_1 + |s_1' t_1| a_2)(|t_2| a_3 + |s_1| a_2)\mathcal{D}^2 \log^2 B,$$

which, in this case, implies

$$-\log|\Lambda| \ll a_1 a_2^2 a_3 (\mathcal{D} \log B)^{8/3},$$

where the implied constant is absolute.

Case 3: $a_3 = \min\{a_1, a_2, a_3\}$

In this last case, we write

$$\delta\Lambda = \delta r_1' b_1' \ell_1 + \delta s_1' b_2' \ell_2 + \delta b_3 \ell_3 = b_1'(r_1\ell_1 - t_1\ell_3) + b_2'(s_1\ell_2 + t_2\ell_3).$$

In this case, [9] gives

$$-\log|\Lambda| \ll (|r_1|a_1 + |t_1|a_3)(|t_1|a_3 + |s_1|a_2)\mathcal{D}^2 \log^2 B,$$

which implies

$$-\log|\Lambda| \ll a_1 a_2 a_3^2 (\mathcal{D}\log B)^{8/3},$$

where the implied constant is again absolute.

In is important to notice that, **in any case**, we have obtained

$$-\log|\Lambda| \ll a_1 a_2 a_3 \times \min\{a_1, a_2, a_3\} \times (\mathcal{D}\log B)^{8/3},$$

where the implied constant is absolute. In particular, when $\min\{a_1, a_2, a_3\}$ is bounded above then we have essentially

$$-\log|\Lambda| \ll a_1 a_2 a_3 \, (\mathcal{D}\log B)^{8/3},$$

with an implied constant depending only on $\min\{a_1, a_2, a_3\}$.

Remark. From the theoretical point of view, the above result is very poor. But, in practice, the problem is with constants and — hopefully — our estimate will lead to good results when compared to the other ones published previously.

5.4. **Some special cases.** We have just seen that the arithmetical nature of the coefficients b_1, b_2 and b_3 is very important for the study of the degenerate case. Here we consider some special situations which, indeed, occur frequently in concrete applications to Diophantine problems. In all these special cases we also assume that we have the relation (ii) with $t_1 t_2 \neq 0$.

S1 : b_1 is prime or equal to one

We have seen that $b_1 = r_1' b_1'$. Here there are at most two possibilities:

- $b_1' = 1$, then $|b_1| = |r_1'| \ll \min\{a_2, a_3\} L^{1/3}$, where the implied constant is absolute.
- $r_1' = 1$, then

$$t_1 b_1 + \delta b_3 = t_2 b_2'.$$

S2 : b_2 is prime or equal to one

We have seen that $b_2 = s_1' b_2'$. Here there are at most two possibilities:

- $b_2' = 1$, then $|b_2| = |s_1'| \ll \min\{a_1, a_3\} L^{1/3}$, where the implied constant is absolute.
- $s_1' = 1$ and

$$t_1 b_1' + \delta b_3 = t_2 b_2.$$

S3 : b_3 is prime or equal to one

Since the roles of b_1 and b_3 are more or less symmetrical, in this case it may be useful to exchange these two coefficients and, simultaneously, α_1 and α_3 (the exchange has to be done from the beginning of the study).

5.5. **A corollary of the main result.** In this Subsection we give a corollary of our main result, which is much easier to use that this general result and we restrict ourselves to light hypotheses.

Proposition 5.1. *We consider three non-zero algebraic numbers α_1, α_2 and α_3, which are either all real and >1 or all complex of modulus one and all $\neq 1$. Moreover, we assume that either the three numbers α_1, α_2 and α_3 are multiplicatively independent, or two of these numbers are multiplicatively independent and the third one is a root of unity. Put*

$$\mathcal{D} = [\mathbb{Q}(\alpha_1,\alpha_2,\alpha_3):\mathbb{Q}] \,\big/\, [\mathbb{R}(\alpha_1,\alpha_2,\alpha_3):\mathbb{R}].$$

We also consider three positive coprime rational integers b_1, b_2, b_3, and the linear form

$$\Lambda = b_2\log\alpha_2 - b_1\log\alpha_1 - b_3\log\alpha_3,$$

where the logarithms of the α_i are arbitrary determinations of the logarithm, but which are all real or all purely imaginary.

And we assume also that

$$b_2|\log\alpha_2| = b_1\,|\log\alpha_1| + b_3\,|\log\alpha_3| \pm |\Lambda|\,.$$

We put

$$d_1=\gcd(b_1,b_2),\quad d_3=\gcd(b_3,b_2),\quad b_2=d_1b_2'=d_3b_2''.$$

Let $\rho\geq e:=\exp(1)$ be a real number. Put $\lambda=\log\rho$. Let a_1, a_2 and a_3 be real numbers such that

$$a_i \geq \rho\,|\log\alpha_i| - \log|\alpha_i| + 2\mathcal{D}\,\mathrm{h}(\alpha_i), \qquad i=1,\,2,\,3,$$

and assume further that

$$\Omega := a_1a_2a_3 \geq 2.5 \quad \textit{and} \quad a:=\min\{a_1,a_2,a_3\}\geq 0.62.$$

Let K and L be positive integers with

$$L\geq 4+\mathcal{D},\quad K=\lfloor m\Omega L\rfloor,\quad \textit{where}\ \ m\geq 3.$$

Ler $\chi>0$ be fixed and ≤ 2. Define

$$c_1=\max\left\{(\chi mL)^{2/3},\,\sqrt{2mL/a}\right\},\quad c_2=\max\left\{2^{1/3}(mL)^{2/3},\,\sqrt{m/a}\,L\right\},\quad c_3=(6m^2)^{1/3}\,L,$$

and then put

$$R_1=\lfloor c_1a_2a_3\rfloor,\ S_1=\lfloor c_1a_1a_3\rfloor,\ T_1=\lfloor c_1a_1a_2\rfloor,\quad R_2=\lfloor c_2a_2a_3\rfloor,\ S_2=\lfloor c_2a_1a_3\rfloor,\ T_2=\lfloor c_2a_1a_2\rfloor,$$

and

$$R_3=\lfloor c_3a_2a_3\rfloor,\quad S_3=\lfloor c_3a_1a_3\rfloor,\quad T_3=\lfloor c_3a_1a_2\rfloor.$$

Let also

$$R=R_1+R_2+R_3+1,\quad S=S_1+S_2+S_3+1,\quad T=T_1+T_2+T_3+1.$$

Define

$$c=\max\left\{\frac{R}{La_2a_3},\,\frac{S}{La_1a_3},\,\frac{T}{La_1a_2}\right\}.$$

Finally assume that

$$(3)\quad \left(\frac{KL}{2}+\frac{L}{4}-1-\frac{2K}{3L}\right)\lambda-(\mathcal{D}+1)\log L-3gL^2c\,\Omega-\mathcal{D}(K-1)\log\tilde{b}-2\log K+2\mathcal{D}\log 1.36\geq 0,$$

where

$$g=\frac{1}{4}-\frac{N}{12RST},\quad b'=\left(\frac{b_1'}{a_2}+\frac{b_2'}{a_1}\right)\left(\frac{b_3''}{a_2}+\frac{b_2''}{a_3}\right),\quad \tilde{b}=\frac{e^3c^2\Omega^2L^2}{4K^2}\times b'.$$

Then either

$$\log|\Lambda| > -\big(KL+\log(3KL)\big)\lambda,$$

or the condition (**C3**) *of Theorem 2 holds.*

Proof. Our first step in this proof is the study of the relationship between Λ and Λ'.

Recall that
$$\Lambda' = |\Lambda| \times \frac{LSe^{LS|\Lambda|/(2b_2)}}{2b_2},$$
so that
$$\Lambda' \le |\Lambda| \times \frac{LSe^{LS|\Lambda|/2}}{2}.$$
First notice that
$$c_1 \le (mL)^{2/3} \times \max\left\{2^{2/3},\ \sqrt{\frac{2}{0.62}} \times (mL)^{-1/6}\right\} = (2mL)^{2/3}$$
and that
$$c_2 = m^{2/3}L \times \max\left\{\left(\frac{2}{L}\right)^{1/3},\ \frac{m^{-1/6}}{\sqrt{a}}\right\} \le \frac{3^{-1/6}}{\sqrt{0.62}} \times m^{2/3}L < 1.058\, m^{2/3}L.$$
Hence,
$$S \le \frac{\Omega}{a}\left((2mL)^{2/3} + 1.058\, m^{2/3}L + (6m^2)^{1/3}L\right) + 1 \le \frac{m^{2/3}L\Omega}{a}\left(\left(\frac{4}{L}\right)^{1/3} + 1.058 + 6^{1/3}\right) + 1,$$
thus
$$S < 6.135\, m^{2/3}L\Omega < 4.26\, mL\Omega,$$
since $L \ge 5$, $m \ge 3$, $\Omega \ge 2.5$ and $a \ge 0.62$. This proves that
$$S \le 4.4\, K.$$
Then, under our present hypotheses, we have
$$\Lambda' \le 3KL\,|\Lambda| \quad \text{if} \quad |\Lambda| \le \exp(-KL).$$
This shows that the lower bound on Λ' given in Theorem 2 implies
$$\log|\Lambda| \ge -\,KL\lambda - \log(3KL).$$
Remark. Under the stronger hypotheses $m \ge 20$, $L \ge 30$ and $a \ge 4$, one easily sees that
$$\Lambda' \le KL\,|\Lambda| \quad \text{if} \quad |\Lambda| \le \exp(-KL).$$

We have already seen that in the present case $\Lambda \notin i\pi\mathbb{Q}$, thus we can 'forget' the condition $|\Lambda| < 2\pi/w$ in the statement. (See the footnote of Theorem 2.)

Now we study the present consequences of the conditions (**C1**) and (**C2**). With our choices of R_1, S_1, …, S_3 and T_3, we see that if (**C1**) or (**C2**) holds then, using our previous upper bounds for c_1 and c_2 we get
$$b_1 \le (2m^2)^{1/3}La_2a_3, \quad b_2 \le (2m^2)^{1/3}La_1a_3, \quad b_3 \le (2m^2)^{1/3}La_1a_2.$$
But a Liouville estimate (see Lemma 3.3) applied to $\alpha_1^{b_1}\alpha_3^{b_3}\alpha_2^{-b_2} - 1$ implies that
$$\log|\Lambda| \ge -(b_1h_1 + b_2h_2 + b_3h_3)\mathcal{D} - \mathcal{D}\log 2 \ge -\frac{3}{2}(2m^2)^{1/3}\Omega L - \mathcal{D} \ge -0.5\, mL^2\,\Omega,$$
(where $h_i = \mathrm{h}(\alpha_i)$ for $i = 1, 2, 3$), since $a_i \ge 2\mathcal{D}h_i$ for $i = 1, 2, 3$, and $L \ge 4 + \mathcal{D}$, $m \ge 3$, $\Omega \ge 1$. This short study proves that, presently, either (**C1**) or (**C2**) implies
$$\log|\Lambda| \ge -KL\lambda.$$

It follows that we can also 'forget' these conditions in the statement.

Notice that, by definition,

$$\eta_0 \cdot \zeta_0 \le \left(\frac{cL}{2}\right)^2 (a_2a_3b_2' + a_1a_3b_1')\,(a_1a_2b_2'' + a_1a_3b_1'') = \left(\frac{cL\Omega}{2}\right)^2 \times b',$$

so that

$$\log b \le \log b' + 2\log\left(\frac{cL\Omega}{2}\right) - 2\log K + 3 - 2\,\frac{\log\left(2\pi K e^{-3/2}\right)}{K-1} + \frac{2 + 6\pi^{-2} + \log K}{3K(K-1)},$$

by Lemma 3.4. This implies (since here $K \ge \lfloor 3\times 5\times 2.5\rfloor \ge 37$)

$$\log b \le \log b' + \log\left(\frac{e^3c^2L^2\Omega^2}{4K^2}\right) - 2\,\frac{\log(1.36\,K)}{K-1}$$

and we see now that condition (2) of Theorem (2) holds when the inequality (3) is satisfied. This ends the proof of the Proposition. □

5.6. **Some explicit estimate.** In this Subsection we give explicit lower bounds for $|\Lambda|$ under some natural hypotheses, but somewhat stronger than just above.

Here, we work under the following hypotheses:

$$a := \min\{a_1, a_2, a_3\} \ge 4, \quad L \ge 30\,\mathcal{D}, \quad \Omega \ge 100.$$

Recall that we choose

$$R_1 = \lfloor c_1a_2a_3\rfloor,\ S_1 = \lfloor c_1a_1a_3\rfloor,\ T_1 = \lfloor c_1a_1a_2\rfloor, \quad R_2 = \lfloor c_2a_2a_3\rfloor\ S_2 = \lfloor c_2a_1a_3\rfloor,\ T_2 = \lfloor c_2a_1a_2\rfloor,$$

and

$$R_3 = \lfloor c_3a_2a_3\rfloor, \quad S_3 = \lfloor c_3a_1a_3\rfloor, \quad T_3 = \lfloor c_3a_1a_2\rfloor, \quad K = \lfloor m\Omega L\rfloor,$$

where now the parameters c_1, c_2 and c_3 satisfy (we take $\chi = 1$ to simplify the study):

$$c_1 = (mL)^{2/3}, \quad c_2 = \max\left\{2^{1/3}(mL)^{2/3}, \sqrt{m/a}\,L\right\} \quad \text{and} \quad c_3 = (6m^2)^{1/3}\,L,$$

and we assume *a priori* that the parameter m satisfies

$$49 \le m \le 60.$$

Notice that this implies

$$K \ge mL\Omega - 1 \ge 146{,}999$$

and

$$\log b \le \log b' + 2\log\left(\frac{e^{3/2}cL\Omega}{2K}\right) - 2\,\frac{\log(\theta K)}{K-1},$$

where

$$\log\theta := \log\left(2\pi K e^{-3/2}\right) - \frac{2 + 6\pi^{-2} + \log K}{6K} > \log 1.4019.$$

We take again

$$R = R_1 + R_2 + R_3 + 1, \quad S = S_1 + S_2 + S_3 + 1, \quad T = T_1 + T_2 + T_3 + 1,$$

and

$$c = \max\left\{\frac{R}{La_2a_3}, \frac{S}{La_1a_3}, \frac{T}{La_1a_2}\right\}.$$

Condition (3) (with θ instead of 1.36) holds when

$$\Phi := \left(\frac{KL}{2} + \frac{L}{4} - 1 - \frac{2K}{3L}\right)\lambda - (\mathcal{D}+1)\log L - 3cgL^2\,\Omega - \mathcal{D}(K-1)\,\log\tilde{b} - 2\log K + 2\mathcal{D}\log\theta$$

is non-negative. We choose

$$L = \left\lceil \frac{5}{\lambda}\mathcal{D}\log\mathcal{B}\right\rceil = \frac{\mu}{\lambda}\mathcal{D}\,\log\mathcal{B}$$

(which defines μ), where

$$\log\mathcal{B} = \max\{10,\, \log\tilde{b}\}.$$

Thus

$$5 \le \mu < \mu_1 := 5 + \frac{\lambda}{\mathcal{D}\,\log\mathcal{B}} \le \frac{5L}{L-1}.$$

Then

$$\begin{aligned}
\Phi &\ge (mL\Omega - 1)\left(\frac{L}{2} - \frac{2}{3L}\right)\lambda + \frac{L-4}{4}\,\lambda - 3cgL^2\,\Omega - \frac{\lambda L}{\mu}(mL\Omega - 2) - 2\log K - (\mathcal{D}+1)\log L + 2\mathcal{D}\log\theta \\
&= L^2\Omega\left(\lambda m\left(\frac{1}{2} - \frac{2}{3L^2} - \frac{1}{\mu}\right) - 3cg\right) - \left(\frac{L+4}{4} + \frac{2}{3L} - \frac{2L}{\mu}\right)\lambda - 2\log K - (\mathcal{D}+1)\log L + 2\mathcal{D}\log\theta \\
&\ge L^2\Omega\left(\lambda m\left(0.499259 - \frac{1}{\mu}\right) - 3cg\right) - L\left(\frac{1}{4} - \frac{2}{\mu} + 0.0326\right)\lambda - 2\log K - (\mathcal{D}+1)\log L + 2\mathcal{D}\log\theta \\
&\ge L^2\Omega\left(\lambda m\left(0.499259 - \frac{1}{\mu}\right) - 3cg\right) + 0.1044\,L - 3\log L - 2\log(m\,\Omega) - \mathcal{D}\,\log\frac{L}{\theta^2} \\
&\ge L^2\Omega\left(\lambda m\left(0.499259 - \frac{1}{\mu}\right) - 3cg\right) - 0.2358\,L - 2\log(m\,\Omega) - \mathcal{D}\,\log\frac{L}{\theta^2}.
\end{aligned}$$

Hence,

$$\frac{\Phi}{L^2\Omega} \ge \lambda m\left(0.499259 - \frac{1}{\mu}\right) - 3cg - \frac{0.2358}{L\Omega} - 2\frac{\log(60\,\Omega)}{L^2\Omega} - 2\frac{1}{30L\Omega}\,\log\frac{L}{\theta^2}.$$

And finally

$$\frac{\Phi}{L^2\Omega} \ge \lambda m\left(0.499259 - \frac{1}{\mu}\right) - 3gc - 3\cdot 10^{-4}.$$

By definition,

$$c \le \frac{c_1 + c_2 + c_3}{L} + \frac{1}{La^2}.$$

Recall that (here)

$$c_1 = (mL)^{2/3}, \quad c_2 = \max\left\{2^{1/3}(mL)^{2/3},\, \sqrt{m/a}\,L\right\}, \quad c_3 = 6^{1/3}m^{2/3}L.$$

Notice that

$$2^{1/3}(mL)^{2/3} \le \sqrt{m/a}L \iff 2\sqrt{ma^3} \le L,$$

and that this last inequality implies $L \ge 2\times 7\times 8 = 112$ since $m \ge 49$ and $a \ge 4$. It is easy to check that

$$30^{-1/3} + (2/30)^{1/3} > 112^{-1/3} + m^{-1/6}/\sqrt{a},$$

hence c satisfies

$$c \le \left(30^{-1/3} + (2/30)^{1/3} + 6^{1/3} + \frac{1}{16\times 30^2\times 49^{2/3}}\right)m^{2/3} < 2.54444\,m^{2/3},$$

and

$$g \le \frac{1}{4} - \frac{0.99999^2m^2}{12c^3} < 0.244942.$$

And we get

$$\frac{\Phi}{\Omega L^2} \geq \lambda\left(0.499259 - \frac{1}{\mu}\right) m - 3\times 10^{-4} - 1.86972\, m^{2/3} > \lambda\left(0.499259 - \frac{1}{\mu}\right) m - 1.86975\, m^{2/3}.$$

We take

$$m = \left(\frac{1.8699}{0.499259 - \frac{1}{\mu}}\right)^3 \cdot \lambda^{-3},$$

thus

$$m_1 := \left(\frac{1.86975}{0.499259 - \frac{1}{\mu_1}}\right)^3 \cdot \lambda^{-3} \leq m \leq \left(\frac{1.86975}{0.499259 - \frac{1}{5}}\right)^3 \cdot \lambda^{-3} < 244\,\lambda^{-3}.$$

It is easy to see that the worst case for the term $m\mu^2$ (occuring in the final estimate) is reached when μ is maximal, *i.e.* when $\mu = \mu_1$, and then $m = m_1$.

We have

$$\tilde b \leq \frac{e^3 c^2}{3.999\, m^2}\, b', \quad \text{where} \quad b' = \left(\frac{b_2'}{a_1} + \frac{b_1'}{a_2}\right)\left(\frac{b_2''}{a_3} + \frac{b_3'}{a_2}\right),$$

and

$$\frac{e^3 c^2}{3.999\, m^2} < 32.5175\, m^{-2/3}.$$

Then — in the non-degenerate case —

$$\log|\Lambda| > -KL\lambda - \log(KL) \geq -\big(KL + \log(KL)\big)\lambda.$$

since $\rho \geq e$, which gives

$$\log|\Lambda| > -1.000004\, KL\lambda \geq -6109.598\,\lambda^{-4} \times \Omega \times \mathcal{D}^2\, \log^2 \mathcal{B}.$$

For example, if we choose $\rho = 5.296$, then $\log\rho = 1.6669518\ldots$ and

$$L \geq \left\lceil \frac{50\mathcal{D}}{\lambda} \right\rceil = 30\,\mathcal{D}, \qquad 5 \leq \mu \leq \frac{155}{30}, \qquad 49.39124 \leq m < 53,$$

as wanted, and then

$$\log|\Lambda| > -790.9478\,\Omega\cdot\mathcal{D}^2\, \log^2\mathcal{B},$$

where

$$\log\mathcal{B} = \max\{10/\mathcal{D}, \log\tilde b\} \quad \text{and} \quad \tilde b \leq 2.4156\, b'.$$

With this choice we take

$$a_i = \max\{4,\, \rho\,\ell_i - \log|\alpha_i| + 2\mathcal{D}h_i\}, \qquad \text{for } i = 1, 2, 3,$$

(with the obvious notation $\ell_i = |\log\alpha_i|$), and then

$$\log|\Lambda| > -6327.59\mathcal{D}^2 \log^2\mathcal{B} \prod_{i=1}^{3} \max\{2, \mathcal{D}h_i + 2.648\,\ell_i\} \geq -307{,}187\mathcal{D}^5 \log^2\mathcal{B} \prod_{i=1}^{3} \max\{0.55, h_i, \ell_i/\mathcal{D}\},$$

where

$$\log\mathcal{B} = \max\{0.882 + \log b',\, 10/\mathcal{D}\}.$$

We are now ready to state our explicit estimate.

Proposition 5.2. *We consider three non-zero algebraic numbers α_1, α_2 and α_3, which are either all real and >1 or all complex of modulus one and all $\neq 1$. Moreover, we assume that either the three numbers α_1, α_2 and α_3 are multiplicatively independent, or two of these numbers are multiplicatively independent and the third one is a root of unity. Put*

$$\mathcal{D} = [\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}] \;/\; [\mathbb{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{R}].$$

We also consider three coprime positive rational integers b_1, b_2, b_3, and the linear form

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1 - b_3 \log \alpha_3,$$

where the logarithms of the α_i are arbitrary determinations of the logarithm, but which are all real or all purely imaginary.

And we assume also that

$$b_2 |\log \alpha_2| = b_1\ |\log \alpha_1| + b_3\ |\log \alpha_3| \pm |\Lambda|\,.$$

We put

$$d_1 = \gcd(b_1, b_2), \quad d_3 = \gcd(b_3, b_2), \quad b_2 = d_1 b_2' = d_3 b_2''.$$

Let a_1, a_2 and a_3 be real numbers such that

$$a_i \geq \max\{4,\, 5.296\, \ell_i - \log|\alpha_i| + 2\mathcal{D}\, \mathrm{h}(\alpha_i)\}, \quad \textit{where}\ \ \ell_i = |\log \alpha_i|\,, \quad i = 1,\, 2,\, 3,$$

and

$$\Omega := a_1 a_2 a_3 \geq 100.$$

Put

$$b' = \left(\frac{b_1'}{a_2} + \frac{b_2'}{a_1}\right)\left(\frac{b_3''}{a_2} + \frac{b_2''}{a_3}\right)$$

and

$$\log \mathcal{B} = \max\bigl\{0.882 + \log b',\, 10/\mathcal{D}\bigr\}, \qquad \Omega = a_1 a_2 a_3.$$

Then either

$$\log|\Lambda| > -\,790.95 \cdot \Omega \cdot \mathcal{D}^2 \log^2 \mathcal{B} > -\,307{,}187 \times \mathcal{D}^5 \log^2 \mathcal{B} \times \prod_{i=1}^{3} \max\{0.55,\, h_i,\, \ell_i/\mathcal{D}\},$$

or the following condition holds:

— **either** *there exist two non-zero rational integers r_0 and s_0 such that*

$$r_0 b_2 = s_0 b_1$$

with

$$|r_0| \leq 5.61\, (\mathcal{D} \log \mathcal{B})^{1/3} a_2 \quad \textit{and} \quad |s_0| \leq 5.61\, (\mathcal{D} \log \mathcal{B})^{1/3} a_1,$$

— **or** *there exist rational integers r_1, s_1, t_1 and t_2, with $r_1 s_1 \neq 0$, such that*

$$(t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2, \qquad \gcd(r_1, t_1) = \gcd(s_1, t_2) = 1,$$

which also satisfy

$$|r_1 s_1| \leq \delta \cdot 5.61\, (\mathcal{D} \log \mathcal{B})^{1/3} a_3, \quad |s_1 t_1| \leq \delta \cdot 5.61\, (\mathcal{D} \log \mathcal{B})^{1/3} a_1, \quad |r_1 t_2| \leq \delta \cdot 5.61\, (\mathcal{D} \log \mathcal{B})^{1/3} a_2,$$

where

$$\delta = \gcd(r_1, s_1).$$

Moreover, when $t_1 = 0$ we can take $r_1 = 1$, and when $t_2 = 0$ we can take $s_1 = 1$.

Proof. The only remaining point is concerned with condition (**C3**) of Theorem 2. First, notice that, for $\chi = 1$,

$$\mathcal{V} := \big((R_1+1)(S_1+1)(T_1+1)\big)^{1/2} \geq c_1^{3/2}\Omega = mL\Omega.$$

Moreover,

$$\max\{R_1, S_1, T_1\} \leq c_1\Omega/a.$$

Thus,

$$\frac{\mathcal{V}}{\max\{R_1, S_1, T_1\}} \geq a\sqrt{c_1} = a(mL)^{1/3},$$

and we have

$$\mathcal{V} > \max\{R_1, S_1, T_1\}.$$

(Remark: it would certainly be more clever to choose some $\chi < 1$, but this does not improve a lot the result. Nevertheless, the freedom given by this parameter may be useful in concrete cases: see the two examples given below where we choose χ in order that the estimate obtained in the non-degenerate case for three logarithms is essentially the same than the estimate obtained in the degenerate case with lower bounds of linear forms in two logarithms. Notice that the first estimate is better when χ is small, whereas the second one increases when when χ decreases.)

We have chosen

$$R_1 = \lfloor c_1 a_2 a_3 \rfloor,$$

hence

$$R_1 + 1 \leq c_1 a_2 a_3 \left(1 + \frac{1}{16(mL)^{2/3}}\right) < 1.0005\, c_1 a_2 a_3,$$

since our choices give $m \geq 49$ and $L \geq 30$. For the same reasons,

$$S_1 + 1 < 1.0005\, c_1 a_1 a_3,$$

which implies[4]

$$B_T := \frac{(R_1+1)(S_1+1)}{\mathcal{V} - \max\{R_1, S_1, T_1\}} < \frac{1.0005^2}{1 - \big(4\sqrt{c_1}\big)^{-1}}\,\sqrt{c_1}\, a_3 < 3.843\, L^{1/3} a_3 < 5.61\,(\mathcal{D}\,\log\mathcal{B})^{1/3} a_3.$$

Similarly,

$$B_R := \frac{(S_1+1)(T_1+1)}{\mathcal{V} - \max\{R_1, S_1, T_1\}} < 3.843\, L^{1/3} a_3 < 5.61\,(\mathcal{D}\,\log\mathcal{B})^{1/3} a_1$$

and

$$B_S := \frac{(R_1+1)(T_1+1)}{\mathcal{V} - \max\{R_1, S_1, T_1\}} < 3.843\, L^{1/3} a_3 < 5.61\,(\mathcal{D}\,\log\mathcal{B})^{1/3} a_2.$$

This proves that our last claims are consequences of condition (**C3**) and this ends the verification of the result. $\square$

It may be interesting to compare the above result with the main theorem of [1], which is the following.

[4]Since the function $x \mapsto \frac{1}{1-(4\sqrt{x})^{-1}}\sqrt{x}$ is increasing for $x > 1$, for the term in the middles the worst case is obtained when c_1 is maximum, *i.e.* for m maximum, and we get an upper bound repleacing m by 53, which implies the second inequality. The last one comes from the definition of L, namely $L = \lceil (5/\lambda)\,\mathcal{D}\,\log\mathcal{B}\rceil$.

Proposition 5.3. *We consider three non-zero rational numbers* α_1*,* α_2 *and* α_3*, which are all* >1 *and multiplicatively independent.*

We also consider three positive rational integers b_1*,* b_2*,* b_3 *with* $\gcd(b_1, b_2, b_3) = 1$*, and the linear form*

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1 - b_3 \log \alpha_3,$$

where the logarithms of the α_i *are the ordinary determinations of the logarithm. We assume that* Λ *is non-zero and that*

$$b_2 |\log \alpha_2| = b_1 \ |\log \alpha_1| + b_3 \ |\log \alpha_3| \pm |\Lambda| \,.$$

Let a_1'*,* a_2' *and* a_3' *be real numbers such that*

$$a_i' \geq \max\{1, \mathrm{h}(\alpha_i)\}, \qquad i = 1,\, 2,\, 3.$$

Put

$$b'' = \left(\frac{b_1}{a_2'} + \frac{b_2}{a_1'}\right)\left(\frac{b_3}{a_2'} + \frac{b_2}{a_3'}\right)$$

and

$$\mathcal{B}' = \max\{\log b'',\, 10\}, \qquad \Omega' = a_1' a_2' a_3'.$$

Then either

$$\log|\Lambda| > -4.5 \times 10^5 \times \Omega' \log^2 \mathcal{B}',$$

or there exists a non trivial relation

$$u_1 b_1 + u_2 b_2 + u_3 b_3 = 0$$

over the rational integers with

$$|u_1|\,,\ |u_3| \leq 10^4 \times \log \mathcal{B}' \times a_1' a_3', \quad \text{and} \quad |u_2| \leq 10^4 \times \log \mathcal{B}' \times a_2' \times \min\{a_1', a_3'\}.$$

With the same hypotheses, our result gives

— **either**

$$\log|\Lambda| > -1.974 \times 10^5 \times h_1 h_2 h_3 \times \log^2 \mathcal{B} \geq -1.974 \times 10^5 \times \Omega' \ \log^2 \mathcal{B}',$$

where

$$\log \mathcal{B} = \max\{10, 0.882 + \log b'\} \leq \max\{10, \log b'' - 0.59\},$$

[take $a_i = 6.296\, h_i$ for $i = 1,\, 2,\, 3$ and notice that $a_i \geq 6.296 \log 2 > 4.364$ for $i = 1,\, 2,\, 3$ and also that $a_1 a_2 a_3 \geq 6.296^3 \log 2 \cdot \log 3 \cdot \log 5 > 300$]

— **or** there exists a non trivial relation

$$u_1 b_1 + u_2 b_2 + u_3 b_3$$

over the rational integers with

$$|u_i| < 36 \left(\max\{10,\, 0.882 + \log b'\}\right)^{1/3} h_i, \quad i = 1,\, 2,\, 3.$$

This result may also be compared to the estimate implied by Matveev's theorem., which gives **unconditionally**

$$\log|\Lambda| > -1.7 \times 10^{10} \times \mathrm{h}_1 h_2 h_3 \times \log(4.08\, B),$$

where

$$B = \max\bigl\{1,\, \max\{b_j h_j / h_i\,;\, 1 \leq j \leq 3\}\bigr\}.$$

6. A first example

Using the previous estimates, we can prove:

Theorem 6.1. *All the solutions of equation*

$$x^n - 2^\alpha 5^\beta y^n = \pm 1$$

in integers $x,\ y \geq 1,\ n \geq 3$ *prime and* $0 \leq \beta < n$, *with* $\alpha = 1,\ 2,\ 3$ *satisfy*

$$n \leq 3.88 \cdot 10^7.$$

Proof. It is clear that $x > y$. First, we give an upper bound for the exponent n using Matveev's estimate.

Let

$$\Lambda_1 = 1 - \frac{2^\alpha 5^\beta y^n}{x^n},$$

so that $|\Lambda_1| = 1/x^n$. Set

$$\Lambda = n \log x/y - \alpha \log 2 - \beta \log 5,$$

then

$$|\Lambda| \leq 2x^{-n}.$$

Matveev's theorem gives (for $x \geq 5$)

$$\log|\Lambda| \geq -5 \cdot 16^5 \cdot 1.5^2 \cdot e^4 \cdot (20.2 + 5.5 \log 3) \cdot \log x \cdot \log 2 \cdot \log 5 \cdot (\log n + 1.41)$$

and we obtain $n \leq 5.36 \cdot 10^{11}$.

We suppose

$$n > 2 \times 10^7,$$

and it is possible to restrict our study to the case (see...)

$$\log x > 5000.$$

For this linear form Λ in three logarithms, we keep the notation of the previous parts. Set

$$\alpha_1 = 2, \quad \alpha_2 = x/y, \quad \alpha_3 = 5.$$

We take $\chi = 0.5$ and

$$L = 100, \quad m = 41.28955, \quad \rho_1 = \rho = 7,$$

$$a_1 = (\rho + 1) \log 2, \quad a_2 = 6(\log 2 + \log 5) + 2 \log x, \quad a_3 = (\rho + 1) \log 5,$$

$$b_1 = \alpha = 1,\ 2,\ 3, \quad b_2 = n, \quad b_3 = \beta,$$

and finally

$$c_1 = 162.133741\ldots, \quad c_2 = 324.267482\ldots, \quad c_3 = 2170.753371\ldots$$

Using these values we get

$$R_1 = \lfloor c_1 a_2 a_3 \rfloor = \lfloor 4176.8434 \log x \rfloor, \quad R_2 = \lfloor c_2 a_2 a_3 \rfloor = \lfloor 8353.6867 \log x \rfloor,$$

and

$$R_3 = \lfloor c_3 a_2 a_3 \rfloor = \lfloor 55922.3320 \log x \rfloor,$$

further

$$S_1 = \lfloor c_1 a_1 a_3 \rfloor = 11575, \quad S_2 = \lfloor c_2 a_1 a_3 \rfloor = 23151, \quad S_3 = \lfloor c_3 a_1 a_3 \rfloor = 154985$$

and finally

$$T_1 = \lfloor c_1 a_1 a_2 \rfloor = \lfloor 1798.8684 \log x \rfloor, \quad T_2 = \lfloor c_2 a_1 a_2 \rfloor = \lfloor 3597.7370 \log x \rfloor,$$

and

$$T_3 = \lfloor c_3 a_1 a_2 \rfloor = \lfloor 24084.4374 \log x \rfloor.$$

Put

$$\mathcal{V} = \big((R_1+1)(S_1+1)(T_1+1)\big)^{1/2},$$

then

$$\chi\mathcal{V} \geq \max\{R_1+S_1+1,\, S_1+T_1+1,\, R_1+T_1+1\},$$

and

$$B_R := \frac{(S_1+1)(T_1+1)}{\chi\mathcal{V} - \max\{S_1,T_1\}} \leq 143, \quad B_T := \frac{(R_1+1)(S_1+1)}{\chi\mathcal{V} - \max\{R_1,S_1\}} \leq 338,$$

independently of x (for $x > e^{5000}$).

We have

$$K = \lfloor Lma_1a_2a_3 \rfloor = \lfloor 589834.7288 \log x \rfloor.$$

As seen above, these choices imply that the conditions (i)–(iv) of Theorem 2 hold. Moreover the above choices have been made so that condition (2) holds.

Thus we have

$$\log|\Lambda| \geq -KL\log\rho - \log(KL) > -114.777 \cdot 10^6 \log x,$$

and we get

$$n < 115 \cdot 10^6.$$

In the cases (**C1**) and (**C2**) we obtain

$$n \leq \max\{S_1, S_2\} < 30000,$$

which is excluded since we assume $n > 10^7$. Moreover, the first case case of condition (**C3**), *i.e.* the case $r_0b_2 = s_0b_1$ cannot hold because of the bound on s_0 (namely $|s_0| \leq B_R$) and the fact that $b_2 = n$ is prime. On supposing that (**C3**) holds then we have necessarily

$$s't'\alpha + r't''n + r's'\beta = 0,$$

where $\alpha = 1$, 2, 3 and the factors of β, n, b_3 are bounded as in the main theorem. We have

$$\gcd(r',s') = \delta, \quad \gcd(r',t') = \gcd(s',t'') = 1$$

and we put

$$r' = \delta r_1', \quad s' = \delta s_1'.$$

With this notation

$$s_1't'\alpha + r_1't''n + \delta r_1's_1'\beta = 0.$$

Then $s_1' \mid t''n$ and using that n is prime, $|s_1'| < n$ and $\gcd(s',t'') = 1$ we get $s_1' = 1$, and thus

$$t'\alpha + r_1't''n + \delta r_1'\beta = 0,$$

where $\gcd(r_1',t') = 1$. Since $\alpha \in \{1,2,3\}$, we have $|r_1'| = 1$ or $|r_1'| = \alpha$. In the first case,

$$\pm t'\alpha + t''n + \delta\beta = 0,$$

and, in the second case,

$$\pm t' + t''n + \delta\beta = 0.$$

Clearly, t'' and δ are not of the same sign and we may assume that $\delta > 0$ and $t'' \leq 0$.

This implies

$$|t''| \leq \delta + \frac{3\,|t'|}{n} < \delta + 1,$$

where

$$0 < |r_1'\delta| \leq 143.$$

We rewrite Λ as a linear form in two logarithms:

$$\Lambda = n\log\left(\left(\frac{x}{y}\right)^{t'} 2^{r_1' t''}\right) - \beta\log\left(5^{t'}2^{-r_1'\delta}\right).$$

Proposition 4.1 (applied four consecutive times with the interpolation radius $\rho_2 = 18$) yields the estimate

$$n < 39 \times 10^6.$$

Thus we have proved that $n \le 1.15 \cdot 10^8$.

From this upper bound for n we iterate four times this process, but with the choices $L = 90$, $\chi = 0.29$, $\rho_1 = 7.1$ and $\rho_2 = 20$. We get a better bound for n, namely, $n < 3.88 \times 10^7$. $\square$

7. A second example

Using the previous estimates, we can also prove:

Theorem 7.1. *All the solutions of equation*

$$2^\alpha x^n - 5^\beta y^n = \pm 1$$

in integers x, $y \ge 1$, where $n \ge 3$ is prime and $0 \le \alpha$, $\beta < n$, satisfy

$$n \le 4.96 \cdot 10^7.$$

if we suppose $\max\{x, y\} > e^{50}$. *Whereas the weaker assumption* $\max\{x, y\} \ge 3$ *leads to*

$$n \le 3.3 \cdot 10^8.$$

Proof. In a first time, we consider more generally the Diophantine equation

$$p^\alpha x^n - q^\beta y^n = 1.$$

We consider the linear form

$$\pm\Lambda = \alpha\log p + n\log(x/y) - \beta\log q.$$

If $x > y$ we write

$$\Lambda = \beta\log q - n\log(x/y) - \alpha\log p$$

and define

$$\alpha_1 = x/y,\ \alpha_2 = q,\ \alpha_3 = p, \qquad b_1 = n,\ b_2 = \beta,\ b_3 = \alpha,$$

according to the conventions of our main theorem.

Whereas, if $x < y$ then we write

$$\Lambda = \alpha\log p - n\log(y/x) - \beta\log q$$

and define

$$\alpha_1 = y/x,\ \alpha_2 = p,\ \alpha_3 = q, \qquad b_1 = n,\ b_2 = \alpha,\ b_3 = \beta,$$

again according to the conventions of our main theorem. Except for notation, this is similar to the first case. In both cases we have

$$\Lambda = \varepsilon\alpha\log p - n\,|\log(x/y)| - \varepsilon\beta\log q,$$

where $\varepsilon = \pm 1$. Changing notation if necessary, we limit our study to the case $\varepsilon = 1$.

We put

$$z = \max\{x, y\}.$$

It is easy to see that
$$|\Lambda| < 2z^{-n}.$$

First, we give an upper bound for the exponent n using Matveev's theorem, which gives (for $z \geq \max\{p, q\}$)
$$\log|\Lambda| \geq -5 \cdot 16^5 \cdot 1.5^2 \cdot e^4 \cdot (20.2 + 5.5 \log 3) \cdot \log z \cdot \log p \cdot \log q \cdot (\log n + 1.41)$$
and — for example — we obtain $n \leq 5.36 \cdot 10^{11}$ when $\{p, q\} = \{2, 5\}$ and $z \geq 5$.

We suppose
$$n > 2 \times 10^7,$$
and we first restrict our study to the case
$$\log z > 50.$$

Now we apply our result on linear forms for $p = 5$ and $q = 2$, taking $\chi = 1$ and
$$L = 100, \quad m = 47.6623398, \quad \rho_1 = \rho = 7,$$
$$a_1 = (\rho - 1)\log p + 2\log z, \quad a_2 = (\rho + 1)\log p, \quad a_3 = (\rho + 1)\log q,$$
$$b_1 = n, \quad b_2 = \alpha, \quad b_3 = \beta,$$
and finally
$$c_1 = 283.2154268\ldots, \quad c_2 = c2 = 356.82907799\ldots, \quad c_3 = 2388.73142356\ldots$$

Using these constants we get
$$R_1 = \lfloor c_1 a_2 a_3 \rfloor = 20220, \quad R_2 = \lfloor c_2 a_2 a_3 \rfloor = 25476, \quad R_3 = \lfloor c_3 a_2 a_3 \rfloor = 170548,$$
further,
$$S_1 = \lfloor c_1 a_1 a_3 \rfloor \leq \lfloor 3444.261 \log z \rfloor, \quad S_2 = \lfloor c_2 a_1 a_3 \rfloor \leq \lfloor 4339.501 \log z \rfloor$$
and
$$S_3 = \lfloor c_3 a_1 a_3 \rfloor \leq \lfloor 29050.101 \log z \rfloor,$$
and finally
$$T_1 = \lfloor c_1 a_1 a_2 \rfloor \leq \lfloor 7997.341 \log z \rfloor, \quad T_2 = \lfloor c_2 a_1 a_2 \rfloor \leq \lfloor 100726.021 \log z \rfloor,$$
and
$$T_3 = \lfloor c_3 a_1 a_2 \rfloor \leq \lfloor 67452.241 \log z \rfloor.$$
Put
$$\mathcal{V} = \big((R_1 + 1)(S_1 + 1)(T_1 + 1)\big)^{1/2},$$
then
$$\chi\mathcal{V} \geq \max\{R_1 + S_1 + 1,\, S_1 + T_1 + 1,\, R_1 + T_1 + 1\},$$
and
$$B_S := \frac{(R_1 + 1)(T_1 + 1)}{\chi\mathcal{V} - \max\{R_1, T_1\}} \leq 220, \quad B_T := \frac{(R_1 + 1)(S_1 + 1)}{\chi\mathcal{V} - \max\{R_1, S_1\}} \leq 94,$$
independently of z (for $z > e^{50}$).

As seen above, these choices imply that the conditions (i)–(iv) hold. Moreover, these choices have been made (with the help of a computer) so that inequality (2) holds.

Thus we have
$$\log|\Lambda| \geq -KL\log\rho - \log(KL) > -145.25 \cdot 10^6 \log z,$$
and
$$n < 145.3 \cdot 10^6.$$

In the cases (**C1**) or (**C2**) we have

$$n \le \max\{R_1, R_2\} < 30000,$$

which is excluded since we assume $n > 10^7$. Moreover, the first case of condition (**C3**), *i.e.* $r_0 b_2 = s_0 b_1$ cannot hold because of the bound on r_0 (namely $|r_0| \le B_S$) and the fact that $b_1 = n$ is prime. On supposing that (**C3**) holds then we necessarily have

$$s' t' n + r' t'' \alpha + r' s' \beta = 0,$$

where the factors of α, n and β are bounded as in the main theorem. We have

$$\gcd(r', s') = \delta, \quad \gcd(r', t') = \gcd(s', t'') = 1$$

and we put

$$r' = \delta r_1', \quad s' = \delta s_1'.$$

With this notation

$$s_1' t' n + r_1' t'' \alpha + \delta r_1' s_1' \beta = 0.$$

Then $r_1' \mid t' n$ and using that n is prime, $|r_1'| < n$ and $\gcd(r_1', t') = 1$ we get $r_1' = 1$, and thus

$$s_1' t' n + t'' \alpha + \delta s_1' \beta = 0,$$

where $\gcd(s_1', t'') = 1$.

This implies

$$|s_1' t'| \le |t''| + |\delta s_1'|\,,$$

where

$$|t''| \le B_S \quad \text{and} \quad |\delta s_1'| \le B_T.$$

Thus

$$|s_1' t'| \le B_S + B_T \le 314,$$

whenever is $z \ge e^{50}$.

We rewrite $t'' \Lambda$ as a linear form in two logarithms:

$$t'' \Lambda = \beta \log\left(5^{t''} \times 2^{\delta s_1'}\right) - n \log\left((x/y)^{\pm t''} \times 2^{-s_1' t'}\right).$$

Proposition 4.1 (applied twice with the choice $\rho_2 = 20$) yields

$$n < 58 \times 10^6.$$

Thus we have proved that $n \le 1.46 \cdot 10^8$.

From this upper bound for n we iterate four times this process, choosing now $L = 90$ and $\chi = 0.65$, but keeping $\rho_1 = 7$ and $\rho_2 = 20$. We get a better bound for n, namely $n < 4.96 \times 10^7$ in the first case.

In the second case, the conclusion is obtained with the choices $L = 90$, $\rho_1 = 7$, $\chi = 0.91$ and $\rho_2 = 6$. $\square$

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