METRIC CONSIDERATIONS CONCERNING THE MIXED LITTLEWOOD CONJECTURE

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Abstract. The main goal of this note is to develop a metrical theory of Diophantine approximation within the framework of the de Mathan-Teulié Conjecture – also known as the 'Mixed Littlewood Conjecture'. Let p be a prime. A consequence of our main result is that, for almost every real number α

$$\liminf_{n\to\infty} n(\log n)^2|n|_p \, \|n\alpha\| = 0 \ .$$

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1. Introduction

The famous Littlewood Conjecture in the theory of simultaneous Diophantine approximation dates back to the 1930's and asserts that for every pair (α, β) of real numbers, we have that

$$\liminf_{n\to\infty} n\|n\alpha\|\|n\beta\| = 0. \tag{1}$$

Here and throughout, $\| \, . \, \|$ denotes the distance to the nearest integer. For background and recent 'progress' concerning this fundamental problem see [11, 19]. However, it is appropriate to highlight the result of Einsiedler, Katok & Lindenstrauss that states that the set of pairs (α, β) for which (1) is not satisfied is of zero Hausdorff dimension; i.e. any exceptional set to the Littlewood Conjecture has to be of zero dimension.

In 1962, Gallagher established a result which implies that if $\psi : \mathbb{N} \to \mathbb{R}$ is a non-negative decreasing function, then for almost every (α, β) the inequality

$$\|n\alpha\|\|n\beta\| \leq \psi(n)$$

has infinitely (resp. finitely) many solutions $n \in \mathbb{N}$ if $\sum_{n\in\mathbb{N}} \psi(n) \log n$ diverges (resp. converges). In particular, it follows that

$$\liminf_{n\to\infty} n \, (\log n)^2 \|n\alpha\|\|n\beta\| = 0 \tag{2}$$

for almost every pair (α, β) of real numbers. Thus from a purely metrical point of view, Gallagher's result enables us to 'beat' Littlewood's assertion (1) by a logarithm squared.

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The main goal of this note is to obtain a Gallagher type theorem within the framework of the recent de Mathan-Teulié Conjecture [18] – also known as the 'Mixed Littlewood Conjecture'. In the following p is a prime number and $|\,.\,|_p$ is the usual p-adic norm. B. de Mathan and O. Teulié conjectured that for every real number α, we have that

$$\liminf_{n\to\infty} n|n|_p\|n\alpha\| = 0\ . \tag{3}$$

Various partial results exist – see [8, 12] and references within. Indeed, Einsiedler & Kleinbock have shown that any exceptional set to the de Mathan-Teulié Conjecture has to be of zero dimension. Furthermore, let $p_1, \ldots, p_k$ be distinct prime numbers. They also deduce, via a theorem of Furstenberg, that if $k \geq 2$ then for every real number α

$$\liminf_{n\to\infty} n|n|_{p_1}\cdots|n|_{p_k}\|n\alpha\| = 0\ . \tag{4}$$

This statement can be strengthened from a metrical point of view. A consequence of our Gallagher type theorem is that for $k \geq 1$,

$$\liminf_{n\to\infty} n\,(\log n)^{k+1}|n|_{p_1}\cdots|n|_{p_k}\|n\alpha\| = 0 \tag{5}$$

for almost every real number α. Thus, just as with Littlewood's conjecture, the metric statement 'beats' the de Mathan–Teulié assertion (3) by a logarithm squared.

Theorem 1. *Let $p_1, \ldots, p_k$ be distinct prime numbers and let $\psi : \mathbb{N} \to \mathbb{R}$ be a non-negative decreasing function. Then, for almost every real number α the inequality*

$$|n|_{p_1}\cdots|n|_{p_k}\|n\alpha\| \leq \psi(n)$$

has infinitely (resp. finitely) many solutions $n \in \mathbb{N}$ if

$$\sum_{n\in\mathbb{N}} (\log n)^k \psi(n)$$

diverges (resp. converges).

This Gallagher type theorem will be deduced as a consequence of our main result.

Theorem 2. *Let $p_1, \ldots, p_k$ be distinct prime numbers and $f_1, \ldots, f_k : \mathbb{R} \to \mathbb{R}$ be positive functions. Furthermore, let $\psi : \mathbb{N} \to \mathbb{R}$ be a non-negative decreasing function. Then, for almost every real number α the inequality*

$$f_1(|n|_{p_1})\cdots f_k(|n|_{p_k})\|n\alpha\| \leq \psi(n) \tag{6}$$

has infinitely (resp. finitely) many solutions $n \in \mathbb{N}$ if

$$\sum_{n\in\mathbb{N}} \frac{\psi(n)}{f_1(|n|_{p_1})\cdots f_k(|n|_{p_k})} \tag{7}$$

diverges (resp. converges).

The final section of the paper is devoted to discussing various related metrical results and open problems.

2. Theorem 2 ⇒ Theorem 1

With reference to Theorem 2, let each of the functions $f_1, \dots, f_k$ be the identity function. Then, Theorem 1 trivially follows from Theorem 2 if we can show that

$$\sum_{n\in\mathbb{N}} (\log n)^k \psi(n) = \infty \quad \Longleftrightarrow \quad \sum_{n\in\mathbb{N}} \frac{\psi(n)}{|n|_{p_1}\cdots|n|_{p_k}} = \infty\,.$$

Actually we will prove the following more general lemma, which will also be needed in Section 4.1.

Lemma 1. *Suppose that $s\in[0,1]$ and that $p_1,\dots,p_k$ are distinct primes. Furthermore, suppose that $\psi:\mathbb{N}\to\mathbb{R}$ is a non-negative decreasing function. If $s<1$ then the sum*

$$\sum_{n\in\mathbb{N}} n\left(\frac{\psi(n)}{n|n|_{p_1}\cdots|n|_{p_k}}\right)^s \tag{8}$$

diverges if and only if

$$\sum_{n\in\mathbb{N}} n^{1-s}\psi(n)^s$$

diverges. If $s=1$, then the divergence of (8) is equivalent to that of

$$\sum_{n\in\mathbb{N}} (\log n)^k \psi(n).$$

Proof. For the duration of the proof let us write $N := p_1\cdots p_k$. For one direction of the proof we begin by using the monotonicity of ψ to deduce that

$$\begin{aligned}\sum_{n\in\mathbb{N}} n\left(\frac{\psi(n)}{n|n|_{p_1}\cdots|n|_{p_k}}\right)^s &= \sum_{a_1,\dots,a_k\geq 0}\ \sum_{\substack{m\in\mathbb{N}\\(m,N)=1}} m^{1-s}p_1^{a_1}\cdots p_k^{a_k}\psi(p_1^{a_1}\cdots p_k^{a_k}m)^s \\ &\geq \sum_{a_1,\dots,a_k\geq 0}\ \sum_{\substack{m\in\mathbb{N}\\(m,N)=1}} m^{1-s} \sum_{\ell=p_1^{a_1}\cdots p_k^{a_k}m}^{p_1^{a_1}\cdots p_k^{a_k}(m+1)-1} \psi(\ell)^s.\end{aligned} \tag{9}$$

Upon interchanging the orders of summation it is apparent that the latter quantity is equal to

$$\sum_{\ell\in\mathbb{N}}\psi(\ell)^s \sum_{\substack{m\in\mathbb{N}\\(m,N)=1}}\ \sum_{\substack{a_1,\dots,a_k\geq 0\\ \ell/(m+1)<p_1^{a_1}\cdots p_k^{a_k}\leq \ell/m}} m^{1-s}.$$

Next by Möbius inversion and partial summation this becomes

$$\begin{aligned}
&\sum_{\ell\in\mathbb{N}}\psi(\ell)^s\sum_{d|N}\mu(d)d^{1-s}\sum_{m\in\mathbb{N}}\sum_{\substack{a_1,\ldots,a_k\geq 0\\ \ell/(md+1)<p_1^{a_1}\cdots p_k^{a_k}\leq \ell/md}} m^{1-s}\\
&\quad=\sum_{\ell\in\mathbb{N}}\psi(\ell)^s\sum_{a_1,\ldots,a_k\geq 0}\sum_{d|N}\mu(d)d^{1-s}\sum_{\substack{m\in\mathbb{N}\\ \ell/(dp_1^{a_1}\cdots p_k^{a_k})-1/d<m\leq \ell/(dp_1^{a_1}\cdots p_k^{a_k})}} m^{1-s}\\
&\quad=\lim_{L\to\infty}\left(\sum_{\ell\leq L}\left(\psi(\ell)^s-\psi(\ell+1)^s\right)\sum_{j=1}^{\ell}\sum_{a_1,\ldots,a_k\geq 0}\sum_{d|N}\mu(d)d^{1-s}\sum_{\substack{m\in\mathbb{N}\\ j/(dp_1^{a_1}\cdots p_k^{a_k})-1/d<m\leq j/(dp_1^{a_1}\cdots p_k^{a_k})}} m^{1-s}\right.\\
&\qquad\qquad\left.+\,\psi(L+1)^s\sum_{j=1}^{L}\sum_{a_1,\ldots,a_k\geq 0}\sum_{d|N}\mu(d)d^{1-s}\sum_{\substack{m\in\mathbb{N}\\ j/(dp_1^{a_1}\cdots p_k^{a_k})-1/d<m\leq j/(dp_1^{a_1}\cdots p_k^{a_k})}} m^{1-s}\right). \qquad (10)
\end{aligned}$$

Now we focus on the sums

$$\sum_{j=1}^{\ell}\sum_{\substack{m\in\mathbb{N}\\ j/(dp_1^{a_1}\cdots p_k^{a_k})-1/d<m\leq j/(dp_1^{a_1}\cdots p_k^{a_k})}} m^{1-s}. \qquad (11)$$

If we write each j in the first sum as $j=idp_1^{a_1}\cdots p_k^{a_k}+r$ with $0\leq i\leq \ell/dp_1^{a_1}\cdots p_k^{a_k}$ and $0\leq r<dp_1^{a_1}\cdots p_k^{a_k}$ then the sum over m is either i^{1-s} or 0, depending on whether or not $r<p_1^{a_1}\cdots p_k^{a_k}$. Thus (11) is equal to

$$\begin{aligned}
&p_1^{a_1}\cdots p_k^{a_k}\sum_{1\leq i<\lfloor \ell/dp_1^{a_1}\cdots p_k^{a_k}\rfloor} i^{1-s}\\
&\quad+\min\left\{p_1^{a_1}\cdots p_k^{a_k},1+\ell-\left\lfloor\frac{\ell}{dp_1^{a_1}\cdots p_k^{a_k}}\right\rfloor dp_1^{a_1}\cdots p_k^{a_k}\right\}\cdot\left\lfloor\frac{\ell}{dp_1^{a_1}\cdots p_k^{a_k}}\right\rfloor^{1-s}. \qquad (12)
\end{aligned}$$

Here we break our analysis into two cases. If $s=1$ then (12) equals

$$\ell/d+O(p_1^{a_1}\cdots p_k^{a_k}),$$

and returning to (10) we find that it is

$$= \lim_{L\to\infty}\left(\sum_{\ell\le L}(\psi(\ell)-\psi(\ell+1))\sum_{\substack{a_1,\dots,a_k\ge 0\\ p_1^{a_1}\cdots p_k^{a_k}\le \ell}}\left(\frac{\varphi(N)\ell}{N}+O\left(\sum_{d|N}|\mu(d)|\,p_1^{a_1}\cdots p_k^{a_k}\right)\right)\right.$$

$$\left.+\psi(L+1)\sum_{\substack{a_1,\dots,a_k\ge 0\\ p_1^{a_1}\cdots p_k^{a_k}\le L}}\left(\frac{\varphi(N)L}{N}+O\left(\sum_{d|N}|\mu(d)|\,p_1^{a_1}\cdots p_k^{a_k}\right)\right)\right).$$

Now note that (since $N := p_1\cdots p_k$ is fixed) the error terms in the inner sums are

$$\ll \sum_{\substack{a_1,\dots,a_k\ge 0\\ p_1^{a_1}\cdots p_k^{a_k}\le \ell}} p_1^{a_1}\cdots p_k^{a_k} \ll \ell(\log \ell)^{k-1}.$$

This inequality can easily be verified by induction on k and we emphasize that the implied constant is dependent only on N. For the main terms we note that

$$\sum_{\substack{a_1,\dots,a_k\ge 0\\ p_1^{a_1}\cdots p_k^{a_k}\le \ell}} 1 \gg (\log \ell)^k,$$

and thus (10) is

$$\gg \lim_{L\to\infty}\left(\sum_{\ell\le L}(\psi(\ell)-\psi(\ell+1))\,\ell(\log \ell)^k+\psi(L+1)(\log L)^k\right)\gg \sum_{n\in\mathbb{N}}(\log n)^k\psi(n).$$

For the case when $0<s<1$ we have that

$$p_1^{a_1}\cdots p_k^{a_k}\sum_{1\le i\le \ell/(dp_1^{a_1}\cdots p_k^{a_k})} i^{1-s} = \frac{\ell^{2-s}}{(2-s)d^{2-s}\left(p_1^{a_1}\cdots p_k^{a_k}\right)^{1-s}}+O\left(\left(\frac{\ell}{d}\right)^{1-s}(p_1^{a_1}\cdots p_k^{a_k})^s\right).$$

It follows from this, (11), and (12) that

$$\begin{aligned}
&\sum_{j=1}^{\ell}\sum_{a_1,\ldots,a_k\geq 0}\sum_{d|N}\mu(d)d^{1-s}\sum_{\substack{m\in\mathbb{N}\\ j/(dp_1^{a_1}\cdots p_k^{a_k})-1/d<m\leq j/(dp_1^{a_1}\cdots p_k^{a_k})}} m^{1-s}\\
&\qquad=\sum_{\substack{a_1,\ldots,a_k\geq 0\\ p_1^{a_1}\cdots p_k^{a_k}\leq\ell}}\sum_{d|N}\mu(d)\,d^{1-s}\sum_{j=1}^{\ell}\sum_{\substack{m\in\mathbb{N}\\ j/(dp_1^{a_1}\cdots p_k^{a_k})-1/d<m\leq j/(dp_1^{a_1}\cdots p_k^{a_k})}} m^{1-s}\\
&\qquad=\sum_{\substack{a_1,\ldots,a_k\geq 0\\ p_1^{a_1}\cdots p_k^{a_k}\leq\ell}}\left(\frac{\varphi(N)\,\ell^{2-s}}{(2-s)N(p_1^{a_1}\cdots p_k^{a_k})^{1-s}}+O\left(2^k\ell^{1-s}(p_1^{a_1}\cdots p_k^{a_k})^s\right)\right). \qquad (13)
\end{aligned}$$

For the error term here we have the trivial upper bound

$$\sum_{\substack{a_1,\ldots,a_k\geq 0\\ p_1^{a_1}\cdots p_k^{a_k}\leq\ell}}\ell^{1-s}(p_1^{a_1}\cdots p_k^{a_k})^s\ll\ell(\log\ell)^k.$$

This shows that the quantity in (13) is bounded below by a positive constant (which depends on N and s) times ℓ^{2-s}, at least for ℓ larger than some fixed bound. Returning to (10) again we have that it is

$$\gg\lim_{L\to\infty}\left(\sum_{\ell\leq L}\left(\psi(\ell)^s-\psi(\ell+1)^s\right)\ell^{2-s}+\psi(L+1)^sL^{2-s}\right)\gg\sum_{n\in\mathbb{N}}n^{1-s}\psi(n)^s.$$

This proves one direction of the lemma. For the other direction we start from the observation that

$$\begin{aligned}
\sum_{n\in\mathbb{N}}n\left(\frac{\psi(n)}{n|n|_{p_1}\cdots|n|_{p_k}}\right)^s&\ll\sum_{a_1,\ldots,a_k\geq 0}\sum_{\substack{m\in\mathbb{N}\\ (m,N)=1}}m^{1-s}\frac{p_1^{a_1}\cdots p_k^{a_k}}{2}\psi(p_1^{a_1}\cdots p_k^{a_k}m)^s\\
&\ll\sum_{a_1,\ldots,a_k\geq 0}\sum_{\substack{m\in\mathbb{N}\\ (m,N)=1}}m^{1-s}\sum_{p_1^{a_1}\cdots p_k^{a_k}(m-1/2)<\ell\leq p_1^{a_1}\cdots p_k^{a_k}m}\psi(\ell)^s.
\end{aligned}$$

Since we are aiming for an upper bound this time we can drop the condition $(m,N)=1$, and this makes things a little simpler than before. Our bound then becomes

$$\begin{aligned}
&\sum_{\ell\in\mathbb{N}}\psi(\ell)^s\sum_{0\leq a_1,\ldots,a_k\leq\max_i(\log_{p_i}2\ell)}\sum_{\substack{m\in\mathbb{N}\\ \ell/p_1^{a_1}\cdots p_k^{a_k}\leq m<\ell/p_1^{a_1}\cdots p_k^{a_k}+1/2}}m^{1-s}\\
&\qquad\ll\sum_{\ell\in\mathbb{N}}\ell^{1-s}\psi(\ell)^s\sum_{0\leq a_1,\ldots,a_k\leq\max_i(\log_{p_i}2\ell)}(p_1^{a_1}\cdots p_k^{a_k})^{s-1}.
\end{aligned}$$

When $s = 1$ this shows that

$$\sum_{n\in\mathbb{N}} \frac{\psi(n)}{|n|_{p_1}\cdots|n|_{p_k}} \ll \sum_{n\in\mathbb{N}} (\log n)^k \psi(n),$$

and when $s < 1$ we have that

$$\sum_{n\in\mathbb{N}} n\left(\frac{\psi(n)}{n|n|_{p_1}\cdots|n|_{p_k}}\right)^s \ll \sum_{n\in\mathbb{N}} n^{1-s}\psi(n)^s.$$

This completes the proof of the lemma. □

3. Proof of main result

The proof of the convergent case of Theorem 2 is an easy application of the Borel-Cantelli Lemma from probability theory. Without loss of generality we can restrict our attention to real numbers α lying within the unit interval $\mathbb{I} := [0, 1]$. It follows that we need to determine the Lebesgue measure $|\,.\,|$ of

$$\limsup_{n\in\mathbb{N}} A_n \qquad \text{where} \qquad A_n := \{\alpha \in \mathbb{I} : (6) \text{ holds}\}\ .$$

It is easily seen that if (7) converges then so does $\sum_{n\in\mathbb{N}} |A_n|$ and the Borel-Cantelli Lemma implies that the associated lim sup set is of zero measure. Note that in proving the convergent case we do not require the function ψ to be monotonic.

The divergent case constitutes the main substance of Theorem 2 and will be established as an application of the Duffin-Schaeffer Theorem.

3.1. Preliminaries.

The following is a consequence of an attempt by R. Duffin and A. Schaeffer to remove the monotonicity assumption from Khintchine's fundamental 'zero-one' law – see [10, Theorem 1] and [15, Chapter 2] for background, proof and further details.

The Duffin-Schaeffer Theorem. *Let $\psi : \mathbb{N} \to \mathbb{R}$ be a non-negative function. Then, for almost every real number α the inequality*

$$|n\alpha - a| \leq \ \psi(n) \qquad (a, n) = 1$$

has infinitely many solutions $(a, n) \in \mathbb{Z} \times \mathbb{N}$ if

$$\sum_{n\in\mathbb{N}} \psi(n) = \infty \quad \textit{and} \quad \limsup_{N\to\infty} \left(\sum_{n\leq N} \frac{\varphi(n)\psi(n)}{n}\right)\left(\sum_{n\leq N}\psi(n)\right)^{-1} > 0. \tag{14}$$

Here and throughout, φ is the Euler phi function. Just for completeness, we mention that the famous and open Duffin-Schaeffer Conjecture corresponds to the above statement with (14) replaced by the single 'natural' condition that $\sum(\varphi(n)\psi(n))/n$ diverges.

In the course of establishing the divergent case of Theorem 2, it will be useful to have the following elementary fact at hand.

Lemma 2. *Let $p_1, \ldots, p_k$ be distinct prime numbers and $N \in \mathbb{N}$. Then*

$$\sum_{\substack{n \le N \\ p_1,\ldots,p_k \nmid n}} \frac{\varphi(n)}{n} = \frac{6N}{\pi^2} \prod_{i=1}^{k} \frac{p_i}{p_i+1} + O_k(\log N).$$

Proof. By well known properties of φ and the Möbius function μ we have that

$$\begin{aligned}
\sum_{\substack{n \le N \\ p_1,\ldots,p_k \nmid n}} \frac{\varphi(n)}{n} &= \sum_{\substack{n \le N \\ p_1,\ldots,p_k \nmid n}} \sum_{d|n} \frac{\mu(d)}{d} = \sum_{\substack{d \le N \\ p_1,\ldots,p_k \nmid d}} \frac{\mu(d)}{d} \sum_{\substack{e \le N/d \\ p_1,\ldots,p_k \nmid e}} 1 \\
&= \sum_{\substack{d \le N \\ p_1,\ldots,p_k \nmid d}} \frac{\mu(d)}{d} \sum_{f | p_1 \cdots p_k} \mu(f) \left(\frac{N}{fd} + O(1) \right) \\
&= N \left(\prod_{i=1}^{k} \frac{\varphi(p_i)}{p_i} \right) \sum_{\substack{d \le N \\ p_1,\ldots,p_k \nmid d}} \frac{\mu(d)}{d^2} + O\left(2^k \sum_{\substack{d \le N \\ p_1,\ldots,p_k \nmid d}} \frac{|\mu(d)|}{d} \right). \qquad (15)
\end{aligned}$$

The Euler product formula for the Riemann zeta function gives us that

$$\sum_{\substack{d \le N \\ p_1,\ldots,p_k \nmid d}} \frac{\mu(d)}{d^2} = \zeta^{-1}(2) \prod_{i=1}^{k} (1 - p_i^{-2})^{-1} + O(N^{-1}).$$

Combining this with (15) completes the proof. □

3.2. **Proof of divergent case of Theorem 2.** We will show that in the divergence case of Theorem 2, the inequality

$$|n\alpha - a| \le \frac{\psi(n)}{f_1(|n|_{p_1}) \cdots f_k(|n|_{p_k})} \qquad (a,n) = 1 \qquad (16)$$

has infinitely many solutions $(a,n) \in \mathbb{Z} \times \mathbb{N}$ for almost every real number α. This clearly implies that (6) has infinitely many solutions for almost every α and thereby completes the proof of Theorem 2.

It is easy to see that the Duffin-Schaeffer Theorem will guarantee infinitely many solutions to (16) for almost every α if, in addition to the divergence of (7), we have that

$$\limsup_{N \to \infty} \left(\sum_{n \le N} \frac{\varphi(n)\psi(n)}{n f_1(|n|_{p_1}) \cdots f_k(|n|_{p_k})} \right) \left(\sum_{n \le N} \frac{\psi(n)}{f_1(|n|_{p_1}) \cdots f_k(|n|_{p_k})} \right)^{-1} > 0. \qquad (17)$$

With this in mind, note that

$$\sum_{n\le N}\frac{\varphi(n)\psi(n)}{nf_1(|n|_{p_1})\cdots f_k(|n|_{p_k})}$$

$$=\sum_{a_1=0}^{\lfloor\log_{p_1}N\rfloor}\cdots\sum_{a_k=0}^{\lfloor\log_{p_k}N\rfloor}\sum_{\substack{n\le N/(p_1^{a_1}\cdots p_k^{a_k})\\ p_1,\ldots,p_k\nmid n}}\frac{\varphi(p_1^{a_1}\cdots p_k^{a_k}n)\psi(p_1^{a_1}\cdots p_k^{a_k}n)}{p_1^{a_1}\cdots p_k^{a_k}nf_1(p_1^{-a_1})\cdots f_k(p_k^{-a_k})}$$

$$=\sum_{a_1=0}^{\lfloor\log_{p_1}N\rfloor}\cdots\sum_{a_k=0}^{\lfloor\log_{p_k}N\rfloor}\left(\prod_{i=1}^{k}\frac{\varphi(p_i)}{p_if_i(p_i^{-a_i})}\right)\sum_{\substack{n\le N/(p_1^{a_1}\cdots p_k^{a_k})\\ p_1,\ldots,p_k\nmid n}}\frac{\varphi(n)\psi(p_1^{a_1}\cdots p_k^{a_k}n)}{n}.$$

To deal with the inner sum we will use partial summation. First write the collection of integers coprime to $p_1\cdots p_k$ in increasing order as $n_1<n_2<\cdots$. Then for any function $\psi':\mathbb{N}\to\mathbb{R}$ and for any $M>1$ we have that

$$\sum_{i\le M}\frac{\varphi(n_i)\psi'(n_i)}{n_i}=\sum_{i\le M}(\psi'(n_i)-\psi'(n_{i+1}))\sum_{j=1}^{i}\frac{\varphi(n_j)}{n_j}$$
$$+\psi'(n_{M+1})\sum_{j=1}^{M}\frac{\varphi(n_j)}{n_j}.$$

It is easy to check that Lemma 2 implies that

$$\sum_{j=1}^{i}\frac{\varphi(n_j)}{n_j}\gg_k i$$

and if ψ' is non-negative and monotonic then we can use this fact in the inner sums of the partial summation to obtain

$$\sum_{i\le M}\frac{\varphi(n_i)\psi'(n_i)}{n_i}\gg_k\sum_{i\le M}i(\psi'(n_i)-\psi'(n_{i+1}))+M\psi'(n_{M+1})=\sum_{i\le M}\psi'(n_i).$$

Since the implied constant here depends at most on k when we return to our above analysis we find that

$$\begin{aligned}
&\sum_{n\le N}\frac{\varphi(n)\psi(n)}{nf_1(|n|_{p_1})\cdots f_k(|n|_{p_k})}\\
&\qquad\gg_k \sum_{a_1=0}^{\lfloor \log_{p_1} N\rfloor}\cdots\sum_{a_k=0}^{\lfloor \log_{p_k} N\rfloor}\left(\prod_{i=1}^{k}\frac{\varphi(p_i)}{p_if_i(p_i^{-a_i})}\right)\sum_{\substack{n\le N/(p_1^{a_1}\cdots p_k^{a_k})\\ p_1,\ldots,p_k\nmid n}}\psi(p_1^{a_1}\cdots p_k^{a_k}n)\\
&\qquad\gg_k \sum_{a_1=0}^{\lfloor \log_{p_1} N\rfloor}\cdots\sum_{a_k=0}^{\lfloor \log_{p_k} N\rfloor}\ \sum_{\substack{n\le N/(p_1^{a_1}\cdots p_k^{a_k})\\ p_1,\ldots,p_k\nmid n}}\frac{\psi(p_1^{a_1}\cdots p_k^{a_k}n)}{f_1(p_1^{-a_1})\cdots f_k(p_k^{-a_k})}\\
&\qquad=\sum_{n\le N}\frac{\psi(n)}{f_1(|n|_{p_1})\cdots f_k(|n|_{p_k})}\ .
\end{aligned}$$

This shows that hypothesis (17) is satisfied and as desired the conclusion of our theorem now follows from the Duffin-Schaeffer Theorem.

4. Related results and open problems

4.1. **The Hausdorff theory.** For $s > 0$, let $\mathcal{H}^s(X)$ denote the s-dimensional Hausdorff measure of a set $X \subseteq \mathbb{R}$ and let $\dim X$ denote its Hausdorff dimension. The Mass Transference Principle [5] allows us to deduce the following Hausdorff measure generalization of Theorem 2.

Theorem 3. *Let $p_1, \ldots, p_k$ be distinct prime numbers and $f_1, \ldots, f_k : \mathbb{R} \to \mathbb{R}$ be positive functions. Furthermore, let $\psi : \mathbb{N} \to \mathbb{R}$ be a non-negative decreasing function and let $W(\psi, \mathbf{p}, \mathbf{f})$ denote the set of real numbers in the unit interval $\mathbb{I} := [0, 1]$ for which inequality (6) has infinitely many solutions. Then, for any $0 < s \le 1$*

$$\mathcal{H}^s\big(W(\psi,\mathbf{p},\mathbf{f})\big)=\begin{cases} 0 \quad if & \displaystyle\sum_{n\in\mathbb{N}} n\left(\frac{\psi(n)}{n\,f_1(|n|_{p_1})\cdots f_k(|n|_{p_k})}\right)^s<\infty\\[2ex] \mathcal{H}^s(\mathbb{I}) \quad if & \displaystyle\sum_{n\in\mathbb{N}} n\left(\frac{\psi(n)}{n\,f_1(|n|_{p_1})\cdots f_k(|n|_{p_k})}\right)^s=\infty\end{cases}.$$

When $s = 1$, the measure $\mathcal{H}^s$ coincides with one dimensional Lebesgue measure $|\ .\ |$ and the above theorem reduces to Theorem 2. In the case that each of the functions $f_1, \ldots, f_k$ is the identity function, let us write $W(\psi, \mathbf{p})$ for $W(\psi, \mathbf{p}, \mathbf{f})$. The following statement is a consequence of Lemma 1 and the fact that $\mathcal{H}^s(\mathbb{I}) = \infty$ when $s < 1$.

Theorem 4. *Let $p_1,\ldots,p_k$ be distinct prime numbers and let $\psi:\mathbb{N}\to\mathbb{R}$ be a non-negative decreasing function. Then, for any* $0<s<1$

$$\mathcal{H}^s\big(W(\psi,\mathbf{p})\big)=\begin{cases} 0 \quad if \quad \displaystyle\sum_{n\in\mathbb{N}} n^{1-s}\,\psi(n)^s<\infty \\ \infty \quad if \quad \displaystyle\sum_{n\in\mathbb{N}} n^{1-s}\,\psi(n)^s=\infty \end{cases}.$$

The fact that $s=1$ is excluded is important on two fronts. The first is trivial, $\mathcal{H}^1\big(W(\psi,\mathbf{p})\big)\le\mathcal{H}^1(\mathbb{I})=1$ and therefore can not possibly be infinite. The other is more interesting. The sum in Theorem 4 at $s=1$ does not coincide with the sum appearing in Theorem 1 which provides the criteria for the 'size' of $W(\psi,\mathbf{p})$ expressed in terms on Lebesgue measure. Thus, it is impossible to unify the Hausdorff and Lebesgue measure statements without appealing to the 'raw' sum in Theorem 3.

A straightforward consequence of Theorem 4 is that

$$\dim W(\psi,\mathbf{p}) \;=\; \inf\{s:\textstyle\sum_{n\in\mathbb{N}} n^{1-s}\,\psi(n)^s<\infty\}\,.$$

In particular, let us consider the case when $\psi(n)=n^{-\tau}$ $(\tau>0)$ and write $W(\tau,\mathbf{p})$ for $W(\psi,\mathbf{p})$. Then, the following statement can be regarded as the 'mixed' analogue of the classical Jarník–Besicovitch Theorem.

Corollary 1. *Let $p_1,\ldots,p_k$ be distinct prime numbers and let $\tau\ge1$. Then,*

$$\dim W(\tau,\mathbf{p})=\frac{2}{\tau+1}\ .$$

For background and further details regarding the general Hausdorff measure theory of metric Diophantine approximation see [2, 4] and references within.

4.2. **Exponents of Diophantine approximation.** Motivated by the 'mixed' analogue of the classical Jarník–Besicovitch Theorem, we introduce the 'mixed' analogue of the classical notion of exact order. For the sake of clarity and simplicity, we restrict our attention to the case of one (fixed) prime p. For a real number ξ, let $\tau_p(\xi)$ denote the supremum of the real numbers τ such that the inequality

$$|n|_p\|n\xi\|\;\le\;n^{-\tau}$$

has infinitely many solutions $n\in\mathbb{N}$. Recall, the *exact order* $\tau(\xi)$ of ξ is defined to be the supremum of the real numbers τ such that the inequality

$$\|n\xi\|\;\le\;n^{-\tau} \tag{18}$$

has infinitely many solutions $n\in\mathbb{N}$. For every real number $t\ge1$, we know that

$$\mathcal{H}^{\frac{2}{t+1}}\big(\{\xi:\tau(\xi)=t\}\big)=\mathcal{H}^{\frac{2}{t+1}}\big(\{\xi:\tau_p(\xi)=t\}\big)=\;\infty \tag{19}$$

and

$$\dim\{\xi:\tau(\xi)=t\}=\dim\{\xi:\tau_p(\xi)=t\}=\frac{2}{t+1}\ . \tag{20}$$

The dimension result for the classical exact order set $\{\xi : \tau(\xi) = t\}$ was first explicitly stated by Güting – see [3] for a 'modern' proof which also implies the measure statement and references within for 'exact order' background. Following the basic principle exploited in [3], it is easy to deduce the measure result for the mixed exact order set $\{\xi : \tau_p(\xi) = t\}$ from Theorem 4 and the fact that

$$W(\tau, p) \setminus W(\psi, p) \subset \{\xi : \tau_p(\xi) = t\}$$

with $\tau := t$ and $\psi(n) := n^{-t}(\log n)^{-(t+1)}$. Note that the Hausdorff measure result implies the lower bound for the Hausdorff dimension statement. The complementary upper bound is a consequence of the fact that $\{\xi : \tau_p(\xi) = t\} \subset W(t + \epsilon, p)$ for any $\epsilon > 0$.

On using the trivial fact that $n^{-1} \leq |n|_p \leq 1$, it follows that for any real number ξ

$$\tau(\xi) \leq \tau_p(\xi) \leq \tau(\xi) + 1 \,.$$

Deeper still, for any given δ in $[0, 1]$ and t sufficiently large, it is possible to adapt the procedure described in [7] to construct explicit real numbers ξ such that

$$\tau(\xi) = t \qquad \text{and} \qquad \tau_p(\xi) = t + \delta \,.$$

Consequently, the set of values taken by the function $\tau_p - \tau$ is precisely the whole interval $[0, 1]$. We suspect that the set of real numbers for which the classical and mixed exact order exponents differ ($\delta > 0$) is of maximal dimension; that is

$$\dim\{\xi : \tau_p(\xi) > \tau(\xi)\} = 1 \,.$$

Currently we are only able to prove that the dimension is positive. Indeed this is a consequence of showing that

$$\dim\{\xi : \tau_p(\xi) = \tau(\xi) + 1\} > 0 \,.$$

The proof relies on being able to construct a Cantor type subset of $\{\xi : \tau_p(\xi) = \tau(\xi) + 1\}$ consisting of real numbers all of whose best rational approximations have a denominator a power of p.

We now turn our attention to the situation for which the classical and mixed exact order exponents are equal ($\delta = 0$) to a given value $t \geq 1$. Let $W(\tau)$ denote the set of real numbers for which inequality (18) has infinitely many solutions and observe that

$$W(\tau) \setminus W(\psi, p) \subset \{\xi : \tau_p(\xi) = \tau(\xi) = t\}$$

with $\tau := t$ and $\psi(n) := n^{-t}(\log n)^{-(t+1)}$. On combining this with Theorem 4 and the classical fact that $\mathcal{H}^{\frac{2}{t+1}}(W(t)) = \infty$, it follows that

$$\mathcal{H}^{\frac{2}{t+1}}\big(\{\xi : \tau_p(\xi) = \tau(\xi) = t\}\big) = \infty \,.$$

In turn it is easy to deduce, that for $t \geq 1$

$$\dim\{\xi : \tau_p(\xi) = \tau(\xi) = t\} = \frac{2}{t+1} \,.$$

Clearly, these results imply our 'opening' results given by (19) and (20). However the opening results do not even imply that $\{\xi : \tau_p(\xi) = \tau(\xi) = t\}$ is non-empty.

4.3. **Removing monotonicity.** The method of proof of Theorem 2 allows us to draw conclusions even when the approximating function ψ is non-monotonic. For example we can prove the following result.

Theorem 5. *Let $p_1, \ldots, p_k$ be distinct prime numbers and $f_1, \ldots, f_k : \mathbb{R} \to \mathbb{R}$ be positive functions. Furthermore, let $\psi : \mathbb{N} \to \mathbb{R}$ be a non-negative function. Then, for almost every real number α the inequality*

$$f_1(|n|_{p_1}) \cdots f_k(|n|_{p_k})\ |n\alpha - a| \le \psi(n) \qquad (a,n) = 1 \tag{21}$$

has infinitely many solutions if there exists $\epsilon > 0$ for which

$$\sum_{n\in\mathbb{N}} \varphi(n) \left(\frac{\psi(n)}{n\, f_1(|n|_{p_1}) \cdots f_k(|n|_{p_k})} \right)^{1+\epsilon} = \infty. \tag{22}$$

We point out that there are examples of non-monotonic ψ for which (7) diverges but (6) has only finitely many solutions almost everywhere. The Duffin-Schaeffer counterexample at the end of [10] can easily be modified to show how this can happen. In other words, disallowing non-reduced solutions and thereby introducing the Euler phi function in Theorem 5 is absolutely necessary when dealing with non-monotonic approximating functions.

Theorem 5 is a trivial consequence of a known result regarding the Duffin-Schaeffer Conjecture. Basically, given ψ we simply apply Corollary 1 of [16] to the approximating function

$$\Psi(n) := \frac{\psi(n)}{f_1(|n|_{p_1}) \cdots f_k(|n|_{p_k})} \ . \tag{23}$$

4.4. **Simultaneous approximation.** So far we have restricted our attention to approximating a single real number $\alpha \in \mathbb{R}$. Clearly, it is natural to develop the theory of 'mixed' simultaneous approximation in which one considers points $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ and the system of inequalities

$$f_1(|n|_{p_1}) \cdots f_k(|n|_{p_k})\ |n\alpha_i - a_i| \le \psi(n) \qquad 1 \le i \le m\ . \tag{24}$$

The following result is a direct consequence of Gallagher's theorem [14] in the classical theory of simultaneous approximation. Indeed, given ψ we simply apply Gallagher's theorem to the approximating function Ψ given by (23).

Theorem 6. *Let $p_1, \ldots, p_k$ be distinct prime numbers and $f_1, \ldots, f_k : \mathbb{R} \to \mathbb{R}$ be positive functions. Furthermore, let $\psi : \mathbb{N} \to \mathbb{R}$ be a non-negative function and $m \ge 2$ be an integer. Then, for almost every $(\alpha_1, \ldots, \alpha_m) \in \mathbb{R}^m$ the system of inequalities given by (24) with $(a_1, \ldots, a_m, n) = 1$ has infinitely (resp. finitely) many solutions $(a_1, \ldots, a_m, n) \in \mathbb{Z}^m \times \mathbb{N}$ if*

$$\sum_{n\in\mathbb{N}} \left(\frac{\psi(n)}{f_1(|n|_{p_1}) \cdots f_k(|n|_{p_k})} \right)^m \tag{25}$$

diverges (resp. converges).

The following is an immediate corollary and generalizes Theorem 2 to the simultaneous setting. Note that it is free of any monotonicity assumption on ψ.

Theorem 7. *Let $p_1, \dots, p_k$ be distinct prime numbers and $f_1, \dots, f_k : \mathbb{R} \to \mathbb{R}$ be positive functions. Furthermore, let $\psi : \mathbb{N} \to \mathbb{R}$ be a non-negative function and $m \geq 2$ be an integer. Then, for almost every $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ the system of inequalities given by (24) has infinitely (resp. finitely) many solutions $(a_1, \dots, a_m, n) \in \mathbb{Z}^m \times \mathbb{N}$ if the sum given by (25) diverges (resp. converges).*

Needless to say, the Mass Transference Principle enables us to place Theorems 6 & 7 within the general setting of Hausdorff measures. In particular, the Hausdorff measure generalization of Theorem 7 extends Theorem 3 to the simultaneous setting.

Theorem 8. *Let $p_1, \dots, p_k$ be distinct prime numbers and $f_1, \dots, f_k : \mathbb{R} \to \mathbb{R}$ be positive functions. Furthermore, let $\psi : \mathbb{N} \to \mathbb{R}$ be a non-negative function and let $W_m(\psi, \mathbf{p}, \mathbf{f})$ denote the set of points in the unit cube $\mathbb{I}^m := [0,1]^m$ for which the system of inequalities given by (24) has infinitely many solutions. Then, for $m \geq 2$ and any $0 < s \leq m$*

$$\mathcal{H}^s\big(W_m(\psi, \mathbf{p}, \mathbf{f})\big) = \begin{cases} 0 \quad if \quad \displaystyle\sum_{n \in \mathbb{N}} n^m \left(\frac{\psi(n)}{n\, f_1(|n|_{p_1}) \cdots f_k(|n|_{p_k})} \right)^s < \infty \\ \mathcal{H}^s(\mathbb{I}^m) \quad if \quad \displaystyle\sum_{n \in \mathbb{N}} n^m \left(\frac{\psi(n)}{n\, f_1(|n|_{p_1}) \cdots f_k(|n|_{p_k})} \right)^s = \infty \end{cases} .$$

4.5. **An intriguing 'multiplicative' problem.** Let $\psi : \mathbb{N} \to \mathbb{R}$ be a non-negative decreasing function. It is natural to attempt to generalize Theorem 2 so as to incorporate approximations of the form

$$f_1(|n|_{p_1}) \cdots f_k(|n|_{p_k}) \|n\alpha_1\| \cdots \|n\alpha_m\| \leq \psi(n) \ , \tag{26}$$

where $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$. We would expect to be able to prove that for almost every $(\alpha_1, \dots, \alpha_m) \in \mathbb{R}^m$ the inequality given by (26) has infinitely (resp. finitely) many solutions $n \in \mathbb{N}$ if

$$\sum_{n \in \mathbb{N}} (\log n)^{m-1} \frac{\psi(n)}{f_1(|n|_{p_1}) \cdots f_k(|n|_{p_k})}$$

diverges (resp. converges). The method which we used to prove Theorem 2 would work for this more general setup if we could establish the following 'multiplicative' generalization of the Duffin-Schaeffer Theorem.

Conjecture. *Let $\psi : \mathbb{N} \to \mathbb{R}$ be a non-negative function and let $m \in \mathbb{N}$. Then, for almost every $(\alpha_1, \dots, \alpha_\ell) \in \mathbb{R}^m$ the inequality*

$$\|n\alpha_1\| \cdots \|n\alpha_m\| \leq \ \psi(n) \tag{27}$$

has infinitely many solutions $n \in \mathbb{N}$ if

$$\sum_{n\in\mathbb{N}} (\log n)^{m-1}\psi(n) = \infty$$

and

$$\limsup_{N\to\infty} \left(\sum_{n\le N} \left(\frac{\varphi(n)}{n}\right)^m (\log n)^{m-1}\psi(n) \right) \left(\sum_{n\le N} (\log n)^{m-1}\psi(n) \right)^{-1} > 0 \ .$$

We are 'morally' able to prove this conjecture. More precisely, we are able to show that the associated lim sup set of points satisfying (27) is of positive measure. The missing piece in our attempted proof is that we have been unable to establish a zero-one law for this lim sup set. Indeed, establishing such a law would be of interest in its own right.

Problem. Let $W^*_m(\psi)$ denote the set of points in the unit cube $\mathbb{I}^m$ for which the inequality given by (27) has infinitely many solutions. Prove that the m-dimensional Lebesgue measure of the lim sup set $W^*_m(\psi)$ is either zero or one.

We make one final comment concerning the conjecture. For $m \geq 2$, the conjecture is likely to be true without imposing the lim sup condition. In other words, the divergent sum condition is all that is required.

Attempting to generalize Theorem 2 as above can be viewed as developing a metrical theory of Diophantine approximation within the framework of the following generalization of the de Mathan-Teulié Conjecture:

$$\liminf_{n\to\infty} n|n|_{p_1}\cdots|n|_{p_k}\|n\alpha_1\|\cdots\|n\alpha_m\| = 0 \qquad \forall\ (\alpha_1,\ldots,\alpha_m)\in\mathbb{R}^m \ . \tag{28}$$

As already mentioned in the introduction the statement is true when $k \geq 2$ – see also Section 4.6 below. Thus, let us assume that $k = 1$ and note that establishing (28) for $m = 1$ (the de Mathan-Teulié Conjecture) trivially implies (28) for all m. However, one could argue that establishing (28) should get easier the larger we take m. Nevertheless, nothing seems to be known.

Conjecture. *Let p be a prime and $m \in \mathbb{N}$. For m sufficiently large we have that*

$$\liminf_{n\to\infty} n|n|_p\|n\alpha_1\|\cdots\|n\alpha_m\| = 0 \qquad \forall\ (\alpha_1,\ldots,\alpha_m)\in\mathbb{R}^m \ .$$

4.6. **Various strengthenings.** In view of the recent work of Bourgain, Lindenstrauss, Michel & Venkatesh [6], it is possible to strengthen (4) and therefore (28) when $k \geq 2$. Indeed, given distinct primes p_1 and p_2, it follows from Theorem 1.8 of [6] that there exists a small positive constant κ such that for every real number α

$$\liminf_{n\to\infty} n(\log\log\log n)^\kappa |n|_{p_1}|n|_{p_2}\|n\alpha\| = 0 \ . \tag{29}$$

To be precise, Theorem 1.8 of [6] can be applied unless α is a Liouville number. However, for Liouville numbers the statement given by (29) trivially holds.

Returning to the original de Mathan-Teulié Conjecture, for quadratic numbers α the stronger statement

$$\liminf_{n\to\infty} n(\log n)|n|_p\|n\alpha\| < +\infty \tag{30}$$

has been established in [18]. Another class of (transcendental) real numbers with bounded partial quotients and for which (30) holds is given in [8, Theorem 1]. It would be highly desirable to determine whether or not there exist α for which (30) is violated. It is shown in [9] that the set of real numbers α for which

$$\liminf_{n\to\infty} n(\log n)^2|n|_p\|n\alpha\| > 0$$

has full Hausdorff dimension. Most recently, a consequence of the main result in [1] is that the set of real numbers α for which

$$\liminf_{n\to\infty} n(\log n)\,(\log\log n)\,|n|_p\|n\alpha\| > 0$$

has full Hausdorff dimension. In all likelihood the full dimension statement is true without the $\log\log n$ term.

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