Algebraic numbers close to 1 in non-archimedean metrics*

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Abstract.

Let $\alpha \neq 1$ be an algebraic number with relatively small height. Recently, many authors, including Amoroso, Dubickas, Mignotte and Waldschmidt, stated sharp lower bounds for the quantity $|\alpha - 1|$. Here, we provide a p-adic analogue of their results. For instance, we give an upper bound for the absolute value of the norm of $\alpha - 1$, and we show that our estimate is rather sharp in terms on the degree of α. Further, we discuss a generalization in several variables of our result.

ALGEBRAIC NUMBERS CLOSE TO 1 IN NON-ARCHIMEDEAN METRICS

Yann Bugeaud (Strasbourg)

1. Introduction

We consider an algebraic number $\alpha \neq 1$ of degree d, with complex conjugates $\alpha = \alpha_1, \alpha_2, \ldots, \alpha_d$ and minimal defining polynomial $a(X-\alpha_1)\ldots(X-\alpha_d)$, where $a>0$. As usual, we denote by $\mathrm{M}(\alpha)$ the Mahler's measure of α :

$$\mathrm{M}(\alpha) = a \prod_{i=1}^{d} \max\{1, |\alpha_i|\}.$$

To provide lower bounds for the quantity $|\alpha-1|$ depending only on the degree of α and its Mahler's measure, one can use Liouville's inequality. It leads to $|\alpha-1| \geq 2^{-d+1}\,\mathrm{M}(\alpha)^{-1}$ and the dependence on $\mathrm{M}(\alpha)$ is sharp (see [9, Section 1] for several examples; one can take $\alpha = (1-1/a)^{1/d}$, for instance). However, this is not the case for the dependence on d as shown by Mignotte [6] in the case of algebraic numbers of small measure. Indeed, he proved that

$$-\log|\alpha-1| \leq 4\sqrt{d}\,\log 4d$$

provided that $\mathrm{M}(\alpha) \leq 2$. Recently, by means of Schneider's method and interpolation determinants, Mignotte & Waldschmidt [9] obtained the following improvement :

$$-\log|\alpha-1| \leq \frac{3}{2}\sqrt{d\,\mu\,\log^+(d/\mu)} + 2\mu + \log^+(D/\mu), \tag{1}$$

where $\mu > 0$, $\mu \geq \log \mathrm{M}(\alpha)$ and $\log^+ x = \max\{\log x, 0\}$. We refer to the survey of Amoroso [2] for a comparison between the two methods and also to the papers of Bugeaud, Mignotte & Normandin [3] and of Dubickas [4] for further refinements concerning the constant term and simpler proofs.

The purpose of this work is to provide a p-adic analogue of the estimate (1). Denoting by N the norm function, we remark that the trivial bound

$$|\mathrm{N}(\alpha-1)| \leq 2^d\,M(\alpha), \tag{2}$$

which may also be seen as a consequence of Liouville's inequality, is rather sharp in terms on $\mathrm{M}(\alpha)$. Indeed, let $a>1$, $d>1$ be two integers such that a is not a d-power, and let α be a root of the irreducible polynomial $X^d + a$. Then, we have $\mathrm{M}(\alpha) = a$ and $|\mathrm{N}(\alpha-1)| = P(1) = a+1$.

However, (2) can be improved in terms of d : the first and up to now single result in this direction is due to Mignotte [8]. Under the extra hypothesis $\mathrm{M}(\alpha) \leq 2$, it implies the upper bound

$$\log \mathrm{N}(\alpha-1) \leq 4\sqrt{d}\,\log 4d. \tag{3}$$

Here, as a consequence of a more general result, we improve (3) by a factor $\sqrt{\log d}$ and we show that

$$\log \mathrm{N}(\alpha - 1) \leq 5\sqrt{d \log d}, \tag{4}$$

provided $\mathrm{M}(\alpha) \leq 2$. Our proof follows the argument of Dubickas [4], based on the use of an auxiliary polynomial which is a Vandermonde determinant.

Further, using an idea of Amoroso [1], we show that (4) is rather sharp and we discuss the problem of simultaneous approximation of $\beta_1 - 1, \ldots, \beta_n - 1$ in a non-archimedean metric, where the β_i's are multiplicatively independent algebraic numbers.

2. Statement of the results

Let α be an algebraic number of degree $d \geq 2$. Put $\mathbf{K} = \mathbf{Q}(\alpha)$ and denote by $\Omega_{\mathbf{K}}$ (resp. Ω_∞, Ω_0) the set of places (resp. infinite, finite places) on $\mathbf{K}$. In every place v we choose a valuation $|\,.\,|_v$ in the following way : if v is infinite and corresponds to an embedding $\sigma : \mathbf{K} \to \mathbf{C}$ then we put, for every $\alpha \in \mathbf{K}$,

$$|\alpha|_v = |\sigma(\alpha)|^{d_v},$$

where $d_v = 1$ or 2 according as $\sigma(\mathbf{K})$ is contained in $\mathbf{R}$ or not; if v is a finite place corresponding to the prime ideal $\mathfrak{p}$ in $\mathbf{K}$ then we put $|0|_v = 0$ and, for $\alpha \in \mathbf{K} \setminus \{0\}$,

$$|\alpha|_v = \mathrm{N}(\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(\alpha)}.$$

We recall that the logarithmic absolute height $\mathrm{h}(x)$ of an algebraic integer $x \in \mathbf{K}$ is defined by

$$\mathrm{h}(\alpha) = \frac{1}{d} \log \mathrm{M}(\alpha) = \sum_{v \in \Omega_{\mathbf{K}}} \max\{0, \log |x|_v\}^{1/d}.$$

First, we deal with the case $\mathrm{M}(\alpha) = 1$. Then, by a theorem of a Kronecker, α is a root of unity, and we can obtain a rather precise estimate of $|\mathrm{N}(\alpha - 1)|$. Indeed, let $\alpha = \exp(2i\pi k/m)$ be a primitive root of unity of degree d and denote by $\Phi_m(X)$ its minimal defining polynomial. Since $\Phi_m(X)$ divides $1 + X + \ldots + X^{m-1}$, we have $|\mathrm{N}(\alpha - 1)| = |\Phi_m(1)| \leq m$. By Theorem 328 of [5], the degree of α satisfies $0.51\, m/\log\log m \leq d \leq m$ if d is large enough. Hence, we get the bound $|\mathrm{N}(\alpha-1)| \leq 2\, d \log\log d$ for large d, which is subsequently better than Liouville's estimate (2). Hence, in the sequel, we always suppose that $\mathrm{M}(\alpha) > 1$.

Throughout this paper, we stand the notation $\log^- x$ for $\min\{0, \log x\}$ and $\log^+ x$ for $\max\{0, \log x\}$. Our main result is the following.

Theorem 1. *Keeping the above notation, let Γ_∞ and Γ_0 be any subsets of Ω_∞ and Ω_0, respectively. Assume there exists an integer $K \geq 2$ such that $|\alpha - 1|_v < \big(3/(K-1)\big)^{d_v}$ for every place $v \in \Gamma_\infty$. Then we have*

$$\sum_{v \in \Gamma_\infty \cup \Gamma_0} \log^- |\alpha - 1|_v \geq -\Big(\sum_{v \in \Gamma_\infty} d_v\Big) \log K - d\,\frac{\log K}{K-1} - \frac{K+1}{3} \log \mathrm{M}(\alpha).$$

Theorem 1 is a generalization of a result of Bugeaud, Mignotte & Normandin [3, Théorème 2] and improves an estimate of Mignotte [8, Proposition 1] in the particular case $P(X) = X - 1$ (see also [7, Proposition 2]).

For any prime number p and any integer x, we denote by $[x]_p$ the p-part of x, i.e. the greatest power of p dividing x. From Theorem 1 we deduce an upper bound for any $[\alpha - 1]_p$ and also for the absolute value of the norm of $\alpha - 1$, improving earlier estimates of Mignotte [8, Corollary 2].

Theorem 2. *Let α be an algebraic number of degree $d \geq 2$, which is not a root of unity. Then we have*

$$\log |\mathrm{N}(\alpha - 1)| \leq \sqrt{6\, d\, \log d\, \log \mathrm{M}(\alpha)} + 4 \log \mathrm{M}(\alpha). \tag{5}$$

In particular, for any prime number p, we obtain

$$\log [\mathrm{N}(\alpha - 1)]_p \leq \sqrt{6\, d\, \log d\, \log \mathrm{M}(\alpha)} + 4 \log \mathrm{M}(\alpha). \tag{6}$$

Moreover, for any $\varepsilon > 0$, there exists a $\delta > 0$, depending only on ε, such that, if $\log \mathrm{M}(\alpha) \leq \delta d$, then

$$\log |\mathrm{N}(\alpha - 1)| \leq \sqrt{(2/3 + \varepsilon)\, d\, \log d\, \log \mathrm{M}(\alpha)}.$$

More generally, we note that the upper bounds occurring in Theorem 2 are still valid for every sum of the shape $\sum_{v \in \Gamma_0} \log^+ |\alpha - 1|_v^{-1}$, where Γ_0 is any subset of Ω_0, and even - but with larger numerical constants - for every sum $\sum_{v \in \Gamma_0 \cup \Gamma_\infty} \log^+ |\alpha - 1|_v^{-1}$, provided that the cardinality of Γ_∞ is less than constant times $\sqrt{d \log \mathrm{M}(\alpha) / \log d}$.

As shown by Amoroso [1], the estimate (1) is rather sharp. Using his method, we construct irreducible polynomials with high degree and relatively small measure in order to prove that the dependence on d in Theorem 2 is also sharp.

Theorem 3. *For any integer $n \geq 1$, there exist algebraic numbers α and β of degree $n^2 + 1$ and $n^2 + 2$, respectively, and with Mahler's measure satisfying*

$$\log \mathrm{M}(\alpha) \leq \frac{1}{\pi^2} (\log n)^2 + 3 \log n + 7,$$

such that $|\mathrm{N}(\alpha - 1)| = 2^{n+1} - 1$ and $[\mathrm{N}(\beta - 1)]_2 = 2^{n+1}$. Hence, the dependence on d in (5) and in (6) cannot be relaxed to $\sqrt{d} / \log^{1+\varepsilon} d$, for any $\varepsilon > 0$.

Very recently, Amoroso [2] managed to generalize (1) for several algebraic numbers, provided these are multiplicatively independent. Following his proof, we state a p-adic analogue of his result. Unfortunately, we can work only with one single place, and, since in Theorem 2 we consider many metrics simultaneously, Theorem 4 below cannot be seen as a multivariable generalization of Theorem 2. It only said that, given n multiplicatively independent algebraic numbers, all of them cannot be very closed to 1 in a fixed non-archimedean metric.

Theorem 4. *Let* $\mathbf{K}$ *be a number field of degree* d *and let* $\alpha_1, \ldots, \alpha_n$ *be* n *multiplicatively independent algebraic numbers in* $\mathbf{K}$. *Set* $h := \mathrm{h}(\alpha_1) \ldots \mathrm{h}(\alpha_n)$. *For any non-archimedean place* v *in* $\mathbf{K}$, *we have*

$$\log^- \max_{j=1,\ldots,n} |\alpha_j - 1|_v \geq -(n+1)\, d \left(h \log^+ \frac{1}{h} \right)^{1/(n+1)} - 2\, d \sum_{j=1}^{n} \mathrm{h}(\alpha_j).$$

Moreover, for any $\varepsilon > 0$, *there exists an effective constant* $c > 0$, *depending only on* ε, *such that*

$$\log^- \max_{j=1,\ldots,n} |\alpha_j - 1|_v \geq -(1+\varepsilon) \left(\frac{2}{3}(n+1) \right)^{n/(n+1)} d \left(h \log^+ \frac{1}{h} \right)^{1/(n+1)} - \frac{d}{6} \sum_{j=1}^{n} \mathrm{h}(\alpha_j).$$

Theorem 1 allows us also to provide some lower bounds for the Mahler's measure of a particular class of algebraic numbers.

Corollary 1. *If the prime number 2 is unramified and splits totally in* $\mathbf{Q}(\alpha)$, *then* $\log \mathrm{M}(\alpha) \geq 0.13\, d$.

Corollary 1 slightly improves a result of Mignotte [7, Théorème 3].

3. Auxiliary results

For any analytic function $f(z_1, \ldots, z_n)$, we define the norm $|\cdot|_1$ by

$$|f|_1 = \max_{|z_1| = \ldots = |z_n| = 1} |f(z_1, \ldots, z_n)|.$$

We will use the following version of Liouville's inequality.

Lemma 1. *Let* $P \in \mathbf{Z}[X_1, \ldots, X_n]$ *be a polynomial with integer coefficients. For* $1 \leq i \leq n$, *denote by* D_i *the partial degree of* P *with respect to the variable* X_i. *Let* $\gamma_1, \ldots, \gamma_n$ *be* n *algebraic numbers of degree* $d_1, \ldots, d_n$, *respectively, and let* d *be the degree of the field* $\mathbf{Q}(\gamma_1, \ldots, \gamma_n)$. *If* $P(\gamma_1, \ldots, \gamma_n) \neq 0$ *and if* Ω *denotes any (finite or infinte) set of places on* $\mathbf{Q}(\gamma_1, \ldots, \gamma_n)$, *then we have*

$$\sum_{v \in \Omega} \log^- |P(\gamma_1, \ldots, \gamma_n)|_v \geq -d \log |P|_1 - \sum_{i=1}^{n} \frac{d}{d_i} D_i \log \mathrm{M}(\gamma_i).$$

Proof : Using Lemma 3.5 of [11], we get

$$\mathrm{h}\big(P(\gamma_1, \ldots, \gamma_n)\big) \leq \log |P|_1 + \sum_{i=1}^{n} D_i\, \mathrm{h}(\gamma_i).$$

Since $\mathrm{h}(x) = \mathrm{h}(x^{-1})$ for every algebraic number $x \neq 0$, it follows that

$$\frac{1}{d} \sum_{v \in \Omega} \log^+ |P(\gamma_1, \ldots, \gamma_n)|_v^{-1} \leq \log |P|_1 + \sum_{i=1}^{n} \frac{D_i}{d_i} \mathrm{M}(\gamma_i),$$

which proves Lemma 1. $\square$

Lemma 2. *Let $K \geq 2$ be an integer. Then the Vandermonde determinant*

$$P(x) = \mathrm{Det}\Big(\big(x^i\big)^j\Big)_{0\leq i,j\leq K-1} = \prod_{0\leq i,j\leq K-1} \big(x^j - x^i\big)$$

is a polynomial of degree $K(K-1)(2K-1)/6$ satisfying $|P|_1 \leq K^{K/2}$. Moreover, P has a zero of order $K(K-1)(K-2)/6$ at $x=0$ and a zero of order $K(K-1)/2$ at $x=1$.

Proof : The second assertion follows from Hadamard's inequality, while the other three are easy calculations. □

We also need a good lower bound for the Mahler's measure of an algebraic number

Lemma 3. *Let α be an algebraic number of degree $d \geq 2$. If α is not a root of unity, then we have*

$$\log \mathrm{M}(\alpha) \geq 2/\big(\log(3d)\big)^3.$$

Proof : This is Corollary 1 of Voutier [10]. □

4. Proofs

Proof of Theorem 1. Following an idea of Dubickas [4], we use a Vandermonde determinant to construct our auxiliary polynomial. Hence, we set

$$P(x) = x^{-K(K-1)(K-2)/6}\,\mathrm{Det}\Big(\big(x^i\big)^j\Big)_{0\leq i,j\leq K-1}.$$

Since α is non-zero and is not a root of unity, we have $P(\alpha) \neq 0$. By Lemma 2, the polynomial P has degree $K(K-1)(K+1)/6$, and, using Lemma 1, we get

$$\sum_{v\in\Omega} \log^- |P(\alpha)|_v \geq -d\,\frac{K}{2}\,\log K - \frac{K}{6}(K-1)(K+1)\log \mathrm{M}(\alpha). \tag{7}$$

If v is a finite place such that $|\alpha - 1|_v < 1$, then $|\alpha|_v = 1$ and, for every positive integer k, we get $|1+\alpha+\ldots+\alpha^k|_v \leq 1$, whence

$$|P(\alpha)|_v = \prod_{0\leq i<j\leq K-1} \big|\alpha^{j-i} - 1\big|_v \big|\alpha^i\big|_v \leq |\alpha - 1|_v^{K(K-1)/2}.$$

Consequently, if v is non-archimedean, we have

$$\log^- |P(\alpha)|_v \leq \frac{K(K-1)}{2}\log^- |\alpha - 1|_v. \tag{8}$$

Consider now the case when v is an infinite place and denote by σ the corresponding complex embedding. Set $\rho = 1 + 3/(K-1)$. By the maximum principle, the analytic function $z \mapsto P(z)/(z-1)^{K(K-1)/2}$ is bounded on the disk of radius ρ by $K^{K/2}\,\rho^{K(K-1)^2/6}\,\big((K-1)/3\big)^{K(K-1)/2}$. Since by hypothesis $|\sigma\alpha| < \rho$, we get

$$|P(\sigma\alpha)| \leq |\sigma\alpha - 1|^{K(K-1)/2}\left(1+\frac{3}{K-1}\right)^{K(K-1)^2/6} K^{K/2}\left(\frac{K-1}{3}\right)^{K(K-1)/2}$$

and, using $\lim\limits_{x\to\infty}(1+1/x)^x = e$ and a direct computation for $K \leq 22$, we infer that

$$\log^{-}|P(\alpha)|_v \leq \frac{K(K-1)}{2}\big(\log^{-}|\alpha-1|_v + d_v \log K\big). \tag{9}$$

And Theorem 1 directly follows from the combination of (7), (8) and (9). $\square$

Proof of Theorem 2. By the product formula in $\mathbf{K}$

$$\prod_{v\in\Omega_\infty}|\alpha-1|_v \cdot \prod_{v\in\Omega_0}|\alpha-1|_v = 1,$$

we obtain

$$|\mathrm{N}(\alpha-1)| \leq \prod_{v\in\Omega_0}\max\big\{1, |\alpha-1|_v^{-1}\big\},$$

hence

$$\log|\mathrm{N}(\alpha-1)| \leq -\sum_{v\in\Omega_0}\log^{-}|\alpha-1|_v.$$

Applying Theorem 1 with $\Gamma_\infty = \emptyset$, $\Gamma_0 = \Omega_0$ and

$$K = \big[\lambda\sqrt{d\,\log d/\log \mathrm{M}(\alpha)}\big] + 1,$$

with $\lambda \geq 1$ (which together with Lemma 3 ensures that $K \geq 2$), we get

$$\log|\mathrm{N}(\alpha-1)| \leq \sqrt{d\,\log d\,\log\mathrm{M}(\alpha)}\left(\sqrt{\frac{d}{\log d\,\log\mathrm{M}(\alpha)}}\,\frac{\log K}{K-1} + \frac{\lambda}{3}\right). \tag{10}$$

By Lemma 3, we infer that for any $\varepsilon > 0$

$$\log|\mathrm{N}(\alpha-1)| \leq \left(\frac{1}{2\lambda} + \frac{\lambda}{3} + \varepsilon\right)\sqrt{d\,\log d\,\log\mathrm{M}(\alpha)},$$

provided d is large enough. The second part of the theorem follows from the choice $\lambda = \sqrt{3/2}$. If $d \leq 4\log\mathrm{M}(\alpha)$, then, using the trivial bound $|\mathrm{N}(\alpha-1)| \leq 2^d\,\mathrm{M}(\alpha)$, we get

$$\log|\mathrm{N}(\alpha-1)| \leq 4\,\log\mathrm{M}(\alpha).$$

Else, putting $\lambda = 2$, we have $K \geq 4$ and $K-1 \geq 1.5\sqrt{d\,\log d/\log\mathrm{M}(\alpha)}$. Since $K \leq d^2$ by Lemma 3, the theorem follows directly from (10). $\square$

Proof of Theorem 3. Let $n \geq 1$ be an integer. Put $F(X) = \prod_{i=1}^{n}(X^{2i-1} - 1)$ and $G(X) = X^{n^2+1} + 2F(X)$. The polynomial $G(X)$ is monic and irreducible, by Eisenstein's criterion. Denote by $-\alpha$ a root of $G(X)$. We have $|\mathrm{N}(\alpha - 1)| = |G(-1)| = 2^{n+1} - 1$. Further, Amoroso [1, proof of Theorem 1.1] proved that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |F(e^{it})| dt \leq \frac{1}{\pi^2}(\log n)^2 + 3\log n + 5.$$

Hence, using $|G(e^{it})| \leq 1 + 2|F(e^{it})| \leq 3\max\{1, |F(e^{it})|\}$ and the equality

$$\log \mathrm{M}(\alpha) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log |G(e^{it})| dt,$$

we get

$$\log \mathrm{M}(\alpha) = \log \mathrm{M}(-\alpha) \leq \frac{1}{\pi^2}(\log n)^2 + 3\log n + 7. \tag{11}$$

Remark that $|\mathrm{N}(\alpha - 1)|$ is composed of many distinct prime numbers. However, we can also construct an algebraic number β, which norm is a power of 2, showing that the estimate (6) is also rather sharp, at last in the particular case $p = 2$. To this end, we consider the monic polynomial $H(X) = X^{n^2+2} + X^{n^2+1} + 2F(X)$. Reasoning modulo 2, we infer that either H is irreducible or H has a linear factor. Since $H(-2)$, $H(-1)$, $H(0)$, $H(1)$ and $H(2)$ are all non-zero, the polynomial H is irreducible. Letting β be the opposite of a root of H, we get $|\mathrm{N}(\beta - 1)| = |H(-1)| = 2^{n+1}$, and the upper bound (11) remains valid for $\mathrm{M}(\beta)$.

The last assertion of Theorem 3 is an easy calculation. $\square$

Proof of Theorem 4. We just describe the general scheme of the proof. For details, we refer the reader to the work of Amoroso [2, Theorem 4.1]. Let $k_1, \ldots, k_n$ be n positive integers to be defined later and set $K = k_1 \ldots k_n$. We consider a natural n-dimensional generalization of the Vandermonde determinant, namely the polynomial in n variables

$$P(x_1, \ldots, x_n) = \prod \big(x_1^{\lambda_1} \ldots x_n^{\lambda_n} - x_1^{\mu_1} \ldots x_n^{\mu_n}\big),$$

where the product is taken over all $(\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_n)$ such that $0 \leq \lambda_i, \mu_i \leq k_i - 1$ and $(\lambda_1, \ldots \lambda_n) \neq (\mu_1, \ldots, \mu_n)$. For convenience, we can suppose that $1 > |\alpha_1 - 1|_v \geq |\alpha_i - 1|_v$ for $1 \leq i \leq n$, and, since $|\alpha_i|_v = 1$, we get

$$|P(\alpha_1, \ldots, \alpha_n)|_v \leq |\alpha_i - 1|_v^{K(K-1)/2},$$

which together with Lemma 1, leads to

$$\log^- |P(\alpha_1, \ldots, \alpha_n)|_v \geq -\frac{2\,d}{K(K-1)} \log |P|_1 - \sum_{i=1}^{n} d \deg_{x_i}(P)\, \mathrm{h}(\alpha_i).$$

Now, we refer to [2] for the calculation and the choice of the parameters k_i. However, we note that the term $1+\log(N/m)$ occurring in the archimedean case (cf [2, inequality (4.5)]) does not appear here : it is due to the use of the maximum principle, which can easily be avoid in the p-adic case. Consequently, we gain $\log(3nd^2)$ in the final estimate. $\square$

Proof of Corollary 1. Applying Theorem 1 with $\Gamma_\infty = \emptyset$ and $\Gamma_0 = \Omega_0$, we get

$$\log \mathrm{M}(\alpha) \geq \frac{3}{K+1}\left(\log|\mathrm{N}(\alpha-1)| - d\,\frac{\log K}{K-1}\right).$$

By hypothesis, we have $|\mathrm{N}(\alpha-1)| \geq 2^d$, hence the result follows from the choice $K=4$. $\square$

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