LINEAR FORMS IN p-ADIC LOGARITHMS AND THE DIOPHANTINE EQUATION $\dfrac{x^n-1}{x-1}=y^q$

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1. Introduction

A longstanding conjecture claims that the Diophantine equation

$$\frac{x^n-1}{x-1} = y^q \quad \text{in integers } x>1, y>1, n>2, q\geq 2 \tag{1}$$

has finitely many solutions, and, maybe, only those given by

$$\frac{3^5-1}{3-1} = 11^2, \quad \frac{7^4-1}{7-1} = 20^2 \quad \text{and} \quad \frac{18^3-1}{18-1} = 7^3.$$

Among the known results, let us mention that Ljunggren [9] solved (1) completely when $q=2$ and that very recently Bugeaud, Mignotte, Roy & Shorey [3] showed that (1) has no solution when x is a square. For more information and in particular for finiteness type results under some extra hypotheses, we refer the reader to Nagell [11], Shorey & Tijdeman [16], [17] and to the recent survey of Shorey [15]. One of the main tools used in most of the proofs is Baker's theory of linear form in archimedean logarithms of algebraic numbers, and especially a dramatic sharpening obtained when the algebraic numbers involved are closed to 1. This was first noticed by Shorey [14], and has been applied in numerous works relating to (1) or to the Diophantine equation

$$(a+1)\,x^n - a\,y^n = 1 \quad \text{in integers } x>1, y>1, n>2, a\geq 2; \tag{2}$$

see for instance [5], [1] and [10]. The purpose of the present work is to prove a similar sharpening for linear forms in p-adic logarithms and to show how it can be applied in the context of (1). Further, following previous investigations by Sander [12] and Saradha & Shorey [13], we derive from our results an irrationality statement for Mahler's numbers.

2. Linear forms in p-adic logarithms

Throughout this paper, we denote by $\mathrm{h}(\alpha)$ the absolute logarithmic height of the algebraic number α, given by $\mathrm{h}(0)=0$ and by

$$\mathrm{h}(\alpha) = \frac{1}{d}\left(\log|a| + \sum_{i=1}^{d} \log\max\{|\alpha_i|, 1\}\right),$$

if $a(X-\alpha_1)\dots(X-\alpha_d)$ is the minimal defining polynomial of α over $\mathbf{Z}$.

Before stating a slight improvement of Théorème 1 of [2], we recall the notation used in that work. Let p be a prime number and denote by $\overline{\mathbf{Q}}_p$ an algebraic closure of the p-adic field $\mathbf{Q}_p$. We equip the field $\overline{\mathbf{Q}}_p$ with the ultrametric absolute value

$|x|_p = p^{-v_p(x)}$, where v_p denotes the unique extension to $\overline{\mathbf{Q}}_p$ of the standard p-adic valuation over $\mathbf{Q}_p$ normalized by $v_p(p) = 1$ (we set $v_p(0) = +\infty$). Let α_1, α_2 be algebraic over $\mathbf{Q}$ and we regard them as elements of the field $\overline{\mathbf{Q}}_p$. We look for a lower bound for the ultrametric absolute value of

$$\Lambda = \alpha_1^{b_1} - \alpha_2^{b_2},$$

where b_1 and b_2 denote positive rational integers.

Our results depend among other things on several parameters related to the field $\mathbf{K}_v = \mathbf{Q}_p(\alpha_1, \alpha_2)$. Hence, we denote by e the ramification index from $\mathbf{Q}_p$ to $\mathbf{K}_v$, by f the residual degree of this extension, and by $t \geq 0$ the integer satisfying the inequalities

$$p^{t-1} \leq \frac{e}{p-1} < p^t.$$

Put

$$D = \frac{[\mathbf{Q}(\alpha_1, \alpha_2) : \mathbf{Q}]}{f}.$$

As in [2], we assume that $v_p(\alpha_1) = v_p(\alpha_2) = 0$, and we denote by g the smallest positive integer such that

$$v_p(\alpha_1^g - 1) > 0 \text{ and } v_p(\alpha_2^g - 1) > 0.$$

We recall from [2] that g divides $p^f - 1$ and that, letting ζ be a fixed g-th primitive root of unity in $\overline{\mathbf{Q}}_p$, we can write uniquely

$$\alpha_1 = \zeta^{m_1} \theta_1 \text{ and } \alpha_2 = \zeta^{m_2} \theta_2, \tag{3}$$

where m_1 and m_2 are determined modulo g, $v_p(\theta_1 - 1) > 0$ and $v_p(\theta_2 - 1) > 0$. Comparing with [2], we make the additional assumption that there exists a real E such that

$$v_p(\alpha_1^g - 1) \geq E > \frac{1}{p-1}.$$

In this particular setting, we can improve Théorème 1 of [2] in the following way.

Theorem 1. *Let $K \geq 3$, $L \geq 2$, R_1, R_2, S_1, S_2 be positive integers and set*

$$R = R_1 + R_2 - 1, \qquad S = S_1 + S_2 - 1, \qquad N = KL,$$
$$\gamma_1 = \frac{R+g-1}{2R} - \frac{gN}{6R(S+g-1)}, \quad \gamma_2 = \frac{S+g-1}{2S} - \frac{gN}{6S(R+g-1)}.$$

Denote by p^u the greatest power of p which divides simultaneously b_1 and b_2 and assume that p does not divide b_2/p^u. Put

$$b = \frac{(R-1)b_2 + (S-1)b_1}{2} \left(\prod_{k=1}^{K-1} k! \right)^{-2/(K^2-K)}.$$

Assume that there exist two residue classes c_1 and c_2 modulo g such that :

$$\begin{aligned} &\mathrm{Card}\left\{\alpha_1^{p^t r}\alpha_2^{p^t s};\, 0 \leq r < R_1,\, 0 \leq s < S_1,\, m_1 r + m_2 s \equiv c_1 \text{ modulo } g\right\} \geq L,\\ &\mathrm{Card}\left\{rb_2 + sb_1;\, 0 \leq r < R_2,\, 0 \leq s < S_2,\, m_1 r + m_2 s \equiv c_2 \text{ modulo } g\right\} > (K-1)L. \end{aligned} \tag{4}$$

Under the condition

$$\begin{aligned} K(L-1)\,(E+t)\log p - (D+2e)\log N - D(K-1)\log b \\ - \gamma_1 p^t DLR\,\mathrm{h}(\alpha_1) - \gamma_2 p^t DLS\,\mathrm{h}(\alpha_2) \ > \ 0. \end{aligned} \tag{5}$$

we have

$$v_p(\Lambda) \le (E+t)\,(KL-1/2) + u.$$

Further, if α_1 *and* α_2 *are rational numbers and if* p *is an odd prime number or if* $p=2$ *and* $v_2(\alpha_2-1) \ge 2$*, the condition*

$$\begin{aligned} K(L-1)E\log p - 3\log N - (K-1)\log b \\ - \gamma_1 LR\,\mathrm{h}(\alpha_1) - \gamma_2 LS\,\mathrm{h}(\alpha_2) \ > \ 0, \end{aligned} \tag{6}$$

implies that

$$v_p(\Lambda) \le E\,(KL-1/2) + u.$$

As we are here interested only in the rational case, we point out the following consequence of Theorem 1. Let x_1/y_1 and x_2/y_2 be two non-zero rational numbers and assume that there exists a real number E such that $v_p\big((x_1/y_1)^g - 1)\big) \ge E > 1/(p-1)$. Theorem 2 below provides an explicit upper bound for the p-adic valuation of

$$\Lambda = \left(\frac{x_1}{y_1}\right)^{b_1} - \left(\frac{x_2}{y_2}\right)^{b_2},$$

where b_1 and b_2 are positive integers. As in [2] and [6], we let $A_1 > 1, A_2 > 1$ be real numbers such that

$$\log A_i \ge \max\{\log|x_i|, \log|y_i|, E\log p\}, \ \ (i=1,2).$$

and we put

$$b' = \frac{b_1}{\log A_2} + \frac{b_2}{\log A_1}.$$

Theorem 2. *With the above notation, if* x_1/y_1 *and* x_2/y_2 *are multiplicatively independent, then we have the upper estimates*

$$v_p(\Lambda) \le \frac{36.1\,g}{E^3\,(\log p)^4}\big(\max\{\log b' + \log(E\log p) + 0.4, 6\,E\log p, 5\}\big)^2 \log A_1 \log A_2$$

and

$$v_p(\Lambda) \le \frac{53.8\,g}{E^3\,(\log p)^4}\big(\max\{\log b' + \log(E\log p) + 0.4, 4\,E\log p, 5\}\big)^2 \log A_1 \log A_2,$$

if p *is odd or if* $p=2$ *and* $v_2(\alpha_2-1) \ge 2$*. Else, we have*

$$v_2(\Lambda) \le 208\,\big(\max\{\log b' + 0.04, 10\}\big)^2 \log A_1 \log A_2.$$

Remark : The novelty in Theorem 2 occurs in the term E. This should be compared with Corollary 3 of Laurent, Mignotte & Nesterenko [6], where a corrected factor denoted by $\log E$ also appears. Roughly speaking, provided that the algebraic numbers involved are closed to 1, their corollary allows them to replace the product of the heights of the algebraic numbers by their sum. As we shall see in the proofs of the following statements, the same also applies in the non-archimedean case, provided that the algebraic numbers are p-adically closed to 1. It seems to be the first time that this has been noticed and it turns out to have some interesting consequences in the context of equation (1).

3. Applications to the Diophantine equation $\frac{x^n - 1}{x - 1} = y^q$

In all what follows, we assume that q is a prime number and that $q \neq 2$, in view of the result of Ljunggren [9].

Maohua Le [7] proved that (1) has only finitely many solutions (x, y, n, q) with x a prime power and $y \equiv 1 \pmod{x}$. His proof is divided in two main steps. Firstly, he gives a sharp upper bound for x in terms of q, provided by an hypergeometric approach. Secondly, he bounds q using linear forms in three archimedean logarithms, together with the result obtained in his first step. Theorem 3 below may favorably replace his second step, in the sense that we obtain a sharp explicit upper estimate for q.

We warn the reader that the results obtained by Le [8] and by Yu & Le [18] remain unproved. Indeed, they all depend on Lemma 3 of [8], which is incorrect; see the comment of Yuan Ping-Zhi [19].

Theorem 3. *Let (x, y, n, q) be a solution of (1). If $x = p^m$ for an integer $m \geq 1$ and a prime p satisfying $\gcd(q, p-1) = 1$, then we have*

$$q \leq 1900.$$

If $x = z^m$ for rational integers $m \geq 1$ and $z \geq 2$ such that every prime divisor of z also divides $y - 1$, then we get

$$q \leq 17000 \log^2 z.$$

Remarks : • We point out that the upper bounds for q are independent of m, and this is achieved thanks to the factor E, appearing in the statement of Theorem 2. Théorème 2 of [2] could also be used in this context, but it is not sharp enough to allow us to drop the dependence on m. Further, we can consider that Theorem 3 is the p-adic analogue of the results of [10] and [1] concerning equation (2), where the above mentioned Corollaire 3 of [6] allows the authors to bound n independently of a. Notice that it has been already observed (and used) that equations (1) and (2) are deeply related.

• An explicit upper bound for q could also be derived from the proof of Theorem 2 of [13], where the authors use estimates for linear forms in three archimedean logarithms. However, the numerical constant obtained by their method would be greater than ours, and the complete resolution of (1) for some fixed x would be hopeless.

• For a solution (x, y, n, q) of (1), the condition "every prime divisor of x also divides $y - 1$" is obviously satisfied if $y \equiv 1 \pmod{x}$, which is the hypothesis made by

Le [7]. This is also satisfied when $\gcd\big(q, \prod_{p|x}(p-1)\big) = 1$. Indeed, rewrite (1) under the form

$$x\,\frac{x^{n-1}-1}{x-1} = (y-1)\,\frac{y^q-1}{y-1}, \tag{7}$$

and let p be a prime divisor of x for which $y \not\equiv 1 \pmod{p}$. Then $y^q \equiv 1 \pmod{p}$, and this can be possible only if q divides $p-1$. By this remark and Lemma 14 of [13], we can check that the hypothesis of Theorem 3 are satisfied by all the integers z in the interval $[2, 20]$, except maybe 11. However, we are unable to prove the next statement.

Problem. *The Diophantine equation*

$$\frac{11^{mn}-1}{11^m-1} = y^5 \qquad \text{in integers } m \geq 1, n > 2, y > 1$$

has only finitely many solutions.

Combining Theorem 2 with Lemma 19 of [13], a sharpening of an above quoted result of Le [7], we obtain the following explicit version of Theorem 2 of [13].

Theorem 4. *If (p^m, y, n, q) satisfies (1), with $q \geq 3$ and p a prime number such that $\gcd(q, p(p-1)) = 1$, then we have*

$$p^m \leq 4063.$$

If $p = q$, then we get $p \leq 1900$ and $p^{m-2} \leq 41000$. Further, if (x^m, y, n, q) is a solution of (1), then either $x^m \leq 6 \cdot 10^{19}$, or there exists a prime p such that p divides x and p does not divide $y-1$.

Remark : The main result of [2] would have been sufficient to prove Theorem 3, however with slightly greater numerical constants. Nevertheless, Theorem 1 is helpful for the complete resolution of some families of equations (1) : this question will be investigated in a further work.

It is rather interesting to compare Theorem 4 with the following recent result of Hirata-Kohno & Shorey [5].

Theorem HKS. *Let $z > 1$ be an integer and $\mu \geq 3$ be a prime number. Assume that*

$$q > 2(\mu-1)(2\mu-3).$$

Then equation (1) with $x = z^\mu$ implies that $\max\{x, y, n, q\}$ is bounded by an effectively computable number depending only on μ.

In their proof, Hirata-Kohno & Shorey [5] use the hypergeometric method to bound $\max\{x, y, n\}$ in terms of q and μ. Then, they apply an estimate for linear forms in archimedean logarithms of algebraic numbers close to 1 to bound q in terms of μ. Here, we use Lemma 1, proved by the hypergeometric method, to bound x, and we conclude by Theorem 2. Notice also that Theorem 4 (*resp.* Theorem HKS) depends on an additional assumption, which means that q is supposed to be sufficiently large compared with the unknown x (*resp.* the unknown μ).

As already mentioned in details in [13], Theorem 4 can be applied to obtain irrationality statements. To this end, we need first to introduce some notations. Let S_1 be the set of all positive integers greater than 1 and composed only by 2 and by the primes of the form 2^a+1, for $a \geq 1$. We remark that S_1 is much bigger than the set also denoted by S_1 in [13]. Let p be a prime number of the form $2^a\,3^b+1$, with integers $a \geq 0$, $b > 0$ and $p \not\equiv 55 \pmod{63}$. Thus $p \equiv 1, 4, 7 \pmod 9$, $p \not\equiv 1 \pmod 7$, and when $p \equiv 1 \pmod 9$, we have $p \not\equiv 6 \pmod 7$. Let f be any non-negative integer satisfying $f \not\equiv 1, 4 \pmod 6$ if either $p \equiv 1 \pmod 9$, $p \equiv 3, 4 \pmod 7$ or $p \equiv 4 \pmod 9$, and $f \not\equiv 2, 5 \pmod 6$ if either $p \equiv 1 \pmod 9$, $p \equiv 2, 5 \pmod 7$ or $p \equiv 7 \pmod 9$. Let S_2 be the set of all numbers of the form $2^f\,p$. Put $S_3 = S_1 \cup S_2$ and notice that S_3 is an infinite set. The next statement directly follows from Theorem 4.

Corollary 1. *Equation (1) has no solution (x, y, n, q), where $x = h^t$, with $h \in S_3$, $h^t \geq 6 \cdot 10^{19}$ and $t \geq 1$.*

Let $g \geq 2$ and $h \geq 2$ be integers. For any integer $m \geq 1$, we define $(m)_h = a_1 \dots a_r$ to be the sequence of digits of m written in h-ary notation, *i.e.* under the form $m = a_1\,h^{r-1} + \dots + a_r$, with $a_1 > 0$ and $0 \leq a_i < h$ for $1 \leq i \leq r$. For a sequence $(n_i)_{i\geq 1}$ of non-negative integers, we put

$$a_h(g) = 0.(g^{n_1})_h\,(g^{n_2})_h \dots$$

and we call Mahler's numbers the real numbers obtained in this way. It is known that $a_h(g)$ is irrational for any unbounded sequence $(n_i)_{i\geq 1}$; see the work of Sander [12] for an account of earlier results in this direction. Sander also considered the case when $(n_i)_{i\geq 1}$ is bounded with exactly two elements occurring infinitely many times, which are called, by definition, limit points. As mentioned in [13], his paper contained an incorrect application of a result of Shorey & Tijdeman [16], hence his Theorem 3 remains unproved. Here, we extend Corollary 3 of [13] as follows.

Theorem 5. *Let $(n_i)_{i\geq 1}$ be a bounded sequence of non-negative integers which is not ultimately periodic and has exactly two limit points $N_1 < N_2$. Let $g \geq 2$ and $h \geq 2$ be integers such that $g \neq 1 + h + \dots + h^{L-1}$ for every integer $L \geq 2$. Assume also that for the integer t given by the inequalities $h^{t-1} \leq g < h^t$, we have $g^{N_2 - N_1} \neq h^t + 1$. If $h \in S_3$ and $h^t \geq 6 \cdot 10^{19}$, then $a_h(g)$ is irrational.*

4. An auxiliary result

When x is a prime power, a first version of Lemma 1 below is due to Le [7]. This result, proved by hypergeometric methods, has then been sharpened and generalized by Saradha & Shorey [13].

Lemma 1. *Let (x, y, n, q) be a solution of (1) such that every prime divisor of x also divides $y - 1$. If $q = 3$ and $\gcd(3, x) = 1$, then $x \leq 1927$. If $q \geq 5$ and $\gcd(q, x) = 1$, then $x \leq \max\{961, 2.1382\,q\}$. If $q \geq 3$ and q divides x, then $x \leq 41000\,q^2$.*

Proof : The first and third assertions follow from Lemma 18 of [13] and the second one is their Lemma 19. Indeed, let p be a prime divisor of x and let $a \geq 1$ such that p^a exactly divides x. We have $y \equiv 1 \pmod p$, whence, since $q \neq p$, $(y^q - 1)/(y - 1) \not\equiv$

$0 \pmod p$. Thus $y \equiv 1 \pmod{p^a}$, and we infer that $y \equiv 1 \pmod x$. It suffices now to follow the proof of Lemma 19 of [13]. $\square$

5. Proofs

Proof of Theorem 1 : We carefully follow the proof of Théorème 1 of [2] and we refer the reader to [2] for the notation. We only recall that we construct a matrix $\mathcal{M}$, and we denote by Δ the determinant of a suitable invertible sub-matrix $N \times N$ of $\mathcal{M}$, whose existence follows from a zero-lemma. The Lemme 11 of [2] provides an upper bound for $v_p(\Delta)$, while Lemme 8 gives a lower bound for $v_p(\Delta)$, assuming that $v_p(\Lambda)$ is not too small. In our context, we do not have to change Lemme 11, but we have to prove the following modified version of Lemme 8.

Claim. *Assume that $v_p(\Lambda) \ge (E+t)\,(N-1/2)+u$. Then we have*

$$v_p(\Delta) \ge \frac{(E+t)\,NK(L-1)}{2} - \frac{N\log N}{\log p}.$$

To prove the claim, we only point out which changes have to be made in the proof of Lemme 8 of [2]. Observe that it suffices to prove the lemma when α_1 and α_2 are both principal units. Indeed, by (3), we have $v_p(\alpha_1^g - 1) = v_p(\theta_1^g - 1) = v_p(\theta_1 - 1)$, since p does not divide g. To deal with the case $u > 0$, we prove, using exactly the same argument and the same notation as in [2], that

$$v_p(\theta_1^{b_1'} - \xi_1\,\theta_2^{b_2'}) \ge (E+t)\,(N-1/2),$$

for a suitable p^t-th root of unity ξ_1. By Lemma 3 of [2], observe that $v_p(\theta_1^{p^t} - 1) \ge E+t$. Then it suffices to replace the lower bound $v_p(\omega) \ge p^t/e > 1/(p-1)$, used in [2], by $v_p(\omega) \ge E+t > 1/(p-1)$. We also remark that the treatment of condition (4) follows the same lines as in [2].

In the rational case, observe that $t = 0$ unless $p = 2$. However, if $p = 2$ and $v_2(\alpha_2 - 1) \ge 2$, then we do not need to raise θ_1 and θ_2 to the power 2 in order to be in the disk of convergence of the 2-adic logarithm. Hence, we have in fact $t = 0$, and we can suppress the terms p^t (*i.e.* 2) in Lemme 11 of [2]. Thus, inequality (6) implies (5). $\square$

Proof of Theorem 2 : If $p = 2$ and $v_2(x_2/y_2 - 1) = 1$, this is Corollaire 2 of [2]. Thus we may assume that p is odd or that $p = 2$ and $v_2(\alpha_2 - 1) = 2$. Put

$$a_i = \frac{\log A_i}{E\,\log p} \ge \max\left\{\frac{\log|x_i|}{E\,\log p}, \frac{\log|y_i|}{E\,\log p}, 1\right\}, \quad (i = 1, 2).$$

and let

$$B \ge \frac{\log b}{E\,\log p}$$

be a positive real number. The condition (6) is fulfilled if our parameters satisfy the inequality

$$K(L-1) > 3\,\frac{\log N}{E\,\log p} + (K-1)B + \gamma_1 LRa_1 + \gamma_2 LSa_2. \tag{8}$$

We observe that (8) corresponds to (2′) of [2]. Hence, we can carefully follow the proof of [2] to derive our numerical estimates, being however aware that the conclusion is now $v_p(\Lambda) \le E(KL - 1/2) + u$, instead of $v_p(\Lambda) \le p(KL - 1/2)/(p-1) + u$. The numerical constants are given by Théorème 4 of [2]. $\square$

Proof of Theorems 3 and 4 : Let (x, y, n, q) be a solution of (1) and put

$$\Lambda = (1 - x) - \left(\frac{1}{y}\right)^q = -x^n\, y^{-q}.$$

We note that $1-x$ and $1/y$ are multiplicatively independent (for a proof, see [13], below inequality (51)). Assume first that $x = p^m$ is a prime power, with $m \ge 1$. If $x = 2$, then (1) becomes $2^n - y^q = 1$, which has no integer solution, see *e.g.* Cassels [4]. If $x = 2^m$, with $m \ge 2$, we derive from (7) and from the fact that q is odd that $y \equiv 1 \pmod{2^m}$. For odd p, we infer from (7) that $y \equiv 1 \pmod{p^m}$ if $p \ne q$ and that $y \equiv 1 \pmod{\max\{p, p^{m-1}\}}$ for $p = q$. Hence, assuming that $q \ne p$ or that $q = p = x$, we get $\log y \ge m \log p$, and we can apply Theorem 2 with $\alpha_1 = (1 - p^m)$, $\alpha_2 = 1/y$, $\log A_1 = m \log p$, $\log A_2 = \log y$, $E = m$ and $b' \le (q+1)/(m \log p)$. Since $g = 1$ and $6\, m \log p \ge 5$, we get

$$v_p(\Lambda) \le \frac{36.1}{m^2\,(\log p)^3} \big(\max\,\{\log(q+1) + 0.4, 6\, m \log p\}\big)^2 \log y. \tag{9}$$

On the other hand, we have $p^{mn} \ge y^q$ and $v_p(\Lambda) = mn$, thus

$$v_p(\Lambda) \ge q\, \frac{\log y}{\log p}. \tag{10}$$

Combining (9) with (10), and using that $m \log p \ge \log 3$, we immediately obtain

$$q \le 1900.$$

If $p = q$ and $m \ge 2$, we apply Theorem 2, but we put $\log A_2 = 2 \log y$, so that $\log A_2$ is greater than $m \log p$. As $4\, m \log p \ge 8 \log 2 \ge 5$, we get

$$v_p(\Lambda) \le \frac{2 \cdot 53.8}{m^2\,(\log p)^3} (4\, m \log p)^2 \log y,$$

whence $q \le 1722$ by (10).

Now, suppose that z is not a prime power and write $z = p_1^{a_1} \dots p_t^{a_t}$, where $p_1 < p_2 < \dots < p_t$ are distinct prime numbers dividing $y - 1$ and the a_i's are strictly positive integers. We deduce from (7) that $v_{p_i}(y-1) \ge a_i\, m > 0$ if $q \ne p_i$ and $v_{p_i}(y-1) \ge \max\{a_i\, m - 1, 1\}$ else, whence $2 \log y \ge a_i \log p_i$ in both cases. Except when $p_1^{a_1\, m} = 2$, we infer from Theorem 2 that

$$v_{p_i}(\Lambda) \le \frac{72.2}{m^2\, a_i^3\, (\log p_i)^4} \big(\max\,\{\log(q+1) + 0.4, 6\, m\, a_i \log p_i\}\big)^2 \log z \log y. \tag{11}$$

If 2 exactly divides x, then we estimate $v_2(\Lambda)$ by applying the last statement of Theorem 2, and we get

$$v_2(\Lambda) \le 208\, \big(\max\{\log(q+1) + 0.41, 10\}\big)^2 \log z \log y. \tag{12}$$

However, we have

$$q \log y \leq m\, n\, (a_1 \log p_1 + \ldots + a_t \log p_t) = \sum_{i=1}^{t} v_{p_i}(\Lambda) \log p_i, \tag{13}$$

and by combining (11), (12) and (13), we get after some calculation that

$$q \leq 17000 \log^2 z,$$

the worst case corresponding to $z = 6$.

Theorem 4 follows from Theorem 3 by applying Lemma 1. □

Proof of Corollary 1 : Let $x = 2^{b_0} (2^{a_1} + 1)^{b_1} \ldots (2^{a_\ell} + 1)^{b_\ell}$ be an element of S_1 with $b_i \geq 0$ for $0 \leq i \leq \ell$. If $b_0 > 0$, then, since q is odd, we deduce from (1) that $y \equiv 1 \pmod 2$. Let $1 \leq i \leq \ell$ such that $b_i > 0$. Working modulo $2^{a_i} + 1$, we obtain $y^q \equiv 1 \pmod{2^{a_i} + 1}$, implying that $y \equiv 1 \pmod{2^{a_i} + 1}$, since q does not divide 2^{a_i}. We conclude by applying Theorem 4. As for the set S_2, we refer the reader to the proof of Corollary 2 of [13]. □

Proof of Theorem 5 : Sander ([12], Theorem 2) proved that $a_h(g)$ is irrational if and only if $g^{N_2 - N_1} \neq (h^{tL} - 1)/(h^t - 1)$ for every integer $L \geq 1$, where t is given in the statement of the Theorem. As noticed in [13], we have $(N_1, N_2) = (0, 1)$ or $N_2 - N_1 \geq 2$. To the first case corresponds the first condition in the statement of Theorem 5, and it is clear that if $g = 1 + h + \ldots + h^{L-1}$ for an integer $L \geq 2$, then $a_h(g)$ is rational. Now we assume $N_2 - N_1 \geq 2$ and $g^{N_2 - N_1} \neq h^t + 1$. Hence, $a_h(g)$ is rational if and only if there is an integer $L \geq 3$ satisfying $g^{N_2 - N_1} = (h^{tL} - 1)/(h^t - 1)$. In view of Corollary 1, this cannot happen since $h \in S_3$ and $h^t \geq 6 \cdot 10^{19}$. □

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