BOUNDS FOR THE SOLUTIONS OF SUPERELLIPTIC EQUATIONS

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1. Introduction

In this work, we study the diophantine equation

$$f(X) = Y^m \tag{1.1}$$

where $m \geq 2$ is an integer and $f(X)$ is a polynomial with coefficients in a number field $\mathbf{K}$. The first important result on this topic is due to Siegel [19], who showed that if $m = 2$ and f has at least three simple roots or if $m \geq 3$ and f has at least two simple roots, then (1.1) has only finitely many integral solutions. Three years later, he proved [20] that if the algebraic curve defined by (1.1) is of positive genus, then (1.1) has only finitely many integral solutions. The p-adic analogue of this theorem was established independently by Lang [9] and LeVeque [12], who showed that, under the same conditions, (1.1) has only finitely many S-integral solutions. After that, LeVeque [13] gave a necessary and sufficient condition for the algebraic curve defined by (1.1) to have positive genus. However, all these results are based on Thue's method, and hence are ineffective.

Using his work on linear forms in logarithms, Baker [1] gave the first effectively computable bound on the size of rational integer solutions of (1.1) in the case $\mathbf{K} = \mathbf{Q}$, under the same hypothesis as Siegel [19]. His results were improved and extended to algebraic number fields by Sprindžuk [23] (see also [24] and the references given there), Brindza [4], Poulakis [16], Schmidt [17] and, more recently, Voutier [26]. We also have to mention an unpublished paper of Bilu [3]. Further, using the p-adic theory of linear forms in logarithms, due to Van der Poorten, generalizations to the case of S-integral solutions were established by Trelina [25], Kotov and Trelina [8] and Brindza [4], among others (see [18] for more references).

The purpose of the present work is to improve and generalize to the p-adic case Voutier's results. We will more or less follow his proofs, however, using some new ideas, we give effective upper bounds for the size of S-integral solutions with a better dependence on m and show that the dependence on the height of the polynomial f is trivial if one takes also its discriminant into account (this was first noticed by Trelina). Our main tools are the new results due to the author and Győry [5], [6], concerning the size of the solutions of S-unit and Thue-Mahler equations.

2. Statement of the results

Let $\mathbf{K}$ be a number field of degree d. Denote by $D_{\mathbf{K}}$ its discriminant, by $h_{\mathbf{K}}$ its class number and by $O_{\mathbf{K}}$ the ring of integers in $\mathbf{K}$. Let S be a finite set of places on $\mathbf{K}$, including the set of infinite places S_∞. Denote by $\mathfrak{p}_1, \dots, \mathfrak{p}_t$ the t prime ideals corresponding to the finite places of S. Further, denote by O_S the ring of S-integers in $\mathbf{K}$. Let $n \geq 2$ be an integer. We consider a monic polynomial

$$f(X) = X^n + a_{n-1}X^{n-1} + \ldots + a_0 \in O_{\mathbf{K}}[X].$$

Let $a \in O_{\mathbf{K}} \setminus \{0\}$ and $m \geq 2$ be an integer, we study the equation

$$f(x) = a\, y^m \quad \text{in } (x, y) \in O_S \times \mathbf{K}. \tag{2.1}$$

Denote by $\alpha_1, \dots, \alpha_r$ the r distinct roots of f and, respectively, by $e_1, \dots, e_r$ their order of multiplicity. For $i = 1, \dots, r$, we define the positive integer

$$m_i := \frac{m}{(e_i, m)}$$

and we reorder the roots such that $m_1 \geq \ldots \geq m_r$.

We assume that the m_i's satisfy the so-called "LeVeque's condition", i.e. that

$$(m_1, \dots, m_r) \neq (2, 2, 1, \dots, 1)$$

and

$$(m_1, \dots, m_r) \neq (t, 1, \dots, 1) \quad \text{where } t \text{ denotes any integer.}$$

Under this condition, it follows from LeVeque [13] that (1) has only finitely many solutions. The purpose of this work is to give a new upper bound for the size of these solutions. We pay particular attention to the dependence on the parameters of the field $\mathbf{K}$ and especially on the height of the polynomial f (for the definition, see Section 3). As in [5] and [6], we denote by $\mathrm{h}(\alpha)$ the absolute multiplicative height of the algebraic number α.

Before stating our theorems, we have to introduce some more notations. We define the polynomial

$$g(X) = (X - \alpha_1) \ldots (X - \alpha_r) \in O_{\mathbf{K}}[X]$$

and denote by $\Delta_g := \prod_{i \neq j} (\alpha_i - \alpha_j)$ its discriminant. Let P be the largest of the rational primes lying below the finite places of S, with the convention that $P = 1$ if $S = S_\infty$. Further, suppose that $|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(a)|$ is at most $A\,(\geq e)$ and that the height of the polynomial f is bounded by $H\,(\geq e^e)$.

Throughout this paper, we stand the notation $\log^* x$ for $\max\{\log x, 1\}$.

Theorem 1. *If $m_i \leq 2$ for each i and there are at least three roots for which $m_i = 2$, then all the solutions of (2.1) satisfy*

$$\mathrm{h}(x) \leq H^2 \, \exp\Big\{ c_1(d,n,t)\, P^{4n^3 d} \, (\log^* P)^{4n^2 dt} \, |D_{\mathbf{K}}|^{15n^2/2} \, A^{3n^2} \, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{12n} \\ \big(\log |A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big)^{6n^2 d} \, \log\log H \Big\}$$

and

$$\mathrm{h}(x) \leq H^2 \, \exp\Big\{ c_2(d,n,t) P^{4n^3 d} \, (\log^* P)^{4tn^3} \, |D_{\mathbf{K}}|^{16n^3} \\ |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{28n^2} \, A^{8n^3} \, \big(\log |A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big)^{8n^3 d} \Big\},$$

where $c_1(d,n,t)$ and $c_2(d,n,t)$ are effectively computable constants.

Remark : The purpose of the first inequality is to give a better estimate in terms of $|D_{\mathbf{K}}|$ than the second one. Further, it is based on Lemma 4, which may be of independent interest.

Theorem 2. *Suppose $m \geq 3$ and there exist $1 \leq i \neq j \leq r$ such that $(m_i, m_j) \geq 3$. If (m_i, m_j) is not a power of 2, let m' be the smallest odd prime number dividing it, otherwise put $m' = 4$. Then all the solutions of (1) satisfy*

$$\mathrm{h}(x) \leq H^{m'+1} \, \exp\{ c_3(d,n,m,t)\, P^{dn^2 m'^3} \, (\log^* P)^{tn^2 m'} \, |D_{\mathbf{K}}|^{5n^2 m'/2} \\ |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{5nm'} \, A^{n^2 m'} \, \big(\log |A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big)^{2dn^2 m'} \},$$

where $c_3(d,n,m,t)$ is an effectively computable constant.

In the particular case $S = S_\infty$, Theorem 2 improves Theorem 2 of [26] in terms of $|D_{\mathbf{K}}|$: we remove a factor mm'. This is mainly due to the following two reasons. On the one hand, we use a case by case analysis which allows us to work in a field $\mathbf{M}$ of degree less than $n^2 m'$ over $\mathbf{K}$ and to derive either S-unit equations, or a Thue-Mahler equation. On the other hand, Lemma 9 (see Section 4), due to the referee, provides us a sharp upper bound for the discriminant of the field $\mathbf{M}$.

From Theorem 2 we deduce the following result.

Theorem 3. *Under LeVeque's condition, suppose that the hypotheses of Theorems 1 and 2 are not fulfilled. Then there exist $1 \leq i \neq j \leq r$ such that $m_i \geq 3$ and $m_j \geq 2$, and all the solutions of (1) satisfy*

$$\mathrm{h}(x) \leq H^{m^2} \, \exp\{ c_4(d,n,m,t)\, P^{d(m^5+4tm^3)/2} \, |D_{\mathbf{K}}|^{m^6/8} \, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{m^6/8} \\ A^{5m^4/8} \, \big(\log |A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big)^{dm^3} \},$$

where $c_4(d,n,m,t)$ is an effectively computable constant.

It is interesting to note that the dependence on n appears only in the constant c_4.

These three theorems considerably improve and generalize the results of Trelina [25] in terms of t, H, A, $|D_{\mathbf{K}}|$ and $|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|$. In particular, the exponents of A, $|D_{\mathbf{K}}|$ and $|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|$ do not depend on t; this is essentially due to the new results concerning the size of the solutions of S-unit and Thue-Mahler equations [5], [6].

Remarks : If (x, y) is a solution of (1), Theorems 1 to 3 give estimates only for the size of x. A bound for the size of y immediately follows, but it also involves the height of a.

If, more generally, the polynomial f has coefficients in $\mathbf{K}$, we easily deduce from our theorems upper bounds for the size of the solutions of (2.1), but we have to take into consideration the denominator and the leading coefficient of f.

Noticing that we can bound $|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|$ by H^{2dn} times a constant depending on d and n (cf [26], Lemma 7), our theorems also provide estimates involving only the height of the polynomial f. However, we point out that the height of f can be arbitrarily large compared with the discriminant of g.

3. Bounds for S-units, S-regulators and linear forms in logarithms

Let $\mathbf{K}$ be an algebraic number field, denote by d its degree and by $M_{\mathbf{K}}$ the set of places on $\mathbf{K}$. Let S be a finite subset of $M_{\mathbf{K}}$ containing the set of infinite places S_∞. Throughout this paper, we will always use the notation $D_{\mathbf{K}}$, $O_{\mathbf{K}}$, $O_{\mathbf{K}}^*$, O_S, O_S^*, R_S and N_S for, respectively, the discriminant of $\mathbf{K}$, the ring of integers in $\mathbf{K}$, the group of units in $\mathbf{K}$, the ring of S-integers in $\mathbf{K}$, the group of S-units in $\mathbf{K}$, the S-regulator and the S-norm (see definitions below). For every place v we choose a valuation $|\,.\,|_v$ in the following way : if v is infinite and corresponds to an embedding $\sigma : \mathbf{K} \to \mathbf{C}$ then we put, for every $\alpha \in \mathbf{K}$,

$$|\alpha|_v = |\sigma(\alpha)|^{d_v},$$

where $d_v = 1$ or 2 according as $\sigma(\mathbf{K})$ is contained in $\mathbf{R}$ or not; if v is a finite place corresponding to the prime ideal $\mathfrak{p}$ in $\mathbf{K}$ then we put $|0|_v = 0$ and, for $\alpha \in \mathbf{K} \setminus \{0\}$,

$$|\alpha|_v = \mathrm{N}(\mathfrak{p})^{-\mathrm{ord}_{\mathfrak{p}}(\alpha)}.$$

The (absolute) height of an algebraic number α contained in $\mathbf{K}$ is defined by

$$\mathrm{h}(\alpha) = \Big(\prod_{v \in \mathbf{M_K}} \max(1, |\alpha|_v) \Big)^{1/d}.$$

This height is independent of the choice of $\mathbf{K}$. Moreover,

$$\sum_{v \in M_{\mathbf{K}}} \big|\log |\alpha|_v\big| = 2d \log \mathrm{h}(\alpha). \tag{3.1}$$

For a polynomial $F(X) = X^l + b_{l-1}X^{l-1} + \ldots + b_0 \in \mathbf{K}[X]$, we define its height $\mathrm{h}(F)$ by

$$\mathrm{h}(F) = \Big(\prod_{v \in \mathbf{M_K}} \max\big\{1, |b_0|_v, \ldots, |b_{l-1}|_v\big\} \Big)^{1/d}.$$

It is well-known (cf. [22], Chapter *VIII*, Theorem 5.9) that

$$2^{-l}\prod_{\alpha \text{ root of } F}\mathrm{h}(\alpha)\le \mathrm{h}(F)\le 2^{l-1}\prod_{\alpha \text{ root of } F}\mathrm{h}(\alpha). \tag{3.2}$$

Let now define the S-norm and the S-regulator. For $\alpha\in\mathbf{K}\setminus\{0\}$, the ideal (α) generated by α can be uniquely written in the form $\mathfrak{a}_1\mathfrak{a}_2$ where the ideal $\mathfrak{a}_1$ (resp. $\mathfrak{a}_2$) is composed of prime ideals outside (resp. inside) S. The S-norm of α, denoted by $\mathrm{N}_S(\alpha)$, is defined as $\mathrm{N}(\mathfrak{a}_1)$, and we put $\mathrm{N}_S(0)=0$. The S-norm is multiplicative, and, for $S=S_\infty$, we have $\mathrm{N}_S(\alpha)=|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\alpha)|$. For any $\alpha\in\mathbf{K}$, we have $\mathrm{N}_S(\alpha)=\prod_{v\in S}|\alpha|_v$ and $\mathrm{N}_S(\alpha)\le\big(\mathrm{h}(\alpha)\big)^d$. Further, if $\alpha\in O_S\setminus\{0\}$, then $\mathrm{N}_S(\alpha)$ is a positive integer.

Let s be the cardinality of S. For $v\in S$, denote by $|\cdot|_v$ the corresponding valuation normalized as above. Let $v_1,\dots,v_{s-1}$ be a subset of S, and let $\{\varepsilon_1,\dots,\varepsilon_{s-1}\}$ be a fundamental system of S-units in $\mathbf{K}$. Denote by R_S the absolute value of the determinant of the matrix $(\log|\varepsilon_i|_{v_j})_{i,j=1,\dots,s-1}$. It is easy to verify that R_S is a positive number which is independent of the choice of $v_1,\dots,v_{s-1}$ and of the fundamental system of S-units $\{\varepsilon_1,\dots,\varepsilon_{s-1}\}$. R_S is called the S-regulator of $\mathbf{K}$. If in particular $S=S_\infty$, then we have $R_S=R_{\mathbf{K}}$, the regulator of $\mathbf{K}$.

We refer to [5] for the proofs of Lemmas 1 – 3 (the first two of them go back to Siegel's well-known paper [21]). We recall that there exists a constant $\delta_d>0$, depending only on d, such that $\log\mathrm{h}(\alpha)\ge\delta_d/d$ for any non-zero algebraic number α with degree $\le d$ unless α is a root of unity. Put

$$c_5=c_5(d,s)=\big((s-1)!\big)^2/(2^{s-2}k^{s-1}),$$

$$c_6=c_6(d,s)=c_5(\delta_d/d)^{2-s},\ c_7=c_7(d,s)=c_5d^{s-1}\delta_d^{-1}.$$

Lemma 1. *There exists in* $\mathbf{K}$ *a fundamental system* $\{\varepsilon_1,\dots,\varepsilon_{s-1}\}$ *of* S*-units with the following properties :*

(i)
$$\prod_{i=1}^{s-1}\log\mathrm{h}(\varepsilon_i)\le c_5R_S;$$

(ii)
$$\log\mathrm{h}(\varepsilon_i)\le c_6R_S,\quad i=1,\dots,s-1;$$

(iii) *the absolute values of the entries of the inverse matrix of* $(\log|\varepsilon_i|_{v_j})_{i,j=1,\dots,s-1}$ *do not exceed* c_7.

Denote by $R_{\mathbf{K}}$, $h_{\mathbf{K}}$ and $r=r_{\mathbf{K}}$ the regulator, class number and unit rank of $\mathbf{K}$. Let $\mathfrak{p}_1,\dots,\mathfrak{p}_t$ be the prime ideals corresponding to the t finite places in S, and denote by P the largest of the rational primes lying below them. Put $c_8=c_8(d,r)=r^{r+1}\delta_d^{-(r-1)}/2$.

Lemma 2. *For every $\alpha \in O_S \setminus \{0\}$ and every integer $n \geq 1$ there exists an S-unit ε such that*

$$\mathrm{h}(\varepsilon^n \alpha) \leq \mathrm{N}_S(\alpha)^{1/d} \exp\bigl\{n(c_8 R_{\mathbf{K}} + t h_{\mathbf{K}} \log^* P)\bigr\}.$$

Lemma 3. *If $t > 0$, then we have*

$$R_S \leq R_{\mathbf{K}} h_{\mathbf{K}} \prod_{i=1}^{t} \log \mathrm{N}(\mathfrak{p}_i) \leq R_{\mathbf{K}} h_{\mathbf{K}} (d \log^* P)^t$$

and

$$R_S \geq R_{\mathbf{K}} \prod_{i=1}^{t} \log \mathrm{N}(\mathfrak{p}_i) \geq c_9 (\log 2)^d \, (\log^* P),$$

where $c_9 = 0.2052$.

Lemma 3 was obtained independently by Bilu ([2], Proposition 1.4.8) and Bugeaud and Győry [5] (see also Pethő [15] and Hajdu [7] for similar results).

Let $\alpha_1, \ldots, \alpha_n$ $(n \geq 2)$ be non-zero algebraic numbers and let $\mathbf{K} = \mathbf{Q}(\alpha_1, \ldots, \alpha_n)$. Let $A_1, \ldots, A_n$ be positive real numbers such that

$$\log A_i \geq \max\Bigl\{\log \mathrm{h}(\alpha_i), \frac{|\log \alpha_i|}{3.3d}, \frac{1}{d}\Bigr\}, \quad i = 1, \ldots, n, \tag{3.3}$$

where log denotes the principal value of the logarithm. Let $b_1, \ldots, b_n$ be rational integers and put $B = \max\bigl\{\,|b_1|, \ldots, |b_n|, 3\bigr\}$. Further, set

$$\Lambda = \alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1.$$

In Proposition 1, it will be convenient to add the following technical conditions :

$$B \geq \log A_n \exp\bigl\{4(n+1)(7 + 3\log(n+1))\bigr\} \tag{3.4}$$

and

$$7 + 3\log(n+1) \geq \log d. \tag{3.5}$$

Proposition 1 is a consequence of Corollary 10.1 of Waldschmidt [27].

Proposition 1. (M. Waldschmidt [27]). *If $\Lambda \neq 0$, $b_n = 1$ and (3.4), (3.5) hold, then*

$$|\Lambda| \geq \exp\Bigl\{-c_{10}(n) d^{n+2} \log A_1 \ldots \log A_n \, \log\Bigl(\frac{2nB}{\log A_n}\Bigr)\Bigr\},$$

where $c_{10}(n) = 1500 \cdot 38^{n+1} (n+1)^{3n+9}$.

In Proposition 2, let $v = v_{\mathfrak{p}}$ be a finite place on $\mathbf{K}$, corresponding to the prime ideal $\mathfrak{p}$ of $O_{\mathbf{K}}$. Let p denote the rational prime lying below $\mathfrak{p}$, and denote by $|\cdot|_v$ the non-archimedian valuation normalized as above. Instead of (3.3), assume now that $A_1, \ldots, A_n$ are positive real numbers such that

$$\log A_i \geq \max\{\log \mathrm{h}(\alpha_i), |\log \alpha_i|/(10d), \log p\}, \quad i = 1, \ldots, n.$$

The following proposition is a simple consequence of the main result of Kunrui Yu [28].

Proposition 2. (Kunrui Yu [28]). *Let*

$$\Phi = c_{11}(n)\big(d/\sqrt{\log p}\big)^{2(n+1)} p^d \log A_1 \dots \log A_n \ \log(10nd \log A),$$

where $c_{11}(n) = 22000(9.5(n+1))^{2(n+1)}$ *and* $A = \max\{A_1, \dots, A_n, e\}$. *If* $\Lambda \neq 0$ *then*

$$|\Lambda|_v \geq \exp\big\{-d(\log p)\Phi \log(dB)\big\}.$$

Further, if $b_n = 1$ *and* $A_n \geq A_i$ *for* $i = 1, \dots, n-1$, *then* A *can be replaced by* $\max\{A_1, \dots, A_{n-1}, e\}$ *and for any* δ *with* $0 < \delta \leq 1$, *we have*

$$|\Lambda|_v \geq \exp\big\{-d(\log p) \max\{\Phi \log(\delta^{-1}\Phi/\log A_n), \delta B\}\big\}.$$

Thanks to the above lemmas and propositions, we are now able to state a generalization of the second part of Lemma 5 of [26] to the case of S-unit equations, which may be of independent interest.

4. Some lemmas

Lemma 4. *Let* $\mathbf{K}$ *be a number field of degree* d *and let* $\mathbf{K}_1$ *and* $\mathbf{K}_2$ *be two subfields of* $\mathbf{K}$. *Let* S *(resp.* S_1, S_2*) a finite set of places on* $\mathbf{K}$ *(resp.* $\mathbf{K}_1$, $\mathbf{K}_2$*) containing the set of infinite places* S_∞. *Denote by* s *(resp.* s_1, s_2*) the cardinality of* S *(resp.* S_1, S_2*) and by* P *the largest of the rational primes lying below the finite places of* S, *with the convention that* $P = 1$ *if* $S = S_\infty$. *Assume that* $O_{S_1}^* \subset O_S^*$ *and* $O_{S_2}^* \subset O_S^*$ *and, for* $i = 1, 2$, *denote by* R_i *the* S_i*-regulator of* $\mathbf{K}_i$. *Let* ν_1, ν_2, ν_3 *be non-zeros elements in* $\mathbf{K}$ *with height at most* H $(H \geq e)$ *and consider the equation*

$$\nu_1\varepsilon_1 + \nu_2\varepsilon_2 + \nu_3\varepsilon_3 = 0 \tag{4.1}$$

in the unknowns $\varepsilon_1 \in O_{S_1}^*$, $\varepsilon_2 \in O_{S_2}^*$ *and* $\varepsilon_3 \in O_S^*$. *Then, for* $i = 1, 2$, *we have the upper bound*

$$\mathrm{h}(\nu_i\varepsilon_i/\nu_3\varepsilon_3) < \exp\Bigg\{c_{12}(d,s)\frac{P^d}{(\log^* P)^2} R_1 R_2 \log^* \max\{R_1, R_2\} \\ \log H \ \log^* \log^* \max\{\mathrm{h}(\varepsilon_1), \mathrm{h}(\varepsilon_2)\}\Bigg\},$$

where $c_{12}(d, s)$ *is an effective constant.*

Remark : In the particular case $S = S_\infty$, this result was first obtained by Voutier ([26], Lemma 5).

Proof. Using an idea of Voutier ([26], Lemma 5), we follow the proof of the Theorem of [5]. The constants $c_{13}, \dots, c_{25}$ in the proof are all effectively computable and depend only on d and s. We recall that there exists a $\delta_d > 0$ such that $\log \mathrm{h}(\alpha) \geq \delta_d/d$ for any non-zero α in $\mathbf{K}$ which is not a root of unity. Let $\{\mu_1, \dots, \mu_{s_1-1}\}$ (resp. $\{\rho_1, \dots, \rho_{s_2-1}\}$)

be a fundamental system of S_1-units (resp. S_2-units) in $\mathbf{K}_1$ (resp. $\mathbf{K}_2$) satisfying the properties specified in Lemma 1. Then we can write

$$\varepsilon_1 = \zeta_1 \mu_1^{b_1} \ldots \mu_{s_1-1}^{b_{s_1-1}} \quad \text{and} \quad \varepsilon_2 = \zeta_2 \rho_1^{d_1} \ldots \rho_{s_2-1}^{d_{s_2-1}}. \tag{4.2}$$

with roots of unity ζ_1, $\zeta_2 \in \mathbf{K}$ and with rational integers $b_1, \ldots, b_{s_1-1}, d_1, \ldots, d_{s_2-1}$.

Put $B = \max\{|b_1|, \ldots, |b_{s_1-1}|, |d_1|, \ldots, |d_{s_2-1}|, 3\}$, it follows from (4.2) that for all $v \in S$ we have

$$\log |\varepsilon_1|_v = \sum_{i=1}^{s_1-1} b_i \log |\mu_i|_v,$$

whence, by (iii) of Lemma 1 and (3.1), we get

$$\max_{1 \le i \le s_1-1} |b_i| \le c_{13} \log \mathrm{h}(\varepsilon_1).$$

In a similar way we can bound d_i for $i = 1, \ldots, s_2 - 1$ and hence we obtain

$$B \le c_{14} \log^* \max\{\mathrm{h}(\varepsilon_1), \mathrm{h}(\varepsilon_2)\}. \tag{4.3}$$

Let $v \in S$ for which $|\varepsilon_3/\varepsilon_1|_v$ is minimal. It follows from the hypotheses that ε_1 and ε_2 are S-units in $\mathbf{K}$. Setting $\alpha_0 = -\zeta_2\nu_2/(\zeta_1\nu_1)$ and $b_0 = 1$, we deduce from (4.1) and (4.2) that

$$\left|\frac{\nu_3\varepsilon_3}{\nu_1\varepsilon_1}\right|_v = \left|\alpha_0 \mu_1^{-b_1} \ldots \mu_{s_1-1}^{-b_{s_1-1}} \rho_1^{d_1} \ldots \rho_{s_2-1}^{d_{s_2-1}} - 1\right|_v. \tag{4.4}$$

We shall derive a lower bound for $|\varepsilon_3/\varepsilon_1|_v$ in order to get an upper bound for $\mathrm{h}(\varepsilon_3/\varepsilon_1)$.

First assume that v is infinite and put

$$\begin{aligned} \log A_i &= \delta_d^{-1} \log \mathrm{h}(\mu_i), && i = 1, \ldots, s_1 - 1, \\ \log A_j &= \delta_d^{-1} \log \mathrm{h}(\rho_j), && j = s_1, \ldots, s_1 + s_2 - 2, \\ \log A_0 &= 2\delta_d^{-1} \log H. \end{aligned}$$

Condition (3.3) is then fulfilled. Indeed, let $\alpha \neq 0$ be in $\mathbf{K}$, we have to check that

$$\log \mathrm{h}(\alpha)\, \delta_d^{-1} \ge |\log \alpha|/(3.3\, d).$$

Write $\alpha = e^{a+ib}$, with $|b| \le \pi$. Then we have

$$|\log \alpha| = (a^2 + b^2)^{1/2} \le (a^2 + \pi^2)^{1/2} \le (\log^2 |\alpha| + \pi^2)^{1/2} \le (\log^2 \mathrm{h}(\alpha) + \pi^2)^{1/2}.$$

From $\log \mathrm{h}(\alpha) \ge \delta_d/d$, it follows that

$$|\log \alpha| \le \log \mathrm{h}(\alpha) \left(1 + \frac{\pi^2 d^2}{\delta_d^2}\right)^{1/2} \le \frac{\log \mathrm{h}(\alpha)}{\delta_d}\, d\,(1 + \pi^2)^{1/2},$$

since $d \geq \delta_d$. Now, it suffices to note that $(1+\pi^2)^{1/2} \leq 3.3$.

Then, we apply Proposition 1 to (4.4) and, using inequality (i) of Lemma 1 as in the proof of the Theorem of [5], we get the upper bound

$$\mathrm{h}\left(\frac{\nu_3\varepsilon_3}{\nu_1\varepsilon_1}\right) \leq \exp\{c_{15} R_1 R_2 \log H \log B\}. \tag{4.5}$$

Next assume that v is finite. To apply Proposition 2, we put now

$$\begin{aligned} \log A_i &= \delta_d^{-1} \log \mathrm{h}(\mu_i) + \log^* P, \qquad && i = 1, \ldots, s_1 - 1, \\ \log A_j &= \delta_d^{-1} \log \mathrm{h}(\rho_j) + \log^* P, \qquad && j = s_1, \ldots, s_1 + s_2 - 2, \\ \log A_0 &= 2\delta_d^{-1} \log H + \log^* P. \end{aligned} \tag{4.6}$$

Exactly as in [5], it follows from (i) of Lemma 1 and the second inequality of Lemma 3 that

$$\begin{aligned} \log A_1 \ldots \log A_{s_1+s_2-2} &\leq c_{16} R_1 (\log^* P)^{s_1-2} R_2 (\log^* P)^{s_2-2} \\ &\leq c_{16} R_1 R_2 (\log^* P)^{s_1+s_2-4}. \end{aligned} \tag{4.7}$$

Let $c_{17} = c_6\big([\mathbf{K}_1 : \mathbf{Q}], s_1\big)$ and $c_{18} = c_6\big([\mathbf{K}_2 : \mathbf{Q}], s_2\big)$. We distinguish two cases. First assume that $\log H < c_{17} R_1 + c_{18} R_2$. Then, by Lemmas 1 and 3, we have

$$\log A := \max_{0 \leq i \leq s_1+s_2-2} \log A_i \leq c_{19} \max\{R_1, R_2\}.$$

We apply now to (4.4) the first part of Proposition 2. Putting

$$\Phi = \frac{P^d}{(\log^* P)^{s_1+s_2}} \log A_0 \log A_1 \ldots \log A_{s_1+s_2-2} \log\big(10(s_1+s_2-1) d \log A\big), \tag{4.8}$$

we get, as in [5], the estimate

$$\mathrm{h}(\nu_3\varepsilon_3/\nu_1\varepsilon_1) \leq \exp\big\{c_{20} \Phi \log^* P \log B\big\}. \tag{4.9}$$

Next assume that $\log H \geq c_{17} R_1 + c_{18} R_2$. Then, by Lemmas 1 and 3, we have $A_0 \geq A_i$ for $i = 1, \ldots, s_1 + s_2 - 2$ and

$$\log A := \max_{1 \leq i \leq s_1+s_2-2} \log A_i \leq c_{19} \max\{R_1, R_2\}.$$

Consider now the above defined Φ with this value of $\log A$.

If $B < \Phi(\log^* P)/(c_{17} R_1 + c_{18} R_2)$ then (4.2) and (ii) of Lemma 1 imply that

$$\begin{aligned} \mathrm{h}\left(\frac{\nu_3\varepsilon_3}{\nu_1\varepsilon_1}\right) &= \mathrm{h}\left(1 + \frac{\nu_2\varepsilon_2}{\nu_1\varepsilon_1}\right) \\ &\leq 2\mathrm{h}(\nu_1)\,\mathrm{h}(\nu_2)\,\mathrm{h}(\varepsilon_1)\,\mathrm{h}(\varepsilon_2) \\ &\leq H^2 \exp\{c_{21}(R_1 + R_2) B\} \\ &\leq \exp\{c_{22} \Phi \log^* P\}. \end{aligned} \tag{4.10}$$

Assume now that $B \geq \Phi(\log^* P)/(c_{17}R_1 + c_{18}R_2)$. We apply the second part of Proposition 2 to (4.4). Putting

$$\delta = \frac{\Phi \log^* P}{B(c_{17}R_1 + c_{18}R_2)}$$

we obtain

$$\mathrm{h}(\nu_3\varepsilon_3/\nu_1\varepsilon_1) \leq \exp\left\{c_{23}\Phi(\log^* P)\log\left(\frac{B(c_{17}R_1 + c_{18}R_2)}{\log^* P \, \log A_0}\right)\right\}.$$

Recalling that $\log H \geq c_{17}R_1 + c_{18}R_2$, we get from (4.6)

$$\mathrm{h}(\nu_3\varepsilon_3/\nu_1\varepsilon_1) \leq \exp\{c_{24}\Phi \, \log^* P \, \log B\}.$$

The definition (4.8) of Φ and estimates (4.7), (4.6) and (4.3) yield

$$\begin{aligned} \mathrm{h}(\nu_3\varepsilon_3/\nu_1\varepsilon_1) \leq \exp\Big\{ & c_{25}\frac{P^d}{(\log^* P)^2} R_1R_2 \, \log^* \max\{R_1, R_2\} \\ & \log H \, \log^* \log^* \max\{\mathrm{h}(\varepsilon_1), \mathrm{h}(\varepsilon_2)\}\Big\}. \end{aligned} \tag{4.11}$$

Since we can bound $\mathrm{h}(\nu_3\varepsilon_3/\nu_2\varepsilon_2)$ in a similar way, the lemma follows from (4.5), (4.9), (4.10) and (4.11). □

Further, we recall some results of [5] and [6].

Let $\mathbf{K}$ be a number field with the same parameters as in Section 3. Let S be a finite set of places on $\mathbf{K}$ containing the set of infinite places S_∞. Denote by t the number of finite places in S and by P the largest of the rational primes lying below the finite places of S, with the convention that $P = 1$ if $S = S_\infty$. Consider the following equation :

$$x_1\varepsilon_1 + x_2\varepsilon_2 + x_3\varepsilon_3 = 0 \quad \text{in } \varepsilon_i \in O_S^* \tag{4.12}$$

where x_1, x_2, $x_3 \in \mathbf{K} \setminus \{0\}$ with $\max_{1\leq i\leq 3} \mathrm{h}(x_i) \leq H \ \ (H \geq e)$.

Proposition 3. *For every solution ε_1, ε_2, ε_3 of (4.12) there is an $\varepsilon \in O_S^*$ such that*

$$\max_{1\leq i\leq 3} \mathrm{h}(\varepsilon\varepsilon_i) < \exp\{c_{26}(d,s)\, P^d \, R_S \, (\log^* R_S)^2 \, (R_{\mathbf{K}} + t h_{\mathbf{K}} \log^* P + \log H)\},$$

where $c_{26}(d, s)$ is effectively computable.

Proof : It is a particular case of the Corollary of [5]. □

Let $\mathbf{M}$ be a finite extension of $\mathbf{K}$ with $[\mathbf{M} : \mathbf{K}] = n \geq 3$. Let $S_{\mathbf{M}}$ be the set of all extensions to $\mathbf{M}$ of the places in S. Denote by $h_{\mathbf{M}}$, $R_{\mathbf{M}}$ and $R_{S_{\mathbf{M}}}$ the class number, regulator and $S_{\mathbf{M}}$-regulator of $\mathbf{M}$, respectively. Let $\alpha \in \mathbf{M}$ such that $\mathbf{M} = \mathbf{K}(\alpha)$ and

$\mathrm{h}(\alpha) \leq A$, with $A \geq e$. Further, let β be a non-zero element of $\mathbf{K}$ with height at most B and with S-norm not exceeding $B^* \, (\geq e)$. Consider the norm form equation

$$\mathrm{N}_{\mathbf{M}/\mathbf{K}}(x + y\alpha) = \beta \quad \text{in } x, y \in O_S. \tag{4.13}$$

Proposition 4. *All the solutions of (4.13) satisfy*

$$\max\{\mathrm{h}(x), \mathrm{h}(y)\} < B^{1/n} \exp\{c_{27}(d,n,s)\, P^{dn(n-1)(n-2)}\, R_{S_{\mathbf{M}}} \\ (\log^* R_{S_{\mathbf{M}}})^2 \left(R_{\mathbf{M}} + t h_{\mathbf{M}} + \log(AB^*)\right)\},$$

where $c_{27}(d,n,s)$ is an effectively computable constant.

Proof : Apply Theorem 2 of [6] with $m = 2$. $\square$

We also need several well-known lemmas, the first of them is due to Minkowski.

Lemma 5. *In every ideal class $\mathcal{C}$ of $\mathbf{K}$, there exists an integral ideal $\mathfrak{a} \in \mathcal{C}$ such that*

$$|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\mathfrak{a})| \leq |D_{\mathbf{K}}|^{1/2}.$$

Proof : cf. [18], Theorem A.1. $\square$

Lemma 6. *Let $\mathbf{K}$ and $\mathbf{M}$ as above. Let a be an integer in $\mathbf{M}$ such that $\mathbf{M} = \mathbf{K}(a)$ and denote by P its minimal defining polynomial over $\mathbf{K}$. Then we have*

$$|D_{\mathbf{M}}| \leq |D_{\mathbf{K}}|^n \, |\mathrm{N}_{\mathbf{M}/\mathbf{Q}}(P'(a))|.$$

Proof : It follows from Narkiewicz (cf. [14], page 160) that the different $\mathrm{diff}_{\mathbf{M}/\mathbf{K}}$ is generated by the $F'(b)$, where b runs through the integral elements of $\mathbf{M}$ satisfying $\mathbf{M} = \mathbf{K}(b)$ and F is the minimal defining polynomial of b over $\mathbf{K}$. Hence, $|\mathrm{N}_{\mathbf{M}/\mathbf{Q}}(\mathrm{diff}_{\mathbf{M}/\mathbf{K}})| \leq |\mathrm{N}_{\mathbf{M}/\mathbf{Q}}(P'(a))|$, and the lemma follows from

$$|D_{\mathbf{M}}| = |D_{\mathbf{K}}|^n \, |\mathrm{N}_{\mathbf{M}/\mathbf{Q}}(\mathrm{diff}_{\mathbf{M}/\mathbf{K}})|. \ \square$$

Lemma 7. *Let $\mathbf{K}$ and $\mathbf{M}$ as above and put $m = [\mathbf{M} : \mathbf{Q}]$. Then there exists an effectively computable constant $c_{28}(m)$ such that*

$$R_{\mathbf{K}} \leq c_{28}(m)\, R_{\mathbf{M}}.$$

Proof : cf. [24], Chapter *II*, Lemma 2.3. $\square$

Lemma 8. *There exists an effective constant $c_{29}(d)$, which depends only on d, such that*

$$R_{\mathbf{K}} h_{\mathbf{K}} \leq c_{29}(d) |D_{\mathbf{K}}|^{1/2} \left(\log^* |D_{\mathbf{K}}|\right)^{d-1}.$$

Proof : See for example [11]. $\square$

Finally, we state a very useful lemma, suggested by the referee.

Lemma 9. *Let* $\mathbf{K}$ *as above and* $a \in \mathbf{K}$. *Let* α *be a root of the polynomial* $P(X) = X^\nu - a$. *Then we have*

$$\mathrm{N}_{\mathbf{K}(\alpha)/\mathbf{Q}}(\mathrm{diff}_{\mathbf{K}(\alpha)/\mathbf{K}}) \le c_{30}(d,\nu)\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}\Big(\prod_{\mathrm{ord}_\mathfrak{p}(a)\neq 0} \mathfrak{p} \Big)^{\nu-1},$$

where $c_{30}(d,\nu)$ *is effectively computable (the product being over the prime ideals of* $\mathbf{K}$ *with the property* $\mathrm{ord}_\mathfrak{p}(a) \neq 0$*).*

Proof : We write D for $\mathrm{N}_{\mathbf{K}(\alpha)/\mathbf{K}}(\mathrm{diff}_{\mathbf{K}(\alpha)/\mathbf{K}})$. Let $\mathfrak{p}$ be a prime ideal of $\mathbf{K}$, ramified in $\mathbf{K}(\alpha)$, and $p = p(\mathfrak{p})$ the underlying rational prime. We know that $\mathfrak{p}$ divides D and, consequently, $\mathfrak{p}$ divides $\mathrm{N}_{\mathbf{K}(\alpha)/\mathbf{K}}\big(P'(\alpha)\big)$. It follows that either $p \le \nu$ or $\mathrm{ord}_\mathfrak{p}(a) \neq 0$.

Denote by $O_{\mathbf{K}(\alpha)}$ the ring of integers of the field $\mathbf{K}(\alpha)$ and let $\mathfrak{p}O_{\mathbf{K}(\alpha)} = \mathfrak{P}_1^{e_1} \dots \mathfrak{P}_k^{e_k}$ be the decomposition of $\mathfrak{p}$ in $O_{\mathbf{K}(\alpha)}$ into prime ideals. Denote by $f_1, \dots, f_k$ the residue degrees of $\mathfrak{P}_1, \dots, \mathfrak{P}_k$, respectively, so that $\mathrm{N}_{\mathbf{K}(\alpha)/\mathbf{K}}(\mathfrak{P}_i) = \mathfrak{p}^{f_i}$ for all i. Notice that $e_1 f_1 + \dots + e_k f_k = [\mathbf{K}(\alpha) : \mathbf{K}] \le \nu$.

By Proposition 6.3 of [14], we have, for $i = 1, \dots, k$,

$$\mathrm{ord}_{\mathfrak{P}_i}(\mathrm{diff}_{\mathbf{K}(\alpha)/\mathbf{K}}) \le e_i + e_i\, \mathrm{ord}_\mathfrak{p}(e_i) - 1,$$

whence

$$\mathrm{ord}_\mathfrak{p}(D) \le \sum_{i=1}^{k} (e_i + e_i\, \mathrm{ord}_\mathfrak{p}(e_i) - 1) f_i. \tag{4.14}$$

If $p > \nu$, then $\mathrm{ord}_\mathfrak{p}(e_i) = 0$ for all i and $\mathrm{ord}_\mathfrak{p}(D) \le (e_1 - 1)f_1 + \dots + (e_k - 1)f_k \le \nu - 1$.

Write $D = D_1 D_2$, where

$$D_1 = \prod_{p(\mathfrak{p}) \le \nu} \mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(D)}, \quad D_2 = \prod_{p(\mathfrak{p}) > \nu} \mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(D)}.$$

It follows from (4.14) that $\mathrm{ord}_\mathfrak{p}(D) \le c_{31}(d,\nu)$, whence $\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(D_1) \le c_{32}(d,\nu)$, with $c_{31}(d,\nu)$ and $c_{32}(d,\nu)$ effectively computable. Finally, since all prime ideals of $\mathbf{K}$ dividing D are ramified in $\mathbf{K}(\alpha)$, we have

$$\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(D_2) \le \mathrm{N}_{\mathbf{K}/\mathbf{Q}}\Big(\prod_{\mathrm{ord}_\mathfrak{p}(a)\neq 0} \mathfrak{p}^{\mathrm{ord}_\mathfrak{p}(D)} \Big) \le \mathrm{N}_{\mathbf{K}/\mathbf{Q}}\Big(\prod_{\mathrm{ord}_\mathfrak{p}(a)\neq 0} \mathfrak{p} \Big)^{\nu-1},$$

and the lemma is proved. □

5. Proofs of the theorems

First we have to introduce some new notations. For $i = 1, \dots, r$ let f_i be the minimal defining polynomial of α_i over $\mathbf{K}$ and denote by Δ_{α_i} its discriminant. Recalling that $g(X) = (X - \alpha_1) \dots (X - \alpha_r)$, we observe that its discriminant, denoted by Δ_g, the resultant of the polynomials g' and f_i, denoted by $\mathrm{Res}(g', f_i)$, and Δ_{α_i} are algebraics integers in $\mathbf{K}$. Further, we will often use (without mentioning it) the fact that $\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_{\alpha_i})$ and $\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\mathrm{Res}(g', f_i))$ divide $\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)$. The constants $c_{33}, \ldots, c_{50}$ are all effectively computable and depend only on d, n and t. The constants $c_{51}, \ldots, c_{93}$ are all effectively computable and depend only on d, n, m and t.

Proof of Theorem 1 :

It follows from the hypothesis of the theorem that $f(X) = f_1(X)^{m/2} f_2(X)^m$, where the polynomial f_1 is monic and has at least three distinct roots with odd multiplicity. If $(x, y) \in O_S \times \mathbf{K}$ is a solution of (2.1), then $a = f_1(x)^{m/2} f_2(x)^m y^{-m}$ must be an $m/2$-th power in $\mathbf{K}$. Hence, there exists $u \in O_{\mathbf{K}}$ such that $a = u^{m/2}$ and $\big(x, y/f_2(x)\big)$ is a solution of the equation $f_1(X) = uY^2$. Further, we have $|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(u)| \leq |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(a)|$ and $|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_{f_1})| \leq |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|$.

Thus, we only have to prove the theorem in the case when $m = 2$ and f has three distinct roots with odd multiplicity. Assuming this hypothesis, let $(x, y) \in O_S \times \mathbf{K}$ be a solution of (2.1).

First step :

The ideal (x) splits uniquely under the form

$$(x) = \mathfrak{a}\,\mathfrak{b}^{-1},$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are relatively prime integer ideals in $O_{\mathbf{K}}$, such that the set of the prime divisors of $\mathfrak{b}$ is contained in S. By Lemma 5, there is an integer ideal $\mathfrak{b}'$ in the same class as $\mathfrak{b}^{-1}$ satisfying $|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\mathfrak{b}')| \leq |D_{\mathbf{K}}|^{1/2}$. Thus we have

$$(x) = (\mathfrak{a}\,\mathfrak{b}') \cdot (\mathfrak{b}'\mathfrak{b})^{-1}.$$

Since the integer ideals $\mathfrak{b}'\,\mathfrak{b}$ and $\mathfrak{a}\,\mathfrak{b}'$ are principal, we can write $x = X/z$, where X, $z \in O_{\mathbf{K}}$ and

$$(X) = \mathfrak{a}\,\mathfrak{b}', \quad (z) = \mathfrak{b}\,\mathfrak{b}'.$$

In particular, $\big((X), (z)\big) = \mathfrak{b}'$.

Clearly, if a power $\mathfrak{p}^l$ of a prime ideal $\mathfrak{p}$ exactly divides (z), then $\mathfrak{p}^l$ divides $\mathfrak{b}'$ or $\mathfrak{p}$ is one of the $\mathfrak{p}_i$'s. Defining the binary form $f(X, z) := z^n f(X/z)$, Equation (2.1) becomes

$$f(X, z) = ay^2 z^n. \tag{5.1}$$

Second step :

It follows from the hypothesis that e_1, e_2 and e_3 are odd. Let $i = 1, \ldots, r$ and put $\mathbf{K}_i := \mathbf{K}(\alpha_i)$. Working in $\mathbf{F}$, the splitting field of f, we have for each root β of f

$$\big((X - \alpha_i z), (X - \beta z)\big) \mid (\beta - \alpha_i)\big((X), (z)\big) \mid g'(\alpha_i)\,\mathfrak{b}'.$$

Let $\mathfrak{p}$ be a prime ideal dividing $X - \alpha_i z$ with an odd exponent. If $\mathfrak{p}$ does not divide $g'(\alpha_i)\mathfrak{b}'$, then it does not divide $X - \alpha_j z$ for all $j \neq i$, and we necessarily have $\mathfrak{p} \mid (a)$. Thus, prime

ideals not appearing in $\mathfrak{b}'\left(ag'(\alpha_i)\right)$ divide $X-\alpha_i z$ with an even exponent, which is also true in the field $\mathbf{K}_i$, and there exist two integer ideals $\mathfrak{a}_i$ and $\mathfrak{b}_i$ in $\mathbf{K}_i$ with $\mathfrak{a}_i$ square-free satisfying
$$(X-\alpha_i z)=\mathfrak{a}_i\,\mathfrak{b}_i^2 \quad\text{and}\quad \mathfrak{a}_i\,O_{\mathbf{F}}\,\big|\,\mathfrak{b}'\left(ag'(\alpha_i)\right)O_{\mathbf{F}}.$$
Let $\alpha_{i_1}=\alpha_i,\dots,\alpha_{i_k}$ be the roots of the polynomial f_i. Since $\mathrm{N}_{\mathbf{F}/\mathbf{Q}}(\mathfrak{a}_{i_j})=\mathrm{N}_{\mathbf{F}/\mathbf{Q}}(\mathfrak{a}_i)$ for $j=1,\dots,k$, we have

$$\begin{aligned}|\mathrm{N}_{\mathbf{F}/\mathbf{Q}}(\mathfrak{a}_i)|^k &\le \left|\mathrm{N}_{\mathbf{F}/\mathbf{Q}}\big(a\mathfrak{b}'\big)^k\prod_{j=1}^{k}\mathrm{N}_{\mathbf{F}/\mathbf{Q}}\big(g'(\alpha_{i_j})\big)\right|\\ &\le \left|\mathrm{N}_{\mathbf{F}/\mathbf{Q}}\big(a\mathfrak{b}'\big)^k\,\mathrm{N}_{\mathbf{F}/\mathbf{Q}}\big(\mathrm{Res}(g',f_i)\big)\right|,\end{aligned}$$

and, noticing that $\mathrm{Res}(g',f_i)\in\mathbf{K}$, we get

$$\begin{aligned}|\mathrm{N}_{\mathbf{K}_i/\mathbf{Q}}(\mathfrak{a}_i)| &\le \left|\big(A\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\mathfrak{b}')\big)^n\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}\big(\mathrm{Res}(g',f_i)\big)\right|\\ &\le A^n\,|D_{\mathbf{K}}|^{n/2}\,\left|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}\big(\mathrm{Res}(g',f_i)\big)\right|.\end{aligned}\tag{5.2}$$

By Lemma 5, there is an integer ideal $\mathfrak{b}_i'$ in the class of $\mathfrak{b}_i$ satisfying $|\mathrm{N}_{\mathbf{K}_i/\mathbf{Q}}(\mathfrak{b}_i')|\le|D_{\mathbf{K}_i}|^{1/2}$. Then we have $(X-\alpha_i z)=(\mathfrak{a}_i\,\mathfrak{b}_i'^2)\cdot(\mathfrak{b}_i/\mathfrak{b}_i')^2$ and, reasoning as above, we obtain $\kappa_i'\in O_{\mathbf{K}_i}$ and $\xi_i'\in\mathbf{K}_i$ such that
$$X-\alpha_i z=\kappa_i'\xi_i'^2.$$
Applying Lemma 7 to the extension $\mathbf{K}\subset\mathbf{K}_i=\mathbf{K}(\alpha_i)$, we get

$$|D_{\mathbf{K}_i}|\le|D_{\mathbf{K}}|^n\,|\mathrm{N}_{\mathbf{K}_i/\mathbf{Q}}(f_i'(\alpha_i))|\le|D_{\mathbf{K}}|^n\,|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_{\alpha_i})|\tag{5.3}$$

and it follows from (5.2) that

$$\begin{aligned}|\mathrm{N}_{\mathbf{K}_i/\mathbf{Q}}(\kappa_i')| &\le |\mathrm{N}_{\mathbf{K}_i/\mathbf{Q}}(\mathfrak{a}_i)|\cdot|\mathrm{N}_{\mathbf{K}_i/\mathbf{Q}}(\mathfrak{b}_i')^2|\\ &\le A^n\,|D_{\mathbf{K}}|^{3n/2}\,\left|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}\big(\mathrm{Res}(g',f_i)\,\Delta_{\alpha_i}\big)\right|.\end{aligned}\tag{5.4}$$

Hence, applying Lemma 2 to the algebraic integer $\kappa_i'\in\mathbf{K}_i$, we obtain $\kappa_i\in O_{\mathbf{K}_i}$ and $\xi_i\in\mathbf{K}_i$ such that $|\mathrm{N}_{\mathbf{K}_i/\mathbf{Q}}(\kappa_i')|=|\mathrm{N}_{\mathbf{K}_i/\mathbf{Q}}(\kappa_i)|$,

$$X-\alpha_i z=\kappa_i\xi_i^2 \quad\text{and}\quad \mathrm{h}(\kappa_i)\le\exp\big\{c_{33}\left(R_{\mathbf{K}_i}+\log|A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\right)\big\}.\tag{5.5}$$

Third step :

We follow very closely the argument of Voutier [26]. For $i=1,2,3$ we fix a square root $\sqrt{\kappa_i}$ of κ_i. For $i,\,j\in\{1,2,3\}$ with $i\neq j$, we define the number fields $\mathbf{K}_{ij}=\mathbf{K}_i(\alpha_j)$ and $\mathbf{L}_{ij}=\mathbf{K}_{ij}\big(\sqrt{\kappa_i\kappa_j}\big)$. Those are subfields of $\mathbf{M}=\mathbf{K}(\alpha_1,\alpha_2,\alpha_3,\sqrt{\kappa_1\,\kappa_2},\sqrt{\kappa_1\,\kappa_3})$, which is a number field with degree less or equal to $4n(n-1)(n-2)d$ over $\mathbf{Q}$. We denote by R_{ij} (resp. h_{ij}) the regulator (resp. the class number) of $\mathbf{L}_{ij}$.

In order to deduce from (5.5) four unit-equations, we set

$$\tau_1 = \kappa_1 \xi_1, \quad \tau_2 = \sqrt{\kappa_1 \kappa_2}\,\xi_2 \quad \text{and} \quad \tau_3 = \sqrt{\kappa_1 \kappa_3}\,\xi_3,$$

and, immediately, it follows that

$$\begin{aligned} \kappa_1(\alpha_2 - \alpha_1)z &= \tau_1^2 - \tau_2^2, \\ \kappa_1(\alpha_3 - \alpha_1)z &= \tau_1^2 - \tau_3^2, \\ \kappa_1(\alpha_2 - \alpha_3)z &= \tau_3^2 - \tau_2^2. \end{aligned} \tag{5.6}$$

For $i \neq j$, let S_{ij} be the set of all extensions to $\mathbf{L}_{ij}$ of the places in S. The algebraic numbers $\tau_1 \pm \tau_2$ belong to the field $\mathbf{L}_{12}$ and are algebraic integers (to see this, consider τ_1^2 and τ_2^2). In the same way, $\tau_1 \pm \tau_3$ (resp. $\sqrt{\kappa_3/\kappa_1}(\tau_2 \pm \tau_3)$) are algebraic integers in $\mathbf{L}_{13}$ (resp. $\mathbf{L}_{23}$). It follows from (5.4) and (5.6) that

$$\begin{aligned} \mathrm{N}_{S_{1j}}(\tau_1 \pm \tau_j) &\leq \exp\bigl\{c_{34}\,\bigl(\log |A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\bigr)\bigr\}, \quad j = 2, 3, \\ \mathrm{N}_{S_{23}}\bigl(\sqrt{\kappa_3/\kappa_1}(\tau_2 \pm \tau_3)\bigr) &\leq \exp\bigl\{c_{35}\,\bigl(\log |A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\bigr)\bigr\}. \end{aligned}$$

Applying Lemma 2 in the fields $\mathbf{L}_{ij}$, we may write

$$\begin{aligned} \tau_1 + \tau_2 &= b_3 \varepsilon_3 \quad \text{and} \quad \tau_1 - \tau_2 = g_3 \delta_3, \\ \tau_1 + \tau_3 &= b_2 \varepsilon_2 \quad \text{and} \quad \tau_1 - \tau_3 = g_2 \delta_2. \\ \sqrt{\kappa_3/\kappa_1}(\tau_2 + \tau_3) &= b_1' \varepsilon_1 \quad \text{and} \quad \sqrt{\kappa_3/\kappa_1}(\tau_2 - \tau_3) = g_1' \delta_1, \end{aligned} \tag{5.7}$$

where, for each permutation (i, j, k) of the indices $(1, 2, 3)$, ε_i and δ_i are S_{jk}-units in $\mathbf{L}_{jk}$. Moreover, setting $b_1 = \sqrt{\kappa_1/\kappa_3}\, b_1'$ and $g_1 = \sqrt{\kappa_1/\kappa_3}\, g_1'$, we have

$$\tau_2 + \tau_3 = b_1 \varepsilon_1 \quad \text{and} \quad \tau_2 - \tau_3 = g_1 \delta_1 \tag{5.8}$$

with

$$\begin{aligned} \max_{1 \leq i \leq 3} \bigl\{\mathrm{h}(b_i), \mathrm{h}(g_i)\bigr\} \leq \exp\bigl\{c_{36}\,\bigl(R_{12} + R_{13} + R_{23} + &(h_{12} + h_{13} + h_{23}) \log^* P + \\ &+ \log |A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\bigr)\bigr\}. \end{aligned} \tag{5.9}$$

The ideal (z) admits the following decomposition into prime ideals in $\mathbf{K}$

$$(z) = \prod_{i=1}^{t} \mathfrak{p}_i^{a_i} \cdot \prod_{j=1}^{u} \mathfrak{q}_j^{b_j},$$

where $\displaystyle\prod_{j=1}^{u} \mathfrak{q}_j^{b_j}$ divides $\mathfrak{b}'$. We make the Euclidean division of a_i by $2h_{\mathbf{K}}$ (recall that $h_{\mathbf{K}}$ is the class number of $\mathbf{K}$) : there exist integers q_i and r_i, with $0 \leq r_i < 2h_{\mathbf{K}}$, such that

$a_i = 2h_{\mathbf{K}}\, q_i + r_i$. Let z_1 be a generator of the principal ideal $\prod_{i=1}^{t} \mathfrak{p}_i^{h_{\mathbf{K}}\, q_i}$ and notice that z_1^{-1} is a S-unit. We have $z = z_1^2 z_2$, where $z_2 \in O_{\mathbf{K}}$ has a norm (over $\mathbf{Q}$) bounded above by $|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\mathfrak{b}')|\, P^{2h_{\mathbf{K}}\, td}$. Applying Lemma 2, with $n = 2$, to the algebraic integer z_2, we obtain a unit $\eta_2 \in O_{\mathbf{K}}$ and $z_3 \in O_{\mathbf{K}}$ such that

$$z_2 = \eta_2^2 z_3 \quad \text{and} \quad \mathrm{h}(z_3) \leq \exp\bigl\{c_{37}\,(R_{\mathbf{K}} + h_{\mathbf{K}} \log^* P + \log|D_{\mathbf{K}}|)\bigr\}. \tag{5.10}$$

Setting $\eta = \eta_2^{-1} z_1^{-1}$, we have $z = \eta^{-2} z_3$ and η is an S-unit.

Let $S_{\mathbf{M}}$ be the set of all extensions to $\mathbf{M}$ of the places in S, we deduce from (5.7) and (5.8) four $S_{\mathbf{M}}$-unit equations, which we multiply by η :

$$\begin{aligned} b_1 \varepsilon_1 \eta - b_2 \varepsilon_2 \eta + g_3 \delta_3 \eta &= 0, \\ b_1 \varepsilon_1 \eta + g_2 \delta_2 \eta - b_3 \varepsilon_3 \eta &= 0, \\ g_1 \delta_1 \eta + b_2 \varepsilon_2 \eta - b_3 \varepsilon_3 \eta &= 0, \\ g_1 \delta_1 \eta - g_2 \delta_2 \eta + g_3 \delta_3 \eta &= 0. \end{aligned} \tag{5.11}$$

Fourth step :

We now prove the first part of the theorem. Before applying Lemma 4 to the equations (5.11), we have to bound the size of the S_{ij}-regulator of $\mathbf{L}_{ij}$, denoted by $R_{S_{ij}}$. The minimal defining polynomial of $\sqrt{\kappa_i\, \kappa_j}$ over $\mathbf{K}_{ij}$ is $X^2 - \kappa_i\, \kappa_j$; hence, by successive applications of Lemma 6 and inequalities (5.3) and (5.4), we get :

$$\begin{aligned} |D_{\mathbf{L}_{ij}}| &\leq |D_{\mathbf{K}_{ij}}|^2\, |\mathrm{N}_{\mathbf{L}_{ij}/\mathbf{Q}}(2\sqrt{\kappa_i\, \kappa_j})| \\ &\leq 2^{2n^2 d}\, |D_{\mathbf{K}_{ij}}|^2\, |\mathrm{N}_{\mathbf{K}_{ij}/\mathbf{Q}}(\kappa_i\, \kappa_j)| \\ &\leq 2^{2n^2 d}\, |D_{\mathbf{K}_i}|^{2n}\, |\mathrm{N}_{\mathbf{K}_i/\mathbf{Q}}(\Delta_{\alpha_j})|^2\, |\mathrm{N}_{\mathbf{K}_{ij}/\mathbf{Q}}(\kappa_i\, \kappa_j)| \\ &\leq 2^{2n^2 d}\, |D_{\mathbf{K}}|^{5n^2}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_{\alpha_i}\, \Delta_{\alpha_j})|^{2n}\, A^{2n^2}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\mathrm{Res}(g', f_i f_j)\, \Delta_{\alpha_i}\, \Delta_{\alpha_j})|^n \\ &\leq 2^{2n^2 d}\, |D_{\mathbf{K}}|^{5n^2}\, A^{2n^2}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{8n}, \end{aligned} \tag{5.12}$$

whence

$$\log|D_{\mathbf{L}_{ij}}| \leq c_{38}\, \bigl(\log|A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\bigr)$$

and, by Lemma 8,

$$h_{ij}\, R_{ij} \leq c_{39}\, |D_{\mathbf{K}}|^{5n^2/2}\, A^{n^2}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{4n}\, \bigl(\log|A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\bigr)^{2n^2 d - 1}. \tag{5.13}$$

Since the number of finite places in S_{ij} is bounded by $2dn(n-1)t$, Lemma 3 and (5.13) lead to the estimate

$$\begin{aligned} \max\{R_{S_{12}}, R_{S_{13}}, R_{S_{23}}\} \leq c_{40}\, &|D_{\mathbf{K}}|^{5n^2/2}\, A^{n^2}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{4n} \\ &\bigl(\log|A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\bigr)^{2n^2 d - 1}\, (\log^* P)^{2dn(n-1)t}. \end{aligned} \tag{5.14}$$

Applying Lemma 4 to the equations (5.11), we obtain from (5.9) and (5.14) the upper bound

$$\max_{i=1,2}\left\{\mathrm{h}\left(\frac{b_i\varepsilon_i}{b_3\varepsilon_3}\right),\mathrm{h}\left(\frac{g_i\delta_i}{b_3\varepsilon_3}\right),\mathrm{h}\left(\frac{b_i\varepsilon_i}{g_3\delta_3}\right),\mathrm{h}\left(\frac{g_i\delta_i}{g_3\delta_3}\right)\right\}\leq\exp\{c_{41}\,T_1\,E\},\tag{5.15}$$

where

$$T_1\leq P^{4n^3d}(\log^*P)^{4dn^2t-1}|D_{\mathbf{K}}|^{15n^2/2}\,A^{3n^2}\,|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{12n}\left(\log|A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\right)^{6n^2d-2},$$
$$E=\log^*\log^*\max\bigl\{\mathrm{h}(\varepsilon_1\eta),\mathrm{h}(\varepsilon_2\eta),\mathrm{h}(\delta_1\eta),\mathrm{h}(\delta_2\eta)\bigr\}.$$

In order to bound E, we notice that, using (5.7), (5.6) and (5.10), we have

$$\begin{aligned}(b_1\varepsilon_1\eta)^2&=\left(\frac{b_1\varepsilon_1}{b_3\varepsilon_3}\right)\left(\frac{b_1\varepsilon_1}{g_3\delta_3}\right)(b_3\varepsilon_3\eta)\,(g_3\delta_3\eta)\\&=\left(\frac{b_1\varepsilon_1}{b_3\varepsilon_3}\right)\left(\frac{b_1\varepsilon_1}{g_3\delta_3}\right)\kappa_1(\alpha_2-\alpha_1)z_3.\end{aligned}\tag{5.16}$$

From (3.2), (5.5), (5.10) and Lemma 7, we get

$$\mathrm{h}(\kappa_1(\alpha_1-\alpha_2)z_3)\leq H\,\exp\bigl\{c_{42}\left(R_{\mathbf{K}_1}+h_{\mathbf{K}}\log^*P+\log|A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\right)\bigr\}$$

and we deduce from (5.15), (5.16) and Lemma 7 that

$$\mathrm{h}(\varepsilon_1\eta)\leq\mathrm{h}(b_1)\mathrm{h}(b_1\varepsilon_1\eta)\leq H\,\exp\{c_{43}\,T_1\,E\}.$$

This bound is still true, with c_{44} instead of c_{43}, for $\mathrm{h}(\varepsilon_2\eta)$, $\mathrm{h}(\delta_1\eta)$ and $\mathrm{h}(\delta_2\eta)$. Consequently, $E\leq\log^*(c_{44}\,T_1\,E)+\log\log H$, and

$$E\leq c_{45}\,\log|A\,P\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|+\log\log H.\tag{5.17}$$

We can now deduce an upper bound for $\mathrm{h}(x)$. Namely, setting $\gamma=b_3\varepsilon_3/g_3\delta_3$, we obtain from (5.7) that

$$2\tau_1=(\tau_1-\tau_2)+b_3\varepsilon_3=(\tau_1-\tau_2)(1+\gamma),$$

and, similarly,

$$2\tau_1=(\tau_1+\tau_2)(1+\gamma^{-1}).$$

Hence, we get the equality

$$4\tau_1^2=(\tau_1^2-\tau_2^2)(1+\gamma)(1+\gamma^{-1}),$$

which, using (5.6) and (5.5), we may write as

$$4(X-\alpha_1z)=(\alpha_2-\alpha_1)z(1+\gamma)(1+\gamma^{-1}).$$

Dividing this equality by z, we infer that

$$x = \alpha_1 + \frac{1}{4}(1+\gamma)(1+\gamma^{-1})(\alpha_2 - \alpha_1). \tag{5.18}$$

Noticing that $\gamma = (b_3\varepsilon_3/g_1\delta_1)(g_1\delta_1/g_3\delta_3)$, we immediately get from (3.2), (5.15), (5.17) and (5.18) the upper bound

$$\begin{aligned} \mathrm{h}(x) \le H^2 \exp\Big\{ & c_{46}\, P^{4n^3d}\,(\log^* P)^{4n^2dt}\,|D_{\mathbf{K}}|^{15n^2/2}\, A^{3n^2}\,|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{12n} \\ & \big(\log |A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big)^{6n^2d} \log\log H \Big\}. \end{aligned} \tag{5.19}$$

Fifth step :

We now prove the second part of the theorem. The aim of Lemma 4, used in the proof of Theorem 1, is to obtain a good dependence in terms of $|D_{\mathbf{K}}|$. Unfortunately, the dependence on H is not very satisfactory and it can be improved by using Proposition 3 instead of Lemma 4. Therefore, the dependence on $|D_{\mathbf{K}}|$ is worse.

We exactly follow the first three steps of the proof and we apply Proposition 3 to the four equations (5.11). Recall that $\mathbf{M} = \mathbf{K}(\alpha_1, \alpha_2, \alpha_3, \sqrt{\kappa_1\kappa_2}, \sqrt{\kappa_1\kappa_3})$ has degree less or equal than $4n(n-1)(n-2)d$. Using Lemma 6, (5.4) and (5.12), we can bound its discriminant :

$$\begin{aligned} |D_{\mathbf{M}}| &\le |D_{\mathbf{L}_{12}}|^{2n}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{4n^2}\, |\mathrm{N}_{\mathbf{M}/\mathbf{Q}}(2\sqrt{\kappa_1\kappa_3})| \\ &\le 2^{4n^3d}\, |D_{\mathbf{K}}|^{10n^3}\, A^{4n^3}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{20n^2}\, |\mathrm{N}_{\mathbf{M}/\mathbf{Q}}(2\sqrt{\kappa_1\kappa_3})| \\ &\le 2^{8n^3d}\, |D_{\mathbf{K}}|^{10n^3}\, A^{4n^3}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{20n^2}\, |\mathrm{N}_{\mathbf{K}_1/\mathbf{Q}}(\kappa_1)|^{2n^2}\, |\mathrm{N}_{\mathbf{K}_3/\mathbf{Q}}(\kappa_3)|^{2n^2} \\ &\le 2^{8n^3d}\, |D_{\mathbf{K}}|^{16n^3}\, A^{8n^3}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{28n^2}. \end{aligned} \tag{5.20}$$

As before, denote by $S_{\mathbf{M}}$ the set of all extensions to $\mathbf{M}$ of the places in S and by $R_{S_{\mathbf{M}}}$ the $S_{\mathbf{M}}$-regulator of $\mathbf{M}$. Applying Proposition 3 to the first two $S_{\mathbf{M}}$-unit equations (5.11), we get, by (5.9) and Lemma 7,

$$\max\left\{ \mathrm{h}\left(\frac{b_1\varepsilon_1}{b_3\varepsilon_3}\right), \mathrm{h}\left(\frac{b_1\varepsilon_1}{g_3\delta_3}\right) \right\} \le \exp\{c_{47}\, T_2\}, \tag{5.21}$$

where

$$T_2 \le P^{4n^3d} R_{S_{\mathbf{M}}}\,(\log^* R_{S_{\mathbf{M}}})^2\, \big(R_{\mathbf{M}} + (h_{12} + h_{13} + h_{23} + h_{\mathbf{M}}) \log^* P + \log |A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big).$$

Recall that we have put $\gamma = b_3\varepsilon_3/g_3\delta_3$. It follows from (5.21) that $\mathrm{h}(\gamma) \le \exp\{c_{48}\, T_2\}$ and from (5.18) that $\mathrm{h}(x) \le H^2 \exp\{c_{49}\, T_2\}$. Finally, we use Lemma 8, (5.12) and (5.20) to bound the quantity T_2 and, after some computations, we get

$$\begin{aligned} \mathrm{h}(x) \le H^2 \Big\{ & c_{50}\, P^{4n^3d}\,(\log^* P)^{4tn^3}\,|D_{\mathbf{K}}|^{16n^3}\,|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{28n^2} \\ & A^{8n^3}\,\big(\log |A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big)^{8n^3d}\big)\Big\}, \end{aligned}$$

as claimed. □

Proof of Theorem 2 : We keep the same notations as in the proof of Theorem 1. By the same reasoning as in the first step of the above proof, letting $f(X,z) := z^n f(X/z)$, equation (2.1) becomes

$$f(X,z) = a\, y^m z^n,$$

in the unknowns $X, z \in O_{\mathbf{K}}$ and $y \in \mathbf{K}$. Further, there is an integral ideal $\mathfrak{b}'$, with $|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\mathfrak{b}')| \le |D_{\mathbf{K}}|^{1/2}$, such that $\big((X),(z)\big) = \mathfrak{b}'$.

We reorder the roots such that $(m_1, m_2) \ge 3$. Arguing as in the proof of Theorem 1, we claim that, for $i = 1, 2$, there exist two integer ideals $\mathfrak{a}_i$ and $\mathfrak{b}_i$ in $O_{\mathbf{K}_i}$, with $\mathfrak{a}_i$ free of m_i-th powers, satisfying

$$(X - \alpha_i z) = \mathfrak{a}_i\, \mathfrak{b}_i^{m_i}$$

and

$$|\mathrm{N}_{\mathbf{K}_i/\mathbf{Q}}(\mathfrak{a}_i)| \le A^n \left|\mathrm{N}_{\mathbf{K}_i/\mathbf{Q}}(\mathfrak{b}')\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}\big(\mathrm{Res}(g', f_i)\big)\right|^{m_i - 1}. \tag{5.22}$$

Further, by (5.22), Lemma 5 and Lemma 2, we obtain $\kappa_i \in O_{\mathbf{K}_i}$, an ideal $\mathfrak{b}_i'$ in $O_{\mathbf{K}_i}$ with $|\mathrm{N}_{\mathbf{K}_i/\mathbf{Q}}(\mathfrak{b}_i')| \le |D_{\mathbf{K}_i}|^{1/2}$ and $\xi_i \in \mathbf{K}_i$ such that $\kappa_i\, O_{\mathbf{K}_i} = \mathfrak{a}_i\, \mathfrak{b}_i'^{m_i}$,

$$\begin{aligned} X - \alpha_i z = \kappa_i \xi_i^{m_i}, \qquad & \mathrm{h}(\kappa_i) \le \exp\big\{c_{51}\left(R_{\mathbf{K}_i} + \log|A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\right)\big\}, \\ & \text{and} \quad |\mathrm{N}_{\mathbf{K}_i/\mathbf{Q}}(\kappa_i)| \le A^n\, |D_{\mathbf{K}}|^{n m_i}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{3m_i/2}. \end{aligned} \tag{5.23}$$

Recall that if (m_1, m_2) is not a power of 2, then m' is the smallest odd prime dividing (m_1, m_2), otherwise $m' = 4$. Further, put $m_1' = m_1/m'$, $m_2' = m_2/m'$. Working in the field $\mathbf{L} = \mathbf{K}(\alpha_1, \alpha_2)$, we deduce from (5.23) the equation

$$(\alpha_2 - \alpha_1) z = \kappa_1^{1-m'} \left(\kappa_1 \xi_1^{m_1'}\right)^{m'} - \kappa_2^{1-m'} \left(\kappa_2 \xi_2^{m_2'}\right)^{m'}. \tag{5.24}$$

In the sequel, we will put for convenience $\tau_1 = \kappa_1 \xi_1^{m_1'}$ and $\tau_2 = \kappa_2 \xi_2^{m_2'}$.

Usually, one works in the field $\mathbf{L}(\kappa_1^{1/m'}, \kappa_2^{1/m'})$ of degree m'^2 (in general) over $\mathbf{L}$. Voutier [26] prefers the field $\mathbf{L}\big((\kappa_1/\kappa_2)^{1/m'}, \zeta_{m'}\big)$, where $\zeta_{m'}$ denotes a primitive m'-th root of unity, but, however, it does not help him to make a numerical improvement. Here, we work either in $\mathbf{L}\big((\kappa_1/\kappa_2)^{1/m'}\big)$ or in $\mathbf{L}(\zeta_{m'})$, and, thus, we remove a factor m'. This idea goes back to Bilu [2]. Further, Lemma 9 provides sharp upper bounds for differents of certain extensions of number fields and allows us to remove a factor m.

Suppose first that $m' \ne 4$. By Theorem 9.1 of Chapter *VIII* of [10], there are two possible cases :

(i) The polynomial $T^{m'} - (\kappa_1/\kappa_2)^{m'-1}$ is irreducible over $\mathbf{L}$.

(ii) There exists an $u \in \mathbf{L}$ such that $(\kappa_1/\kappa_2)^{m'-1} = u^{m'}$.

Case (i) :

Let $v \in \mathbf{C}$ be a root of $T^{m'} - (\kappa_1/\kappa_2)^{m'-1}$ and consider the field $\mathbf{M} = \mathbf{L}(v)$, it follows from (5.24) that

$$\mathrm{N}_{\mathbf{M}/\mathbf{L}}(\tau_1 - v\tau_2) = \kappa_1^{m'-1} (\alpha_2 - \alpha_1) z. \tag{5.25}$$

Recall that there exist non-negative integers $a_1, \dots, a_t$ and an ideal $\mathfrak{b}''$ which divides $\mathfrak{b}'$ such that $zO_{\mathbf{K}} = \mathfrak{b}''\mathfrak{p}_1^{a_1} \dots \mathfrak{p}_t^{a_t}$. Let $\pi_1, \dots, \pi_t \in O_{\mathbf{K}}$ be generators of the principal ideals $\mathfrak{p}_1^{h_{\mathbf{K}}}, \dots, \mathfrak{p}_t^{h_{\mathbf{K}}}$, respectively. Using Euclidean divisions, it is easy to see that we can write $z = z''(\pi_1^{b_1} \dots \pi_t^{b_t})^{m'}$, where the b_i's are non-negative integers and $z'' \in O_{\mathbf{K}}$ satisfies $|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(z'')| \le |D_{\mathbf{K}}|^{1/2} P^{tdh_{\mathbf{K}}m'}$. By Lemma 2, there exists a unit $\varepsilon \in O_{\mathbf{K}}$ such that $z'' = z'\varepsilon^{m'}$,

$$\mathrm{N}_S(z') \le |D_{\mathbf{K}}|^{1/2} \quad \text{and} \quad \mathrm{h}(z') \le \exp\{c_{52}\,(R_{\mathbf{K}} + h_{\mathbf{K}} \log^* P + \log^* |D_{\mathbf{K}}|)\}. \tag{5.26}$$

Equation (5.25) now becomes

$$\mathrm{N}_{\mathbf{M}/\mathbf{L}}\Big(\frac{\tau_1}{\varepsilon\pi_1^{b_1} \dots \pi_t^{b_t}} - v\frac{\tau_2}{\varepsilon\pi_1^{b_1} \dots \pi_t^{b_t}}\Big) = \kappa_1^{m'-1}\,(\alpha_2 - \alpha_1)z'. \tag{5.27}$$

Let $S_{\mathbf{L}}$ (resp. $S_{\mathbf{M}}$) be the set of all extensions to $\mathbf{L}$ (resp. $\mathbf{M}$) of the places in S and denote by $R_{S_{\mathbf{M}}}$ the $S_{\mathbf{M}}$-regulator of $\mathbf{M}$. Further, observe that the number of finite places in $S_{\mathbf{L}}$ is not greater than tn^2. In order to apply Proposition 4 to the Thue-Mahler equation

$$\mathrm{N}_{\mathbf{M}/\mathbf{L}}(X_0 - vY_0) = \kappa_1^{m'-1}\,(\alpha_2 - \alpha_1)z' \quad \text{in} \quad X_0, Y_0 \in O_{S_{\mathbf{L}}},$$

we need the following upper bounds, which can be deduced from (3.2), (5.23) and (5.26) :

$$\begin{aligned} \mathrm{h}\big(\kappa_1^{m'-1}(\alpha_2 - \alpha_1)z'\big) &\le H\,\exp\big\{c_{53}\,\big(R_{\mathbf{K}_1} + h_{\mathbf{K}}\,\log|A\,P\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big)\big\}, \\ \mathrm{N}_{S_{\mathbf{L}}}\big(\kappa_1^{m'-1}(\alpha_2 - \alpha_1)z'\big) &\le \exp\big\{c_{54}\,\log|A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big\}, \\ \mathrm{h}(v) &\le \exp\big\{c_{55}\,\big(R_{\mathbf{K}_1} + R_{\mathbf{K}_2} + \log|A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big)\big\}. \end{aligned} \tag{5.28}$$

By Lemma 3, we have

$$R_{S_{\mathbf{M}}} \le R_{\mathbf{M}}\,h_{\mathbf{M}}(m'n^2d\log^* P)^{tn^2m}. \tag{5.29}$$

Apply Proposition 4 to the equation (5.27). Using Lemma 7, (5.28) and (5.29), we obtain

$$\mathrm{h}\big(\tau_i/(\varepsilon\pi_1^{b_1} \dots \pi_t^{b_t})\big) \le H\,\exp\big\{c_{56}\,T_3\big\}, \quad i = 1, 2, \tag{5.30}$$

where

$$T_3 \le P^{dn^2m'(m'-1)^2}\,(\log^* P)^{tn^2m'}\,(R_{\mathbf{M}}h_{\mathbf{M}})^2\,\big(\log^*(R_{\mathbf{M}}h_{\mathbf{M}})\big)^2\,\log\big|AD_{\mathbf{K}}\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)\big|.$$

We deduce from Lemma 7 and (5.26) that

$$\mathrm{h}\big(z/(\varepsilon\pi_1^{b_1} \dots \pi_t^{b_t})^{m'}\big) = \mathrm{h}(z') \le \exp\{c_{57}\,T_3\big\},$$

and, since $X - \alpha_1 z = \kappa_1^{1-m'}\,\tau_1^{m'}$, we infer from (5.30) that

$$\mathrm{h}\big(X/(\varepsilon\pi_1^{b_1} \dots \pi_t^{b_t})^{m'}\big) \le H^{m'+1}\,\exp\{c_{58}\,T_3\big\}.$$

Thus, we get the upper bound

$$\mathrm{h}(x) = \mathrm{h}(X/z) \le H^{m'+1} \exp\{c_{59}\, T_3\}. \tag{5.31}$$

Now, we have to bound the quantity $R_{\mathbf{M}} h_{\mathbf{M}}$; for this, in view of Lemma 8, it is sufficient to bound $|D_{\mathbf{M}}|$. Recall that $v \in \mathbf{C}$ is a root of $T^{m'} - (\kappa_1/\kappa_2)^{m'-1}$ and that $\mathbf{M} = \mathbf{L}(v)$. In order to apply Lemma 9, which leads to

$$|D_{\mathbf{M}}| \le c_{60}\, |D_{\mathbf{L}}|^{m'}\, \mathrm{N}_{\mathbf{L}/\mathbf{Q}}\Big(\prod_{\mathrm{ord}_{\mathfrak{p}}(\kappa_1\kappa_2)\neq 0} \mathfrak{p}\Big)^{m'-1}, \tag{5.32}$$

observe that the prime ideals in $O_{\mathbf{L}}$ dividing $\kappa_1\kappa_2$ belong to one of the following two groups :

(a) those dividing $\mathfrak{a}_1\, \mathfrak{a}_2\, O_{\mathbf{L}}$;

(b) those dividing $\mathfrak{b}_1'\, \mathfrak{b}_2'\, O_{\mathbf{L}}$.

Let $i = 1, 2$ and recall that $\mathfrak{a}_i\, O_{\mathbf{K}_i}$ divides $a\big(\mathfrak{b}'\, g'(\alpha_i)\big)^{m_i-1} O_{\mathbf{K}_i}$. Denoting by $\mathbf{F}$ the splitting field of f, it follows from

$$\big|\mathrm{N}_{\mathbf{F}/\mathbf{Q}}\big(g'(\alpha_i)\big)\big|^{[\mathbf{K}(\alpha_i):\mathbf{K}]} \le \big|\mathrm{N}_{\mathbf{F}/\mathbf{Q}}\big(\mathrm{Res}(g', f_i)\big)\big|$$

that

$$\big|\mathrm{N}_{\mathbf{K}_i/\mathbf{Q}}\big(g'(\alpha_i)\big)\big| \le |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|.$$

Consequently, we have

$$\begin{aligned} \mathrm{N}_{\mathbf{L}/\mathbf{Q}}\Big(\prod_{\mathfrak{p}|\kappa_1\kappa_2} \mathfrak{p}\Big) &\le \big|\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(a\,\mathfrak{b}') \cdot \mathrm{N}_{\mathbf{K}_1/\mathbf{Q}}(\mathfrak{b}_1')^n \cdot \mathrm{N}_{\mathbf{K}_2/\mathbf{Q}}(\mathfrak{b}_2')^n \cdot \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)^{2n}\big| \\ &\le A^{n^2}\, |D_{\mathbf{K}}|^{n^2/2}\, |D_{\mathbf{K}_1}|^{n/2}\, |D_{\mathbf{K}_2}|^{n/2}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{2n}, \end{aligned} \tag{5.33}$$

and, by (5.32) and

$$\begin{aligned} |D_{\mathbf{K}_i}| &\le |D_{\mathbf{K}}|^n\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|, \quad i = 1, 2, \\ |D_{\mathbf{L}}| &\le |D_{\mathbf{K}}|^{n^2}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{2n}, \end{aligned} \tag{5.34}$$

we obtain

$$\begin{aligned} |D_{\mathbf{M}}| &\le c_{61}\, |D_{\mathbf{L}}|^{m'}\, A^{n^2 m'}\, |D_{\mathbf{K}}|^{n^2 m'/2}\, |D_{\mathbf{K}_1}|^{nm'/2}\, |D_{\mathbf{K}_2}|^{nm'/2}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{2nm'} \\ &\le c_{61}\, |D_{\mathbf{K}}|^{5n^2 m'/2}\, A^{n^2 m'}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{5nm'}. \end{aligned} \tag{5.35}$$

Finally, (5.31), (5.35) and Lemma 8 lead to the bound

$$\begin{aligned} \mathrm{h}(x) \le H^{m'+1} \exp\big\{ & c_{62}\, P^{dn^2 m'^3}\, (\log^* P)^{tn^2 m'}\, |D_{\mathbf{K}}|^{5n^2 m'/2} \\ & |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{5nm'}\, A^{n^2 m'}\, \big(\log |A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big)^{2dn^2 m'} \big\}. \end{aligned} \tag{5.36}$$

Case (ii) :

Let ζ be a primitive m'-th root of unity and consider the field $\mathbf{M}_1 = \mathbf{L}(\zeta)$, which is of degree $\leq m'-1$ over $\mathbf{L}$. Equation (5.24) now becomes

$$\kappa_1^{m'-1}(\alpha_2-\alpha_1)z = \prod_{k=1}^{m'}(\tau_1-\zeta^k u\tau_2).$$

Let S_1 be the set of all extensions to the field $\mathbf{M}_1$ of the places in S and R_{S_1} be the S_1-regulator of $\mathbf{M}_1$. Observe that the number of finite places in S_1 is not greater than $m'n^2t$. Clearly, it follows from (5.23) that

$$\mathrm{N}_{S_1}\big(\kappa_2^{m'}\,\kappa_1^{m'-1}(\alpha_2-\alpha_1)z\big) \leq \exp\big\{c_{63}\,\log|A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big\},$$

and the same upper bound is also valid for $\mathrm{N}_{S_1}\big(\kappa_2(\tau_1-\zeta^k u\tau_2)\big)$, $k=1,\dots,m'$. Thus, noticing that $\kappa_2(\tau_1-\zeta^k u\tau_2)$ is an algebraic integer and applying Lemma 2, we can write

$$\kappa_2(\tau_1-\zeta^k u\tau_2) = b_k\varepsilon_k, \tag{5.37}$$

where ε_k is an S_1-unit in $\mathbf{M}_1$ and $b_k\in\mathbf{M}_1$ satisfies

$$\mathrm{h}(b_k) \leq \exp\big\{c_{64}\,\big(R_{\mathbf{M}_1}+h_{\mathbf{M}_1}\log^*P+\log|A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big)\big\}. \tag{5.38}$$

Using (5.37), we get the following S_1-unit equations in $\mathbf{M}_1$:

$$\big((\zeta^k-\zeta^2)b_1\big)\varepsilon_1+\big((\zeta-\zeta^k)b_2\big)\varepsilon_2+\big((\zeta^2-\zeta)b_k\big)\varepsilon_k=0, \tag{5.39}$$

for $k=3,\dots,m'$. The height of the algebraic numbers $\zeta^k-\zeta^{k'}$, $1\leq k<k'\leq m'$, is bounded by an absolute constant depending only on m'. So, applying Proposition 3 to the equations (5.39) and using (5.38), there exist S_1-units $\eta_3,\dots,\eta_{m'}$ in $\mathbf{M}_1$ such that, for $k=3,\dots,m'$ and $i\in\{1,2,k\}$,

$$\begin{gathered}\mathrm{h}(\varepsilon_i/\eta_k)\leq\exp\{c_{65}\,T_4\},\quad\text{where}\\ T_4=P^{n^2d(m'-1)}\,R_{S_1}\,\big(\log^*R_{S_1}\big)^2\,\big(R_{\mathbf{M}_1}+h_{\mathbf{M}_1}\log^*P+\log|AD_{\mathbf{K}}\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big).\end{gathered} \tag{5.40}$$

It follows from (3.2), (5.40) and

$$\kappa_2^{m'}\,\kappa_1^{m'-1}(\alpha_2-\alpha_1)\cdot\frac{z}{\eta_3^3\eta_4\dots\eta_{m'}} = b_1\frac{\varepsilon_1}{\eta_3}\cdot b_2\frac{\varepsilon_2}{\eta_2}\cdot\prod_{k=3}^{m'}b_k\frac{\varepsilon_k}{\eta_k}$$

that

$$\mathrm{h}\big(z/(\eta_3^3\eta_4\dots\eta_{m'})\big)\leq H\,\exp\{c_{66}\,T_4\}. \tag{5.41}$$

Further, by eliminating $u\tau_2$ from the two equalities

$$\kappa_2(\tau_1+\zeta u\tau_2)=b_1\varepsilon_1 \quad \text{and} \quad \kappa_2(\tau_1+\zeta^2 u\tau_2)=b_2\varepsilon_2,$$

and using again (5.40), we infer that

$$\mathrm{h}(\tau_1/\eta_k)\leq \exp\{c_{67}\,T_4\}, \qquad k=3,\ldots,m',$$

whence, from $X-\alpha_1 z=\kappa_1^{1-m'}\tau_1^{m'}$, we get

$$\mathrm{h}\bigl(X/(\eta_3^3\eta_4\ldots\eta_{m'})\bigr)\leq H^2\exp\{c_{68}\,T_4\}. \tag{5.42}$$

In order to bound the quantity T_4, we use the estimate $R_{S_1}\leq R_{\mathbf{M}_1}\,h_{\mathbf{M}_1}\,(m'n^2d\log^* P)^{m'n^2t}$ given by Lemma 3, the bound $|D_{\mathbf{M}_1}|\leq |D_{\mathbf{L}}|^{m'-1}$ (see [26], Equation (28)) and (5.34). Hence, from Lemma 8, (5.40), (5.41) and (5.42), it easily follows that

$$\begin{aligned}\mathrm{h}(x)=\mathrm{h}(X/z)\leq H^3\,\exp\bigl\{&c_{69}\,P^{dn^2m'}\,(\log^* P)^{m'n^2t}\,|D_{\mathbf{K}}|^{n^2m'}\\ &|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{2nm'}\,\bigl(\log|A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\bigr)^{2dn^2m'}\bigr\}.\end{aligned} \tag{5.43}$$

Now we deal with the case when $(m_1,m_2)=2^{l+2}$, where $l\geq 0$. Equation (5.24) becomes

$$\kappa_1^3(\alpha_2-\alpha_1)z=\bigl(\kappa_1\xi_1^{m_1'}\bigr)^4-(\kappa_1/\kappa_2)^3\,\bigl(\kappa_2\xi_2^{m_2'}\bigr)^4. \tag{5.44}$$

By Theorem 9.1 of Chapter *VIII* of [10], there are three possible cases :

(iii) The polynomial $T^4-(\kappa_1/\kappa_2)^3$ is irreducible over $\mathbf{L}$.
(iv) There exists $u\in\mathbf{L}$ such that $-4(\kappa_1/\kappa_2)^3=u^4$.
(v) There exists $u\in\mathbf{L}$ such that $(\kappa_1/\kappa_2)^3=u^2$.

Case (iii) :

We exactly follow the argument of Case (i) and we get the same bound, namely

$$\begin{aligned}\mathrm{h}(x)\leq H^{m'+1}\,\exp\bigl\{&c_{70}\,P^{dn^2m'^3}\,(\log^* P)^{tn^2m'}\,|D_{\mathbf{K}}|^{5n^2m'/2}\\ &|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{5nm'}\,A^{n^2m'}\,\bigl(\log|A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\bigr)^{2dn^2m'}\bigr\}.\end{aligned} \tag{5.45}$$

Case (iv) :

We work in the field $\mathbf{M}_2=\mathbf{L}(i,\sqrt{2})$. Equation (5.44) can be rewritten as

$$\kappa_1^3(\alpha_2-\alpha_1)z=\prod_{k=1}^{4}(\tau_1-i^k u\tau_2/\sqrt{2}).$$

Noticing that $\sqrt{2}\kappa_2\bigl(\tau_1-i^k u\tau_2/\sqrt{2}\bigr)$, where $1\leq i\leq 4$, are algebraic integers, we proceed as in the proof of Case (ii). By estimates (5.34) and $|D_{\mathbf{M}_2}|\leq c_{71}|D_{\mathbf{L}}|^4$, we get

$$\begin{aligned}\mathrm{h}(x)\leq H^3\,\exp\bigl\{&c_{72}\,P^{4dn^2}\,(\log^* P)^{4tn^2+3}\,|D_{\mathbf{K}}|^{4n^2}\\ &|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{8n}\,\bigl(\log|A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\bigr)^{8dn^2+3}\bigr\}.\end{aligned} \tag{5.46}$$

Case (v) :

Let $v \in \mathbf{C}$ such that $v^2 = u$, in the field $\mathbf{M}_3 = \mathbf{L}(i, v)$, Equation (5.44) becomes

$$\kappa_1^3(\alpha_2 - \alpha_1)z = \prod_{k=1}^{4}(\tau_1 - i^k v\tau_2).$$

We have to estimate the discriminant of the field $\mathbf{M}_3$. For this, observe that there exists some $u' \in \mathbf{L}$ such that $\kappa_2\kappa_1^3 = u'^2$ and, if $v' \in \mathbf{C}$ satisfies $v'^2 = u'$, we have $\mathbf{L}(v) = \mathbf{L}(v')$. Noticing that u' and $\kappa_1\kappa_2$ have exactly the same prime divisors, we apply Lemma 9 in order to bound $|D_{\mathbf{L}(v')}|$, and, using (5.33), we get

$$|D_{\mathbf{L}(v')}| \leq c_{73}\, |D_{\mathbf{L}}|^2\, A^{n^2}\, |D_{\mathbf{K}}|^{n^2/2}\, |D_{\mathbf{K}_1}|^{n/2}\, |D_{\mathbf{K}_2}|^{n/2}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{2n}, \tag{5.47}$$

whence, by (5.34),

$$|D_{\mathbf{L}(v')}| \leq c_{73}\, |D_{\mathbf{K}}|^{7n^2/2}\, A^{n^2}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{7n}.$$

Thus, we obtain the estimate

$$|D_{\mathbf{M}_3}| \leq c_{74}\, |D_{\mathbf{K}}|^{7n^2}\, A^{2n^2}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{14n}.$$

Repeating the same reasoning as in the proof of Case (ii), we get the bound

$$\begin{aligned} \mathrm{h}(x) \leq H^3\, \exp\big\{c_{75}\, P^{4dn^2}\, (\log^* P)^{4tn^2}\, |D_{\mathbf{K}}|^{7n^2}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{14n} \\ A^{2n^2}\, \big(\log|A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big)^{8dn^2+3}\big\}. \end{aligned} \tag{5.48}$$

Comparing the estimates (5.36), (5.43), (5.45), (5.46) and (5.48) obtained in the cases (i) to (v), we see that the bound

$$\begin{aligned} \mathrm{h}(x) = \mathrm{h}(X/z) \leq H^{m'+1}\, \exp\big\{c_{76}\, P^{dn^2m'^3}\, (\log^* P)^{tn^2m'}\, |D_{\mathbf{K}}|^{5n^2m'/2} \\ |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{5nm'}\, A^{n^2m'}\, \big(\log|A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big)^{2dn^2m'}\big\} \end{aligned} \tag{5.49}$$

is always valid, and the proof of Theorem 2 is complete. □

Preliminary to the proof of Theorem 3 : In order to proof Theorem 3, we need a variant of Theorem 2, in which $|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|$ does not appear. Keeping the same notations and the same arguments as in the proof of Theorem 2, we state a new estimate for $\mathrm{h}(x)$, using the parameters of the field $\mathbf{L} = \mathbf{K}(\alpha_1, \alpha_2)$.

Denote by $d_{\mathbf{L}}$ the degree of $\mathbf{L}$ and by $t_{\mathbf{L}}$ the number of finite places in $S_{\mathbf{L}}$. We do the same case by case analysis as in the proof of Theorem 2.

Cases (i) and (iii) :

Using the estimate $R_{S_{\mathbf{M}}} \leq R_{\mathbf{M}}\, h_{\mathbf{M}}(m'd_{\mathbf{L}}\, \log^* P)^{t_{\mathbf{L}}\, m'}$ given by Lemma 3, we proceed as in the proof of Theorem 2 to get, instead of (5.31),

$$\mathrm{h}(x) \leq H^{m'+1}\, \exp\big\{c_{77}\, T_3'\big\}, \tag{5.50}$$

where

$$T_3' \leq P^{d_{\mathbf{L}}\, m'(m'-1)^2} (\log^* P)^{t_{\mathbf{L}}\, m'} \, (R_{\mathbf{M}} h_{\mathbf{M}})^2 \left(\log^*(R_{\mathbf{M}} h_{\mathbf{M}})\right)^2 \log\bigl| A D_{\mathbf{K}} \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)\bigr|.$$

Instead of (5.33), we use the estimate

$$\mathrm{N}_{\mathbf{L}/\mathbf{Q}}\Bigl(\prod_{\mathrm{ord}_{\mathfrak{p}}(\kappa_1 \kappa_2) \neq 0} \mathfrak{p} \Bigr) \leq \bigl| \mathrm{N}_{\mathbf{L}/\mathbf{Q}}(a\, \mathfrak{b}'\, \mathfrak{b}_1'\, \mathfrak{b}_2')\bigr| \cdot \mathrm{N}_{\mathbf{L}/\mathbf{Q}}\Bigl(\prod_{\mathrm{ord}_{\mathfrak{p}}(g'(\alpha_1) g'(\alpha_2)) \neq 0} \mathfrak{p} \Bigr),$$

and, in view of $|D_{\mathbf{K}_i}|^{[\mathbf{L}:\mathbf{K}_i]} \leq |D_{\mathbf{L}}|$ for $i = 1, 2$, we get

$$\mathrm{N}_{\mathbf{L}/\mathbf{Q}}\Bigl(\prod_{\mathrm{ord}_{\mathfrak{p}}(\kappa_1 \kappa_2) \neq 0} \mathfrak{p} \Bigr) \leq A^{n^2}\, |D_{\mathbf{K}}|^{n^2/2}\, |D_{\mathbf{L}}| \prod_{\mathfrak{p} | \Delta_g} \bigl(\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p})\bigr)^{n^2}. \tag{5.51}$$

Finally, (5.32), (5.50), (5.51) and Lemma 8 lead to the bound

$$\begin{aligned} \mathrm{h}(x) \leq H^{m'+1} \exp\Bigl\{ & c_{78}\, P^{d_{\mathbf{L}}\, m'(m'-1)^2} (\log^* P)^{t_{\mathbf{L}}\, m'}\, |D_{\mathbf{L}}|^{2m'}\, |D_{\mathbf{K}}|^{m' n^2/2}\, A^{m' n^2} \\ & \prod_{\mathfrak{p} | \Delta_g} \bigl(\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p})\bigr)^{m' n^2} \Bigl(\Bigl(\log A\, |D_{\mathbf{L}}| \prod_{\mathfrak{p} | \Delta_g} \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p})\Bigr)^{2 d_{\mathbf{L}}\, m'} + \log |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)| \Bigr) \Bigr\}. \end{aligned} \tag{5.52}$$

Cases (ii) and (iv) :

Using $|D_{\mathbf{M}_1}| \leq |D_{\mathbf{L}}|^{m'-1}$ and $|D_{\mathbf{M}_2}| \leq c_{79}\, |D_{\mathbf{L}}|^4$, we get in both cases the estimate

$$\begin{aligned} \mathrm{h}(x) \leq H^3 \exp\Bigl\{ & c_{80}\, P^{d_{\mathbf{L}}\, m'} (\log^* P)^{t_{\mathbf{L}}\, m'+3}\, |D_{\mathbf{L}}|^{m'} \\ & \bigl((\log A\, |D_{\mathbf{L}}|)^{2 d_{\mathbf{L}}\, m'+3} + \log |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)| \bigr) \Bigr\}. \end{aligned} \tag{5.53}$$

Case (v) :

Instead of (5.47), we have

$$|D_{\mathbf{L}(v')}| \leq c_{81}\, |D_{\mathbf{L}}|^2\, A^{n^2}\, |D_{\mathbf{K}}|^{n^2/2}\, |D_{\mathbf{L}}| \prod_{\mathfrak{p} | \Delta_g} \bigl(\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p})\bigr)^{n^2},$$

whence, after some computations,

$$\begin{aligned} \mathrm{h}(x) \leq H^3 \exp\Bigl\{ & c_{82}\, P^{4 d_{\mathbf{L}}} (\log^* P)^{4 t_{\mathbf{L}}}\, |D_{\mathbf{L}}|^6\, |D_{\mathbf{K}}|^{n^2}\, A^{2n^2} \\ & \prod_{\mathfrak{p} | \Delta_g} \bigl(\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p})\bigr)^{2n^2} \bigl((\log A\, |D_{\mathbf{L}}|)^{8 d_{\mathbf{L}}+3} + \log |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)| \bigr) \Bigr\}. \end{aligned} \tag{5.54}$$

Comparing the estimates (5.52), (5.53) and (5.54) obtained in the cases (i) to (v), we see that the bound

$$
\begin{aligned}
\mathrm{h}(x) \le H^{m'+1} \exp\Big\{ & c_{83}\, P^{d_{\mathbf{L}}\, m'(m'-1)^2} (\log^* P)^{t_{\mathbf{L}}\, m'} \, |D_{\mathbf{L}}|^{2m'}\, |D_{\mathbf{K}}|^{m' n^2/2}\, A^{m' n^2} \\
& \prod_{\mathfrak{p}|\Delta_g} \left(\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p})\right)^{m' n^2} \Big(\Big(\log A\, |D_{\mathbf{L}}| \prod_{\mathfrak{p}|\Delta_g} \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\mathfrak{p})\Big)^{2 d_{\mathbf{L}}\, m'} + \log |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)| \Big) \Big\}.
\end{aligned}
\tag{5.55}
$$

is always valid.

Proof of Theorem 3 : We can suppose $m_1 \ge m_2$, with $m_1 \ge 3$ and $m_2 \ge 2$ and we claim that $\alpha_1 \in \mathbf{K}$. Indeed, if $f(X)$ has a root $\alpha_i \notin \mathbf{K}$ for which $m_i \ge 3$, then there exists $j \ne i$ such that α_j is a conjugate of α_i over $\mathbf{K}$, hence we have $m_i = m_j \ge 3$ and the hypothesis of Theorem 2 is satisfied, in contradiction with our assumption. Similarly, if α_2 lies in an extension of degree ≥ 3 over $\mathbf{K}$, then there exist distinct i and j such that $i \ne 2, j \ne 2$ and both α_i and α_j are conjugate to α_2 over $\mathbf{K}$, and we have $m_i = m_j = m_2 \ge 2$. But this case is covered by Theorem 1 or Theorem 2, in contradiction with our assumption. Whence we deduce $[\mathbf{K}(\alpha_2) : \mathbf{K}] \le 2$ (all this is due to Voutier [26]).

Let $(x, y) \in O_S \times \mathbf{K}$ be a solution of (2.1) and put $\mathbf{L} = \mathbf{K}(\alpha_2)$. Keeping the same notations and arguing as in the proofs of Theorems 1 and 2, we get the two equations

$$
\begin{aligned}
X - \alpha_1 z &= \kappa_1 \xi_1^{m_1} \\
X - \alpha_2 z &= \kappa_2 \xi_2^{m_2},
\end{aligned}
\tag{5.56}
$$

where

$$
\begin{aligned}
|\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(\kappa_1)| &\le A^2\, |D_{\mathbf{K}}|^{2m_1}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{2m_1}, \\
|\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(\kappa_2)| &\le A^2\, |D_{\mathbf{K}}|^{2m_2}\, |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{3m_2/2} \\
\text{and} \quad \mathrm{h}(\kappa_i) &\le \exp\big\{c_{84}\big(R_{\mathbf{L}} + \log|A\, D_{\mathbf{K}}\, \mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\big)\big\} \ \text{ for } \ i = 1, 2.
\end{aligned}
\tag{5.57}
$$

As before, we deduce from (5.56) the equation

$$
(\alpha_1 - \alpha_2) z = \kappa_2 \xi_2^{m_2} - \kappa_1 \xi_1^{m_1},
\tag{5.58}
$$

which can be viewed as a superelliptic equation with coefficients in $O_{\mathbf{L}}$.

More precisely, using Euclidean divisions as in the proof of Theorem 1 (see after (5.9)), we infer that there exist an S-unit η with $\eta^{-1} \in O_{\mathbf{K}}$ and $z' \in O_{\mathbf{K}}$ such that

$$
\begin{aligned}
z = z'\, \eta^{-m_1 m_2}, \quad & |\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(z')| \le |D_{\mathbf{K}}|\, P^{2 t d h_{\mathbf{K}}\, m_1 m_2} \\
\text{and} \quad & \mathrm{h}(z') \le \exp\big\{c_{85}\, (R_{\mathbf{K}} + h_{\mathbf{K}} \log^* P + \log^* |D_{\mathbf{K}}|)\big\}.
\end{aligned}
\tag{5.59}
$$

Together with (5.58), it yields

$$
\kappa_2^{m_2 - 1} \kappa_1 \big(\xi_1 \eta^{m_2}\big)^{m_1} = \big(\kappa_2 \xi_2 \eta^{m_1}\big)^{m_2} - \kappa_2^{m_2 - 1} (\alpha_1 - \alpha_2) z',
$$

and, denoting by $S_{\mathbf{L}}$ the set of all extensions to the field $\mathbf{L}$ of the places in S, we remark that $\left(\kappa_2\xi_2\eta^{m_1}, \xi_1\eta^{m_2}\right) \in O_{S_{\mathbf{L}}} \times \mathbf{L}$ is a solution to the superelliptic equation

$$X_0^{m_2} - \kappa_2^{m_2-1}(\alpha_1 - \alpha_2)z' = \kappa_2^{m_2-1}\kappa_1 Y_0^{m_1}, \tag{5.60}$$

to which we may apply the estimate (5.55). The purpose of this estimate is to get an upper bound with no $h_{\mathbf{K}}$ in the exponent of P. Indeed, if we apply Theorem 2 to (5.60), a factor $P^{h_{\mathbf{K}}}$ due to $|\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(\Delta_{f_0})|$ occur (see (5.62) and (5.63) after).

Let β be a root of the polynomial $f_0(X) := X_0^{m_2} - \kappa_2^{m_2-1}(\alpha_1 - \alpha_2)z'$ and ζ be a primitive m_2-th root of unity. Here the field $\mathbf{L}$ (resp. $\mathbf{L}' := \mathbf{L}(\beta, \zeta)$) plays the role of $\mathbf{K}$ (resp. $\mathbf{L}$) occurring in (5.55).

First, observe that the prime ideals in $O_{\mathbf{L}}$ dividing the algebraic integer $\beta^{m_2} = \kappa_2^{m_2-1}(\alpha_2 - \alpha_1)z'$ belong to one of the following four groups :

(a) those dividing κ_2;

(b) those dividing $\alpha_1 - \alpha_2$;

(c) those dividing $\mathfrak{b}'$;

(d) those belonging to $S_{\mathbf{L}}$.

Arguing as in the proof of Theorem 2 and using the same notations, we get

$$\begin{aligned} \mathrm{N}_{\mathbf{L}/\mathbf{Q}}\Bigg(\prod_{\mathrm{ord}_{\mathfrak{p}}(\beta^{m_2})\neq 0} \mathfrak{p}\Bigg) &\leq \left|\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(a\,\mathfrak{b}'\,\mathfrak{b}_2'\,\Delta_g)\right| P^{2dt} \\ &\leq A^2\,|D_{\mathbf{K}}|\,|D_{\mathbf{L}}|^{1/2}\,|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^2\,P^{2dt}. \end{aligned} \tag{5.61}$$

By Lemma 9, (5.61), (5.34) and $|D_{\mathbf{L}}| \leq |D_{\mathbf{K}}|^2\,|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|$, we can estimate the discriminant $|D_{\mathbf{L}(\beta)}|$ of the field $\mathbf{L}(\beta)$:

$$|D_{\mathbf{L}(\beta)}| \leq c_{86}\,A^{2m_2}\,|D_{\mathbf{K}}|^{4m_2}\,|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{7m_2/2}\,P^{2dtm_2},$$

whence we get

$$|D_{\mathbf{L}'}| \leq c_{87}\,A^{2m_2^2}\,|D_{\mathbf{K}}|^{4m_2^2}\,|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{7m_2^2/2}\,P^{2dtm_2^2}.$$

The polynomial $f_0(X_0) = \prod_{l=0}^{m_2-1}(X_0 - \zeta^l\beta)$ is squarefree and its discriminant, denoted by Δ_{f_0}, satisfies

$$|\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(\Delta_{f_0})| \leq c_{88}|\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(\beta^{m_2})|^{m_2-1} \tag{5.62}$$

and, using (5.61), we have

$$\begin{aligned} \prod_{\mathfrak{p}|\Delta_{f_0}}\left(\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(\mathfrak{p})\right) &\leq c_{89}\prod_{\mathfrak{p}|\beta^{m_2}}\left(\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(\mathfrak{p})\right) \\ &\leq c_{89}\,A^2\,|D_{\mathbf{K}}|\,|D_{\mathbf{L}}|^{1/2}\,|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^2\,P^{2dt}. \end{aligned}$$

In view of (5.57) and (5.59), we infer that

$$\begin{aligned}|\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(\beta^{m_2})| &\le |\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(\kappa_2)|^{m_2-1}\,|\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(z')|\,|\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(\Delta_g)| \\ &\le \exp\bigl\{c_{90}\,h_{\mathbf{K}}\,\log|A\,P\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\bigr\}.\end{aligned}\tag{5.63}$$

Moreover, we have

$$\mathrm{h}(\beta) \le H\,\exp\bigl\{c_{91}\,\bigl(R_{\mathbf{L}} + h_{\mathbf{K}}\log^* P + \log|A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\bigr)\bigr\}$$

and

$$|\mathrm{N}_{\mathbf{L}/\mathbf{Q}}(\kappa_2^{m_2-1}\kappa_1)| \le A^{2m_2}\,|D_{\mathbf{K}}|^{2m_1+2m_2(m_2-1)}\,|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{2m_1+3m_2(m_2-1)/2}.$$

Using the above estimates together with $|D_{\mathbf{L}}| \le |D_{\mathbf{K}}|^2\,|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|$, $m_1 \ge 3$ and $m_2 \ge 2$, we apply (5.55) to the equation (5.60) and we get after some calculation

$$\mathrm{h}(\kappa_2\xi_2\eta^{m_1}) \le H^{m_1+1}\exp\bigl\{c_{92}\,T_5\bigr\},\tag{5.64}$$

where

$$\begin{aligned}T_5 = P^{2dm_2^2m_1^3+6dtm_1m_2^2}\,(\log^* P)^{2tm_1m_2^2}\,A^{5m_1m_2^3}\,|D_{\mathbf{K}}|^{2m_1^2m_2^4} \\ |\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{2m_1^2m_2^4}\,\bigl(\log|A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\bigr)^{4dm_1m_2^2}.\end{aligned}$$

Hence, using $m_2 \le m/2$ (otherwise, $m_1 = m_2 = m$ and we could apply Theorem 2, in contradiction with our assumption), we get

$$\begin{aligned}T_5 \le P^{d(m^5+4tm^3)/2}\,|D_{\mathbf{K}}|^{m^6/8}\,|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{m^6/8} \\ A^{5m^4/8}\,\bigl(\log|A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\bigr)^{dm^3}.\end{aligned}\tag{5.65}$$

Finally, we infer from (5.56) and (5.59) that

$$x = \frac{X}{z} = \frac{\kappa_2\xi_2^{m_2}}{z} + \alpha_2 = \frac{(\kappa_2\xi_2\eta^{m_1})^{m_2}}{\kappa_2^{m_2-1}z'} + \alpha_2,$$

which, with (5.57), (5.59), (5.64), (5.65) and $m_2 \le m/2$, yields

$$\begin{aligned}\mathrm{h}(x) \le H^{m^2}\,\exp\Bigl\{c_{93}\,P^{d(m^5+4tm^3)/2}\,|D_{\mathbf{K}}|^{m^6/8}\,|\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|^{m^6/8} \\ A^{5m^4/8}\,\bigl(\log|A\,D_{\mathbf{K}}\,\mathrm{N}_{\mathbf{K}/\mathbf{Q}}(\Delta_g)|\bigr)^{dm^3}\Bigr\},\end{aligned}$$

and the proof is complete. □

Acknowledgements : I am indebted to the referee for his numerous valuable remarks and, in particular, for Lemma 9. Also, I would like to thank Maurice Mignotte and Kálmán Győry for their constant encouragements.

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