Anticyclic operads and Auslander-Reiten translation

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IN BRIEF

1: Theory of operads (algebraic topology)

2: Representation theory

Build some cyclic actions

$$\tau \in \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}^N) \text{ with } \tau^n = \mathrm{Id}, \tag{1}$$

for some N and n.

Observe that these two actions are closely related.

Propose some conjectural explanation for this link.

Operads: Basics and examples

(Non-symmetric) Operads in the category of Abelian groups.

Definition 1 *An operad $\mathcal{P}$: DATA of*

- *a sequence $\{\mathcal{P}(n)\}_{n\in\mathbb{N}}$ of Abelian groups,*

- *a distinguished element $\mathbf{1} \in \mathcal{P}(1)$,*

- *composition maps $\circ_i$ from $\mathcal{P}(n) \otimes_{\mathbb{Z}} \mathcal{P}(m)$ to $\mathcal{P}(n+m-1)$ for each n, m and each $1 \leq i \leq n$.*

This data must satisfy some AXIOMS, modelled after the properties of the first example below:

- *unity,*

- *associativity of nested compositions,*

- *commutativity of disjoint compositions.*

Let us give some examples.

Example 1: the endomorphism operads

Pick any free Abelian group V of finite rank. Let $\mathcal{P}(n) = \mathrm{Hom}_{\mathbb{Z}}(V^{\otimes n}, V)$. Let $\mathbf{1}$ be the identity map in $\mathcal{P}(1)$. Let $\circ_i$ be the composition of multilinear maps defined, for $f \in \mathcal{P}(n)$ and $g \in \mathcal{P}(m)$, by

$$\begin{aligned}&(f \circ_i g)(x_1, \ldots, x_{m+n-1})\\ &= f(x_1, \ldots, g(x_i, \ldots, x_{i+m-1}), \ldots, x_{m+n-1}).\end{aligned} \tag{2}$$

These data define the so-called endomorphism operad of V.

Example 2: the associative operad

Let $\mathrm{Assoc}(n)$ be the free Abelian group of rank 1 with basis b^n. Let $\mathbf{1}$ be b^1 and let

$$b^n \circ_i b^m = b^{n+m-1}. \tag{3}$$

The axioms are easily checked and this defines the associative operad Assoc.

Example 3: the diassociative operad (Loday)

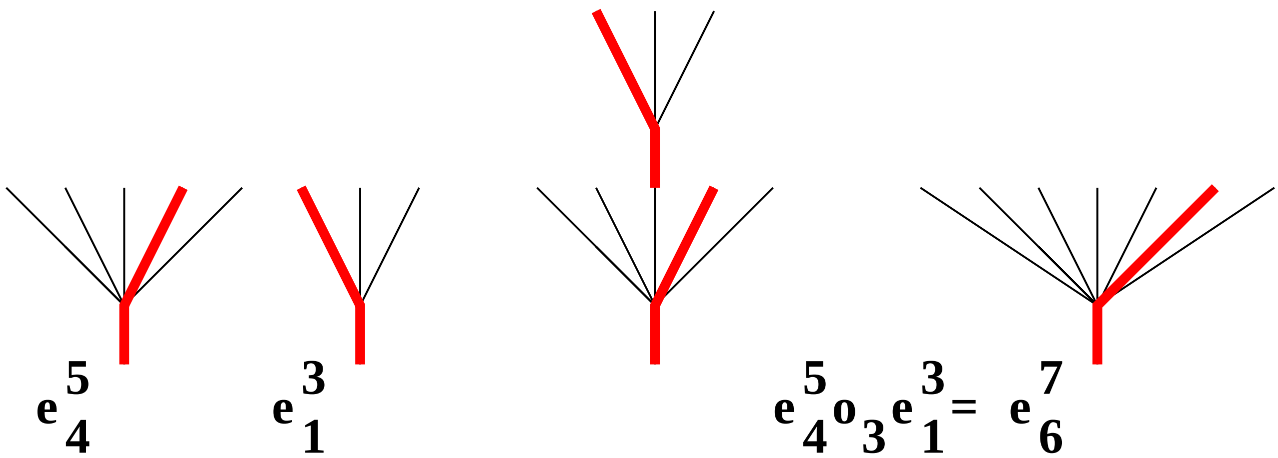
Let Dias(n) be the free Abelian group of rank n with basis $\{e_1^n, \ldots, e_n^n\}$. Let $\mathbf{1}$ be e_1^1. The composition maps are defined by

$$e_k^n \circ_i e_\ell^m = e_j^{n+m-1}, \tag{4}$$

where j is given by the following rule:

$$\begin{cases} k \text{ if } i > k, \\ k + \ell - 1 \text{ if } i = k, \\ k + n - 1 \text{ if } i < k. \end{cases} \tag{5}$$

GRAPHICAL DESCRIPTION using corollas (Follow the red line from the bottom to the top)



- free operads
- ideal in an operad
- quotient of an operad by an ideal

Hence one can speak of a presentation by generators and relations of an operad.

Let us give some examples of such presentations.

Example 2: the associative operad The operad Assoc is generated by $b^2 \in \mathrm{Assoc}(2)$. One can compute that

$$b^2 \circ_1 b^2 = b^2 \circ_2 b^2 = b^3. \tag{6}$$

The operad Assoc is presented by the generator b^2 and the relation

$$b^2 \circ_1 b^2 = b^2 \circ_2 b^2. \tag{7}$$

Example 3: the diassociative operad

The operad Dias is generated by Dias(2) = $\mathbb{Z}\{e_1^2, e_2^2\}$. One can compute (using the graphical description of Dias) that

$$e_1^3 = e_1^2 \circ_1 e_1^2 = e_1^2 \circ_2 e_1^2 = e_1^2 \circ_2 e_2^2, \tag{8}$$

$$e_2^3 = e_2^2 \circ_2 e_1^2 = e_1^2 \circ_1 e_2^2, \tag{9}$$

$$e_3^3 = e_2^2 \circ_2 e_2^2 = e_2^2 \circ_1 e_1^2 = e_2^2 \circ_1 e_2^2. \tag{10}$$

This provides a presentation of the operad Dias.

Anticyclic operads

Definition 2 *An anticyclic operad $\mathcal{P}$ is an operad $\mathcal{P}$ together with the data of endomorphisms τ_n of $\mathcal{P}(n)$ satisfying*

$$\tau_1(\mathbf{1}) = -\mathbf{1}, \tag{11}$$
$$\tau_n^{n+1} = \mathrm{Id}, \tag{12}$$
$$\tau_{n+m-1}(x \circ_n y) = -\tau_m(y) \circ_1 \tau_n(x), \tag{13}$$
$$\tau_{n+m-1}(x \circ_i y) = \tau_n(x) \circ_{i+1} y, \tag{14}$$

where $x \in \mathcal{P}(n)$, $y \in \mathcal{P}(m)$ and $1 \le i \le n-1$.

This notion has been introduced by Getzler and Kapranov.

AIM: show that the operad Dias
is an anticyclic operad.

HOW: define τ on generators,
then check against relations.

Let us define τ_2 by

$$\tau_2(e^2_1) = -e^2_1 + e^2_2, \tag{15}$$

$$\tau_2(e^2_2) = -e^2_1. \tag{16}$$

Thus the matrix of τ_2 in the basis e^2 is

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}. \tag{17}$$

Theorem 3 *The operad* Dias *is an anticyclic operad with τ_2 as above. The matrix of τ_n in the basis e^n is*

$$\begin{bmatrix} -1 & -1 & \dots & -1 \\ 1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \ddots & 1 & 0 \end{bmatrix}. \tag{18}$$

Let us give an example of computation for τ_3:

$$\begin{aligned} \tau_3(e_2^3) &= \tau_3(e_2^2 \circ_2 e_1^2) && (19)\\ &= -\tau_2(e_1^2) \circ_1 \tau_2(e_2^2) && (20)\\ &= (e_2^2 - e_1^2) \circ_1 (-e_1^2) && (21)\\ &= e_3^3 - e_1^3. && (22) \end{aligned}$$

You may check that using $e_2^3 = e_1^2 \circ_1 e_2^2$ instead leads to the same value.

Therefore we have defined an action of the cyclic group $\mathbb{Z}/(n+1)\mathbb{Z}$ on the Abelian group $\mathbb{Z}^n$ for each $n \geq 1$.

Let us now define a similar action in a completely different way.

Algebras and Auslander-Reiten translation

Λ an algebra of finite dimension over a field k.

Assume that Λ has finite global dimension.

$\operatorname{Mod}\Lambda$ category of finite-dimensional modules.

$D\operatorname{Mod}\Lambda$ bounded derived category of $\operatorname{Mod}\Lambda$.

AUSLANDER-REITEN THEORY:

self-equivalence τ of $D\operatorname{Mod}\Lambda$

This is the Auslander-Reiten translation.

This functor τ descends on the Grothendieck group $K_0(\operatorname{Mod}\Lambda) = K_0(D\operatorname{Mod}\Lambda)$ and defines a bijective linear map, still denoted by τ, on the Grothendieck group. This map is sometimes called the Coxeter transformation.

This theory has some nice applications to path algebras of quivers.

Choose any Dynkin diagram of finite type, in the usual list $(\mathbb{A}_n)_{n\geq 1}$,$(\mathbb{D}_n)_{n\geq 4}$, $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$.

Picking any orientation of this Dynkin diagram defines a quiver Q.

Let $\mathrm{Mod}\, kQ$ be the Abelian category of representations of Q.

classical results (Gabriel ; Gelfand & Ponomarev):

$\mathrm{Mod}\, kQ$ has a finite number of isomorphism classes of indecomposable modules,

in bijection with positive roots of the associated root system.

action of τ on the Grothendieck group is exactly the action of a Coxeter element in the corresponding Weyl group.

Hence τ has finite order h, the Coxeter number.

Let us look at the case of the equioriented quiver of type $\mathbb{A}_n$:

$$n \to \cdots \to 2 \to 1. \tag{23}$$

$\operatorname{Mod} \mathbb{A}_n$ the category of modules on this quiver

S_i the simple module on the vertex i.

The action of τ in the basis $\{S_1, S_2, \ldots, S_n\}$ of $K_0(\operatorname{Mod} \mathbb{A}_n)$ has the following matrix

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & \ddots & \ddots \\ \vdots & \vdots & \ddots & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}. \tag{24}$$

This is clearly the **transposed** matrix of the map τ_n that was defined purely in terms of operads before.

Now, it is possible to dualize the anticyclic operad Dias into an anticyclic **cooperad** Dias*. Then the cyclic group actions become exactly the same. This should be the proper setting.

Second example

more complicated,

more interesting.

another operad and

another family of algebras.

Binary trees

A planar binary tree is a

graph drawn in the plane,

which is connected and simply connected,

has vertices of valence 1 or 3 only,

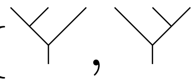
together with the data of a distinguished vertex of valence 1 called the root.

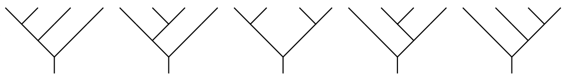
The other vertices of valence 1 are called the leaves.

The root is drawn at the bottom.

Let Y_n be the set of planar binary trees with $n+1$ leaves.

$$Y_1 = \{\mathsf{Y}\} \tag{25}$$

$Y_2 = \{$  $\}$ (26)

$Y_3 = \{$  $\}$ (27)

The cardinality of Y_n is the Catalan number

$$c_n = \frac{1}{n+1}\binom{2n}{n} \tag{28}$$

Then (Loday) there exists an operad Dend such that $\mathrm{Dend}(n) = \mathbb{Z}Y_n$. We will not describe the composition maps $\circ_i$ here. The unit $\mathbf{1}$ is the unique element of Y_1.

This operad is generated by the two trees $[(\cdot\cdot)\cdot]$ and $[\cdot(\cdot\cdot)]$ in Y_2.

The relations are as follows:

$$[(\cdot\cdot)\cdot] \circ_2 [(\cdot\cdot)\cdot] = [(\cdot\cdot)\cdot] \circ_1 [(\cdot\cdot)\cdot] + [(\cdot\cdot)\cdot] \circ_1 [\cdot(\cdot\cdot)], \tag{29}$$

$$[(\cdot\cdot)\cdot] \circ_2 [\cdot(\cdot\cdot)] = [\cdot(\cdot\cdot)] \circ_1 [(\cdot\cdot)\cdot], \tag{30}$$

$$[\cdot(\cdot\cdot)] \circ_2 [(\cdot\cdot)\cdot] + [\cdot(\cdot\cdot)] \circ_2 [\cdot(\cdot\cdot)] = [\cdot(\cdot\cdot)] \circ_1 [\cdot(\cdot\cdot)]. \tag{31}$$

Theorem 4 *There exists a unique structure of anticyclic operad on* Dend *such that*

$$\tau([\cdot(\cdot\cdot)]) = [(\cdot\cdot)\cdot] \quad \textit{and} \quad \tau([(\cdot\cdot)\cdot]) = -([\cdot(\cdot\cdot)] + [(\cdot\cdot)\cdot]). \tag{32}$$

Let us display the matrix of τ_3 in the basis Y_3 of Dend(3):

$$\begin{bmatrix} -1 & 0 & 1 & 1 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{bmatrix}. \tag{33}$$

In general, the map τ_n seems quite complicated.

One knows that

$$\tau_n^{n+1} = \mathrm{Id}\,. \tag{34}$$

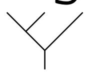
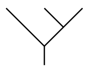
But what exactly are the eigenvalues of τ_n ?

Tamari posets

a partial order $\leq$ on Y_n,

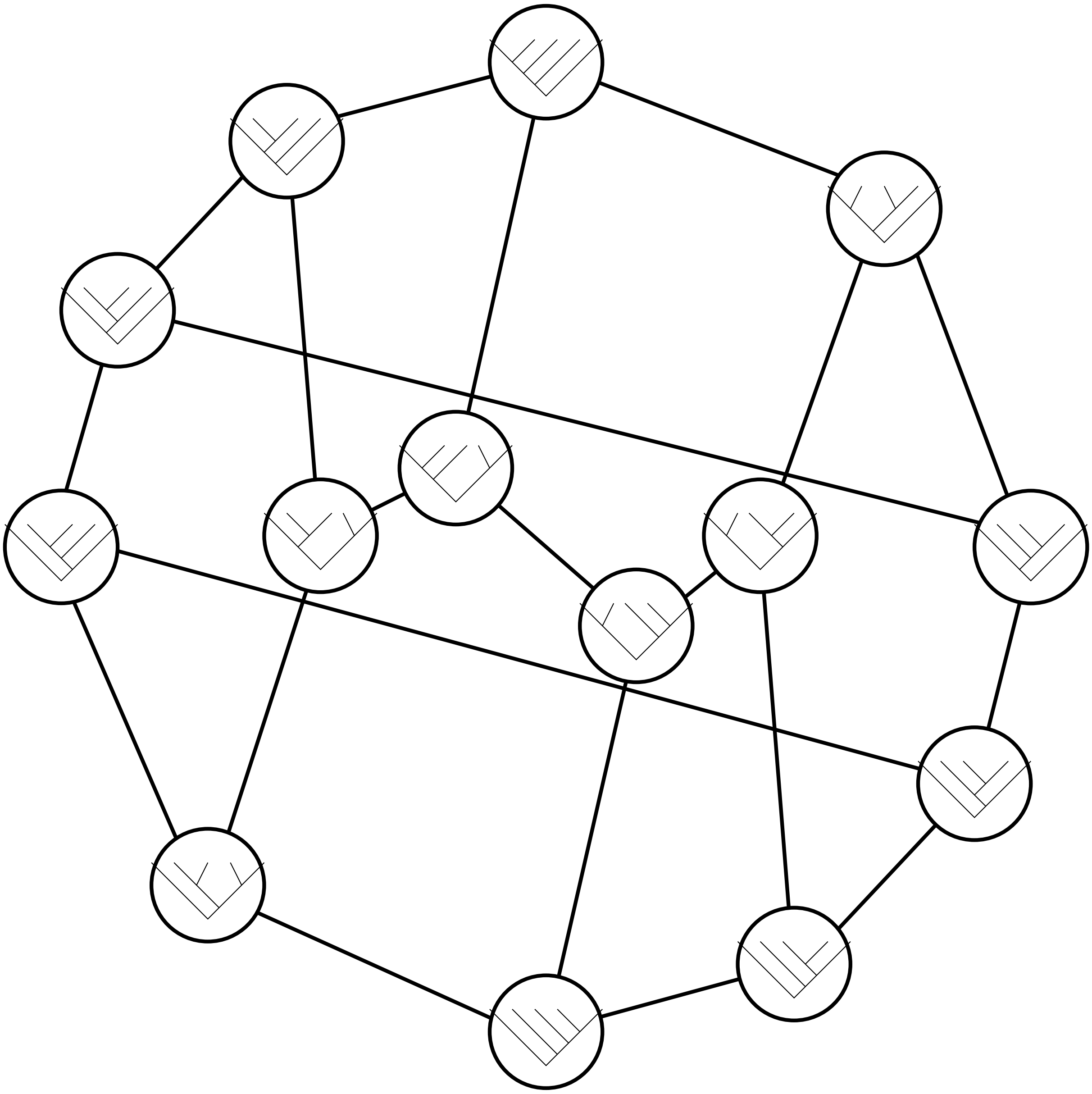
called the Tamari order or Tamari lattice.

The order relation $\leq$ is the transitive closure of some covering relations.

A tree S is covered by a tree T if they differ only in some neighborhood of an edge by the replacement of the configuration  in S by the configuration  in T.

Λ_n = incidence algebra of the poset $(Y_n, \leq)$

finite dimensional algebra of finite global dimension.



THE TAMARI LATTICE T_3

a.k.a. the third associahedron...

Auslander-Reiten theory gives:

Coxeter transformation θ

acting on the Grothendieck group of Λ_n.

This Grothendieck group has a basis coming from simple modules, which are labelled by Y_n.

Hence one can identify $K_0(\Lambda_n)$ with $\mathrm{Dend}(n)$.

Theorem 5 *On the Abelian group* $\mathrm{Dend}(n)$*, one has the relation*

$$\tau_n = (-1)^n \theta^2. \tag{35}$$

expected explanation:

appropriate functors

$$\circ_i : \mathrm{Mod}\,\Lambda_n \otimes \mathrm{Mod}\,\Lambda_m \longrightarrow \mathrm{Mod}\,\Lambda_{n+m-1}, \tag{36}$$

satisfying, together with the Auslander-Reiten translation, some version of the axioms of an anticyclic operad.