Operadic point of view on the Hopf algebra of rooted trees

F. Chapoton

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joint work with M. Livernet

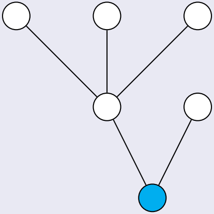
Hopf algebras in renormalization

History : the use of Hopf algebra in renormalization appears in Connes and Kreimer work.
First with a Hopf algebra of rooted trees, then with Hopf algebras H of Feynman diagrams.
Feynman rules and **dimensional regularisation** leads to an algebra map $\varphi : H \longrightarrow \mathbb{C}((\varepsilon))$.
Then one has to do **Birkhoff decomposition** of φ to get φ_+ and φ_-.
This gives counterterms and renormalized values.
This story is going on..
I will speak only of the Hopf algebra of rooted trees, not about the Hopf algebras of Feynman diagram.

The Connes-Kreimer Hopf algebra of rooted trees

Definition

A **rooted tree** is a connected and simply-connected finite graph, together with a distinguished vertex: the root.



Let H_{CK} be the polynomial algebras on rooted trees.
It is graded by the number of vertices.
The basis is indexed by forests (i.e. sets) of rooted trees.
The product is just disjoint union.

Coproduct

The coproduct is defined by **pruning** trees (cutting branches).
One has to introduce the notion of "admissible cut" (omitted here).
Then one defines Δ of a tree t as:

$$\Delta(t) = \sum_c R_c(t) \otimes P_c(t),$$

where the sum runs over the set of admissible cuts.
$R_c(t)$ is a tree (the pruned tree)
$P_c(t)$ is a forest (the fallen branches)
Then this definition is extended to forests by multiplicativity:

$$\Delta(f\,f') = \Delta(f)\Delta(f'). \tag{1}$$

The associated group

The Hopf algebra H_{CK} is a commutative and non-cocommutative graded Hopf algebra.
As such, one can consider the associated "group scheme".
For each ring R, the set of characters $H_{CK} \to R$ (the set of R-points) is a group.
This is known in numerical analysis as the **Butcher group** (group of Butcher series).

An interesting object

The usual description insists on this property of H_{CK}:

Hochschild cocycle

It has a universal property with respect to Hochschild cohomology.

I would rather like to emphasize that:

Pre-Lie algebras

It has a natural description using free pre-Lie algebras.

and that:

Nap operad

It appears in the study of the NAP (non-associative permutative) operad.

Description starting from pre-Lie algebras

Definition

A pre-Lie algebra is a vector space V and a bilinear map $\triangleleft$ from $V \otimes V \to V$ such that

$$(x \triangleleft y) \triangleleft z - x \triangleleft (y \triangleleft z) = (x \triangleleft z) \triangleleft y - x \triangleleft (z \triangleleft y). \tag{2}$$

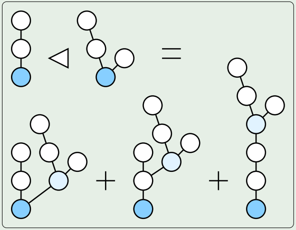
This notion has been studied by Gerstenhaber, Vinberg, Koszul and others.

It is related to the geometry of affine structures on manifolds and to left-invariant affine or symplectic structures on groups.

Free pre-Lie algebras

There is a nice description of the free Pre-Lie algebra on one generator PL:
A basis of PL is indexed by rooted trees.
The product $S \triangleleft T$ is given by the sum of all possible **graftings** of the root of T on a vertex of S.

Example



Then PL is a **Lie algebra** for the bracket $[x, y] = x \triangleleft y - y \triangleleft x$.
(Jacobi identity follows from the 4-term axiom of pre-Lie algebras)

Enveloping algebra

Let $U(PL)$ be the enveloping algebra of the Lie algebra PL.

Theorem

There is an isomorphism of $U(PL)$-modules $PL \simeq \mathbb{Q} \otimes U(PL)$.

The Hopf algebra $U(PL)$, also known as the Grossman-Larsson Hopf algebra, has a basis indexed by forests.
The product $*$ is given by the sum of all possible **grafting or falling**: $f * f'$ is the sum of all possible addition of edges to $f \sqcup f'$ from a root of f' to a vertex of f.
The coproduct is given by unshuffling.

Theorem

There is an isomorphism $H_{CK} \simeq U(PL)^*$ (graded dual).

This is essentially just an identification. The natural basis of $U(PL)$ is (up to symmetry factors) the dual basis of the usual basis of H_{CK}.

Remarks

- All this works just the same with a set of decorations: use decorated trees and forests, etc.
- This point of view naturally gives an action of another group on the Butcher group (cf D. Manchon talk).
- In some sense, the whole combinatorics of trees is contained in the definition of pre-Lie algebras, just as the notion of word is contained in the associative axiom $(xy)z = x(yz)$.

Operads in one slide

Definition

A **species** is a functor from the groupoid (finite sets,bijections) to the category (sets,maps). This defines a category of species. There is a (non-symmetric) monoidal structure $\circ$ on this category.

Definition

An **operad** is a monoid in the monoidal category (Species, $\circ$).

In more concrete terms: an operad P is the data of

- for each finite set I, a set $P(I)$, defined using natural constructions (not using in any way the nature of the elements of I)
- for each partition $I = \sqcup_{\ell \in L} I_\ell$, a map

$$P(L) \times \prod_{\ell \in L} P(I_\ell) \to P(I).$$

These **composition maps** have to be "associative".

Starting from an operad P (under some mild condition : augmented and basic), one can define **two groups**.

First construction: direct definition, using invariants and composition maps.

This is called G_P.

Second construction: in two steps, from operad to posets and from posets to commutative graded Hopf algebra H_P (incidence Hopf algebra).

Theorem

These two constructions are related : the second group is contained in the first.

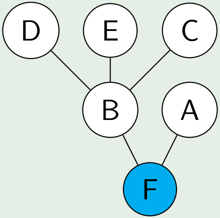
$$spec(H_P) \subset G_P$$

This can also be stated as a quotient map of Hopf algebras from $\mathbb{Q}[G_P]$ to H_P.

The NAP species

Let I be a finite set.
The set $\mathrm{NAP}(I)$ is the set of rooted trees on vertex set I.
This defines a species.
If $\#I = n$ then $\#\,\mathrm{NAP}(I) = n^{n-1}$.

Example



Example of labelled rooted tree on the set $\{A, B, C, D, E, F\}$.

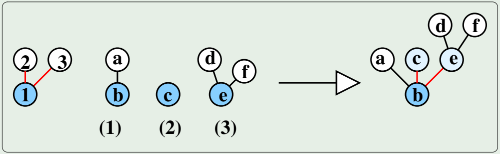
The NAP operad

To define an operad on the species NAP, one needs maps

$$\mathrm{NAP}(L) \times \prod_{\ell \in L} \mathrm{NAP}(I_\ell) \to \mathrm{NAP}(I).$$

These maps are given as follows: fix $(s, (t_l)_{\ell \in L})$.
Then consider the disjoint union of all trees t_ℓ and add edges between their roots according to the pattern given by s.

Example



Example of composition : the pattern is given by the leftmost tree.

There is a general construction of a poset starting from an operad. Let us present this in the case of NAP.

Let I be a finite set. Let $\Pi_{\mathrm{NAP}}(I)$ be the species of forests on the vertex set I.

Then one says that $f \leq f'$ if one can obtain f' by a composition map on a subforest of f.

This defines a partial order, graded by the number of connected components.

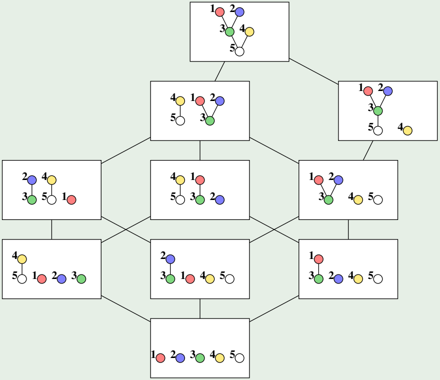
The covering relations are given by grafting the root of a tree on the root of another tree in the forest.

There is a unique minimal element.

Maximal elements are rooted trees.

A maximal interval

Example



Here is an interval in the poset of forests on $\{1, 2, 3, 4, 5\}$.

Stability of intervals and Schmitt construction

Theorem

In this collection of posets $\Pi_{\mathrm{NAP}}(I)$, every interval is isomorphic to a product of maximal intervals.

This is exactly the starting point needed to use the construction (due to W. Schmitt) of an **incidence Hopf algebra**.
Let us define the incidence Hopf algebra H_{NAP}.
It has a basis indexed by isomorphism classes of products of maximal intervals.
The product is just given by the product of intervals.
The coproduct is given by the following rule:

$$\Delta[x,z] = \sum_{x \leq y \leq z} [x,y] \otimes [y,z].$$

Warning

It is not a free algebra on the maximal intervals ! But it's free..

Incidence Hopf algebra H_{NAP}

One can describe precisely this Hopf algebra.
As a vector space, it has a basis indexed by forests.
The product is the disjoint union of forests.
The coproduct is given by admissible cuts.

Theorem

The incidence Hopf algebra H_{NAP} is isomorphic to Connes-Kreimer Hopf algebra H_{CK}.

Once again, the isomorphism is trivial: natural bases on both sides can just be identified.

One gets more than just finding again the Hopf algebra H_{CK}.
As told before, there is another group G_{NAP}, which contains the Butcher group $spec(H_{\mathrm{NAP}})$.
Equivalently, there is a bigger Hopf algebra, with H_{CK} as a quotient.
Combinatorics of this Hopf algebra is still given by rooted trees.