Quadrangulations,
Stokes posets
and serpent nests

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July 2015
Triangulations

Start with classical objects: **triangulations** of regular polygons, already considered by Leonhard Euler.

A triangulation of an heptagon chosen from the 42 possible ones

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\text{The number of triangulations in the polygon with } n + 2 \text{ sides is the Catalan number } \mathcal{C}_n = \frac{1}{n+1} \binom{2n}{n}.
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A famous sequence of numbers, named after Eugène Catalan.
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Flips between triangulations

Triangulations can be connected by flips, replacing just one interior edge by another one:

This gives a regular graph, the flip graph of triangulations.
Triangulations and associahedra

the flip graph of triangulations of the hexagon
14 vertices $\leftrightarrow$ 14 triangulations
Edges are flips
Triangulations and associahedra

The flip graph of triangulations of the hexagon
14 vertices $\leftrightarrow$ 14 triangulations
Edges are flips

It is also known that the flip graph can be realized using vertices and edges of a polytope called the associahedra, introduced by Jim Stasheff.
Oriented flips and Tamari lattices

By a standard bijection,

- triangulations $\leftrightarrow$ planar binary trees,
- flips $\leftrightarrow$ “rotation” of trees.

This allows to orient the edges of the flip graph, into the Hasse diagram of a poset, the Tamari lattice (Dov Tamari).

The Tamari lattice of triangulations of the hexagon

14 vertices $\leftrightarrow$ 14 triangulations $\leftrightarrow$ 14 planar binary trees

Edges are flips oriented from top to bottom.
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In particular, it provides a construction (due to Nathan Reading)

Finite Coxeter group $W \rightarrow$ Cambrian lattices for $W$

such that the special case of type $\mathbb{A}$ is

Symmetric group $\rightarrow$ Tamari lattice (and other lattices)
Modern point of view: cluster combinatorics

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such that the special case of type \( A \) is

Symmetric group \( \rightarrow \) Tamari lattice (and other lattices)

There are also polytopes called generalized associahedra.
A trilogy for each integer $n$

Back to some other classical combinatorial objects.

Three classical families of objects counted by the Catalan numbers.

A triangulations
flip graph, lattice, polytope

B noncrossing partitions (Germain Kreweras)
graded lattice
M"obius numbers

C Dyck paths (named after Walther von Dyck)
distributive lattice
H-triangle

They are also tied by more refined enumerative properties.
Each triangle is a two-variable generating polynomial; they are related by rational change-of-variables transformations.
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Three classical families of objects counted by the Catalan numbers.

A triangulations
- flip graph, lattice, polytope $\rightarrow F$-triangle counting faces

B noncrossing partitions (Germain Kreweras)
- graded lattice $\rightarrow M$-triangle for Möbius numbers

C Dyck paths (named after Walther von Dyck)
- distributive lattice $\rightarrow H$-triangle

They are also tied by more refined enumerative properties.
Each triangle is a two-variable generating polynomial; they are related by rational change-of-variables transformations.
An example, for concreteness

\( F \)-triangle for triangulations of the pentagon:

\[
\begin{pmatrix}
1 & 0 & 0 \\
2 & 2 & 0 \\
1 & 3 & 2
\end{pmatrix}
\]

\( M \)-triangle for the noncrossing partitions lattice of size 5:

\[
\begin{pmatrix}
2 & -3 & 1 \\
-3 & 3 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

\( H \)-triangle for Dyck paths with 3 up and 3 down steps:

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 2 & 0 \\
1 & 1 & 0
\end{pmatrix}
\]
A trilogy for each Weyl group $W$

All three of them are now understood as “being of type $A$” and have been generalized to all finite Weyl groups, some even to finite Coxeter groups or complex reflexion groups.

- Triangulations $\rightarrow W$-clusters
  - Flip graph, Cambrian lattices, polytopes

- $W$-noncrossing partitions (Bessis, Brady-Watt)
  - Graded lattice

- $W$-nonnesting partitions, ideals in the root poset (Postnikov)
  - Distributive lattice with still the same enumerative relations between $F$-triangle, $M$-triangle and $H$-triangle.
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more precisely, previous construction was a Coxeter element $c$ in Coxeter group $W \to$ a Cambrian lattice

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B The noncrossing side of the trilogy is still missing.

Again A and C are expected to have the same cardinality and the same enumerative relations between $F$-triangle and $H$-triangle.
Quadrangulations of regular polygons

set of lines between vertices cutting the polygon into parts with 4 sides
Quadrangulations of regular polygons

The number of sides of the polygon must be even, say $2n + 2$. The number of quadrangulations is then $\frac{1}{2n+1} \binom{3n}{n}$, called a Fuss-Catalan number (named after Nikolaus Fuss).
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Quadrangulations can also be depicted like that: as a tree-like union of quadrilaterals along their edges.
Quadrangulations and ambiguous flips

One can flip quadrangulations, but there are two ways to replace any given edge.
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This has something to do with 2-cluster categories and 2-Cambrian lattices. Not exactly the subject of this talk.
Yuliy Baryshnikov has defined, for every quadrangulation \( Q \) a polytope called the **Stokes polytope**. Some of them are associahedra! His motivation came from the study of bifurcation diagrams of quadratic differentials (singularity theory, geometry).
Stokes polytopes

Yuliy Baryshnikov has defined, for every quadrangulation $Q$ a polytope called the **Stokes polytope**. Some of them are associahedra! His motivation came from the study of bifurcation diagrams of quadratic differentials (singularity theory, geometry). To define these polytopes, he introduced a **compatibility** relation between quadrangulations.

**Theorem (Baryshnikov)**

Let $Q$ be a quadrangulation. There exists a polytope $St_Q$ with

- vertices $\longleftrightarrow$ $Q$-compatible quadrangulations,
- edges $\longleftrightarrow$ flips between them.

For $Q$ in the $2n + 2$-sided polygon, the dimension of $St_Q$ is $n - 1$. 
Let $Q$ be fixed. Let us now describe compatibility with $Q$.

- color vertices of $Q$ by alternating black and white,
- rotate $Q$ by an angle of $\frac{2\pi}{4}n + \pi$ and color it blue,
- orient all edges of $Q$ from white vertices to black vertices.
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Let $Q$ be fixed. Let us now describe compatibility with $Q$.

- consider another quadrangulation $Q'$ (color it red)
- color vertices of $Q'$ by alternating black and white as before
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Compatibility of quadrangulations

Let $Q$ be fixed. Let us now describe compatibility with $Q$.

- consider another quadrangulation $Q'$ (color it red)
- color vertices of $Q'$ by alternating black and white as before
- orient all edges of $Q'$ from white vertices $\circ$ to black vertices $\bullet$
- superpose rotated $Q$ and non-rotated $Q'$

Compatibility: at every crossing, $(\text{red}, \text{blue})$ has orientation $\circ$
No need to look closely at the boundary: compatibility only needs to be checked at interior crossings.
Q-compatible quadrangulations

No need to look closely at the boundary: compatibility only needs to be checked at interior crossings.

There are always at least two $Q$-compatible quadrangulations: $Q$ itself (not rotated) and $Q$ rotated by $\frac{2\pi}{2n+2}$. 
Q-compatible quadrangulations

The number of Q-compatible quadrangulations depends on Q:

Not a full table. Distinct Q can have the same number.
Let $Q$ be fixed.

**Statement**

Let $Q'$ be a $Q$-compatible quadrangulation. Given any edge $e$ of $Q'$, there exists a unique other edge $e'$ such that $Q - e + e'$ is a $Q$-compatible quadrangulation.

So one can always flip, and without having to choose! Exactly one of the two possible flips is allowed.
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So one can always flip, and without having to choose! Exactly one of the two possible flips is allowed.

This gives a regular graph $\text{St}_Q$ of $Q$-compatible quadrangulations.

This is the graph of edges and vertices of the Stokes polytopes.
Oriented flips of Q-compatible quadrangulations

One can in fact orient the flips in a natural way and get a directed graph $\overrightarrow{\text{St}_Q}$.

Theorem (C.)

The directed graph $\overrightarrow{\text{St}_Q}$ is the Hasse diagram of a poset.
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**Theorem (C.)**

*The directed graph $\overrightarrow{St}_Q$ is the Hasse diagram of a poset.*

Conjecturally, all these posets are lattices.
Tamari lattice as a special case

One finds the Tamari lattices for the following quadrangulations

because in this case

\( Q \)-compatible quadrangulations \( \leftrightarrow \) planar binary trees,
flips \( \leftrightarrow \) rotation (and orientations agree).
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One finds the Tamari lattices for the following quadrangulations because in this case

$Q$-compatible quadrangulations $\leftrightarrow$ planar binary trees,
flips $\leftrightarrow$ rotation (and orientations agree).

It is moreover expected that one can also recover all the Cambrian lattices of type $\mathbb{A}$, from appropriate ("ribbon") quadrangulations.
Here comes the second part
Next step, the other side of the story
or why serpents are never crossing bridges

A Q-compatible quadrangulations (analogs of triangulations)
flip graph, posets, polytopes,
Next step, the other side of the story
or why serpents are never crossing bridges

A  Q-compatible quadrangulations  (analogs of triangulations)
flip graph, posets, polytopes,

C  serpent nests in Q  (analogs of Dyck paths)
graded set with duality
Next step, the other side of the story
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<table>
<thead>
<tr>
<th>A</th>
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</tr>
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<tbody>
<tr>
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This is supposed to go that way: from polytopes one can go to fans and toric varieties. The other side of the story is supposed to be related to the cohomology of the toric variety. Just a motivation, no clear statement so far.
Serpent nests: serpent

Serpent = another word for snake

Fix a background quadrangulation $Q$.

A serpent (in $Q$) is a path joining two square centers (with steps at square centers) and turning either left or right at every step.
Serpent nests: serpent

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A serpent (in $Q$) is a path joining two square centers (with steps at square centers) and turning either left or right at every step.

Never cross a square by going straight to the opposite side!
Serpent nests: definition

A **serpent nest** is a set of serpents + some conditions and modulo some equivalence relation
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**Condition:** no two ends can share both the same square center and the same exit side:

```
+---+---+
|   |   |
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**Equivalence:** at every edge of $Q$, one can change arbitrarily the connections between half-serpents crossing this edge (so one does no longer know which head goes with which tail!)
Serpent nests: properties

In any quadrangulation $Q$, there is only a finite number of serpent nests. This number depends on $Q$. 

**Conjecture**

For any quadrangulation $Q$, the number of serpent nests in $Q$ is equal to the number of $Q$-compatible quadrangulations. This equality can easily be checked for many small examples and for some families.
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For example, in this kind of quadrangulation, serpent nests $\leftrightarrow$ Dyck paths (hence counted by Catalan numbers).
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Duality of serpent nests

Serpent nests form a graded set, by the number of serpents, which runs from 0 (empty serpent nest) to \( n - 1 \) (one serpent by edge).

There exists an involution mapping degree \( k \) to degree \( n - 1 - k \).
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The grading allows to define the $h$-vector. It seems that $h(-1)$ is (up to sign) the number of self-dual serpent nests. One can define an $H$-triangle by counting “simple” serpents.
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One can define an $H$-triangle by counting “simple” serpents.

There should also be refined enumerative relations between $F$-triangle of $Q$-compatible quadrangulations and $H$-triangle of serpent nests in $Q$. 
Bridges and factorisation

A **bridge** is a square that only has two neighbor squares on opposite sides:

![Diagram of a bridge](image.png)
A **bridge** is a square that only has two neighbor squares on opposite sides:

One can show that when there is a bridge,
- the Stoke poset is a product of two Stokes posets,
- the set of serpent nests is also a product.
This last part is because serpents cannot cross the bridges!
Twisting quadrangulations
analog of changing the Coxeter element

Twisting along an edge: operation on quadrangulations

defined by cutting in two parts along one edge,
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analog of changing the Coxeter element

Twisting along an edge: operation on quadrangulations
defined by cutting in two parts along one edge,
taking the mirror image of one part and gluing it back.
Twisting quadrangulations
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Twisting along an edge: operation on quadrangulations
defined by cutting in two parts along one edge, taking the mirror image of one part and gluing it back.

- this does not change the set of serpent nests (easy bijection)
- It is expected that this does not change the flip graph $St_Q$.

But the Stokes poset $\overrightarrow{St}_Q$ does change, like Cambrian lattices for different Coxeter elements.
A nice family of examples

There is a nice family of quadrangulations $L_n$ with $2n$ squares:

(available in $L_8$)

called the Lucas quadrangulations (after Édouard Lucas).
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Their number of serpent nests is given by a Lucas sequence:

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\ell_0 = 0 \quad \ell_1 = 2 \quad \ell_{n+2} = 6\ell_{n+1} + 3\ell_n,
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starting $2, 12, 78, 504, 3258, 21060, 136134, 879984, 5688306, \ldots$
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Maybe the quadrangulations (of even size and with no bridge) with the smallest number of serpent nests.
Open quadrangulations
half-turn symmetry and open serpents

It makes sense, when $Q$ is invariant under half-turn rotation, with an edge sent to itself by the half-turn, to speak about invariant $Q$-compatible quadrangulations.

This should give some type $\mathbb{B}$ objects (flip graphs, posets, polytopes) including the type $\mathbb{B}$ Cambrian lattices.

One can do the same for serpent nests.

These half-turn invariant serpent nests can be considered as “open” and one can glue them back by pairs.
Natural context: cluster categories, quiver representations and also study of derived categories of modules over posets.

- Quadrangulations are **objects** in the derived category of modules over the Tamari lattices.
- The posets $\vec{\text{St}}_Q$ should describe some **morphisms** between these objects.
- Twisting should not change the derived category of modules over $\vec{\text{St}}_Q$.

Moreover, two operads are involved in the story.
To every quadrangulation $Q$, one associates
- a poset and a polytope, called Stokes poset, Stokes polytope
- a graded set with a duality: serpent nests (but no partial order)

For some specific $Q$, one recovers type $\text{A}$ cluster combinatorics.
In general, many new generalized “flip graphs”.
Some things are lost, for example all the nice product formulas.
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Open

- All the Stokes posets are lattices ?
- Same cardinality for $Q$-compatible quad. and serpent nests in $Q$ ?
Conclusion

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Questions (mysteries)

- Is there something like cluster variables in this setting ?
- What would be the missing noncrossing side of the story ?
The End

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