Illinois Journal of Mathematics
Volume 59, Number 3, Fall 2015, Pages 597–621
S 0019-2082

EXPANSIVE ACTIONS OF COUNTABLE AMENABLE GROUPS, HOMOCLINIC PAIRS, AND THE MYHILL PROPERTY

TULLIO CECCHERINI-SILBERSTEIN AND MICHEL COORNAERT

To Étienne Ghys on his 60th anniversary

ABSTRACT. Let X be a compact metrizable space equipped with a continuous action of a countable amenable group G. Suppose that the dynamical system (X,G) is expansive and is the quotient by a uniformly bounded-to-one factor map of a strongly irreducible subshift. Let $\tau\colon X \to X$ be a continuous map commuting with the action of G. We prove that if there is no pair of distinct G-homoclinic points in X having the same image under τ then τ is surjective.

1. Introduction

Let G be a countable group and A a finite set. Consider the set A^G consisting of all maps $u\colon G \to A$. This set is called the set of *configurations* over the group G and the *alphabet* A. Equip A^G with its *prodiscrete topology*, that is, the topology of pointwise convergence. The *shift action* of G on A^G is the continuous action defined by $gu(h) := u(g^{-1}h)$ for all $g,h \in G$ and $u \in A^G$. A *cellular automaton* over A^G is a continuous map $\tau\colon A^G \to A^G$ that is G-equivariant, that is, satisfies $\tau(gu) = g\tau(u)$ for all $g \in G$ and $u \in A^G$. Two configurations $u,v \in A^G$ are said to be *almost equal* if they coincide outside of a finite subset of G. A cellular automaton $\tau\colon A^G \to A^G$ is said to be *pre-injective* if there exist no distinct configurations $u,v \in A^G$ that are almost equal and satisfy $\tau(u) = \tau(v)$. The celebrated *Garden of Eden theorem*, originally established by Moore and Myhill in the early 1960s, states that a cellular automaton over the group $\mathbb{Z}$ of integers is surjective if and only if it is pre-injective. Actually, the implication surjective $\Rightarrow$ pre-injective was first

Received October 27, 2015; received in final form December 22, 2015.
2010 *Mathematics Subject Classification.* 37D20, 37B40, 37B10, 43A07.

established by Moore [23] and, shortly after, Myhill [24] proved the converse implication. The Moore-Myhill Garden of Eden theorem was extended to all amenable groups in [10]. It follows from a result of Bartholdi [3] that if a group G is non-amenable then there exist cellular automata over G that are surjective but not pre-injective. Thus, the Garden of Eden theorem yields a characterization of amenability for groups.

The goal of the present paper is to extend the Myhill implication in the Garden of Eden theorem to certain dynamical systems (X,G), consisting of a compact metrizable space X equipped with a continuous action of a countable amenable group G. Our motivations come from Gromov (cf. [20, Section 8.H]) who mentions the possibility of extending the Garden of Eden theorem to a "suitable class of hyperbolic dynamical systems".

Before stating our main result, let us briefly recall some additional definitions (see Section 2 for more details). A closed G-invariant subset of A^G is called a *subshift*. A subshift $\Sigma \subset A^G$ is said to be *strongly irreducible* if there is a finite subset $\Delta \subset G$ satisfying the following property: if Ω_1 and Ω_2 are finite subsets of G such that there exists no element $g \in \Delta$ such that the right-translate of Ω_1 by g meets Ω_2, then, given any two configurations $u_1, u_2 \in \Sigma$, there exists a configuration $u \in \Sigma$ which coincides with u_1 on Ω_1 and with u_2 on Ω_2.

Let (X,G) be a dynamical system consisting of a compact metrizable space X equipped with a continuous action of the group G. Two points $x, y \in X$ are said to be *homoclinic* with respect to the action of G on X, or more briefly *G-homoclinic*, if for every $\varepsilon > 0$ there exists a finite subset $F \subset G$ such that $d(gx, gy) < \varepsilon$ for all $g \in G \setminus F$ (here d denotes any metric on X that is compatible with the topology). We say that a continuous G-equivariant map $\tau \colon X \to X$ is *pre-injective* with respect to the action of G if there is no pair of distinct G-homoclinic points in X having the same image under τ. When $X = A^G$ and G acts on X by the shift, this definition is equivalent to the one given above (see Proposition 2.5). We say that the dynamical system (X,G) has the *Myhill property* if every G-equivariant continuous map $\tau \colon X \to X$ that is pre-injective with respect to the action of G is surjective.

Our main result is the following.

THEOREM 1.1. *Let X be a compact metrizable space equipped with a continuous action of a countable amenable group G. Suppose that the dynamical system (X,G) is expansive and that there exist a finite set A, a strongly irreducible subshift $\Sigma \subset A^G$, and a uniformly bounded-to-one factor map $\theta \colon \Sigma \to X$. Then the dynamical system (X,G) has the Myhill property.*

As the shift action on every subshift $\Sigma \subset A^G$ is expansive, we deduce from Theorem 1.1 (by taking $\theta := \mathrm{Id}_\Sigma$, the identity map on Σ) that if G is a countable amenable group and A is a finite set, then every strongly irreducible subshift $\Sigma \subset A^G$ has the Myhill property. This last result had been already established by the authors [6, Theorem 1.1]. In the particular case when $\Sigma = A^G$ is the full subshift, it yields the Myhill implication in the Garden of Eden theorem for cellular automata over amenable groups established in [10].

Theorem 1.1 had been previously obtained by the authors [8, Theorem 1.1] when $G = \mathbb{Z}$ and $\Sigma \subset A^{\mathbb{Z}}$ is a topologically mixing subshift of finite type. Actually, it is well known that a subshift of finite type $\Sigma \subset A^{\mathbb{Z}}$ is strongly irreducible if and only if it is topologically mixing. On the other hand, there exist strongly irreducible subshifts $\Sigma \subset A^{\mathbb{Z}}$ that are not of finite type and even not sofic (see, e.g., [7]), so that the above result for $G = \mathbb{Z}$ is stronger than the one in [8].

According to Gromov [19, Section 5], a dynamical system (X, G) is *hyperbolic* (or *finitely presented* [16]) if it is expansive and a factor of some subshift of finite type. Thus, if the dynamical system (X, G) satisfies the hypotheses of Theorem 1.1 with Σ of finite type, then (X, G) is hyperbolic in the sense of Gromov. However, as already mentioned above, there are strongly irreducible subshifts over $\mathbb{Z}$ that are not sofic and hence not finitely presented. Consequently, there are dynamical systems (X, G) satisfying all the hypotheses of Theorem 1.1 without being hyperbolic in the sense of Gromov.

Note also that there exist dynamical systems (X, G) satisfying all the hypotheses of Theorem 1.1 that do not have the *Moore property*, that is, admitting continuous surjective G-equivariant maps $\tau \colon X \to X$ that are not pre-injective (cf. [8]). In the case $G = \mathbb{Z}$, an example of such a dynamical system is provided by the even subshift $X \subset \{0, 1\}^{\mathbb{Z}}$ (see [14, Section 3]).

Following a terminology introduced by Gottschalk [17] (cf. [20]), let us say that a dynamical system (X, G) is *surjunctive* if every injective G-equivariant continuous map $\tau \colon X \to X$ is surjective.

COROLLARY 1.2. *Every dynamical system (X, G) satisfying the hypotheses of Theorem 1.1 is surjunctive.*

Proof. Injectivity trivially implies pre-injectivity. □

The paper is organized as follows. Section 2 contains basic definitions and preliminary results. The proof of Theorem 1.1 is given in Section 3. It relies on an entropic argument. In Section 4, we prove that, in a non-trivial dynamical system (X, G) that is the quotient by a finite-to-one factor map of a strongly irreducible subshift, all G-homoclinicity classes are infinite (Corollary 4.2). This applies in particular to any non-trivial dynamical system satisfying the hypotheses of Theorem 1.1 but requires neither expansiveness of the system nor amenability of the acting group. On the other hand, extending a result previously obtained by Schmidt [28] in the case $G = \mathbb{Z}^d$, we show that if G is a countable amenable group and (X, G) is a dynamical system with positive topological entropy that is the quotient by a uniformly bounded-to-one factor map of a subshift of finite type, then, for every integer $n \geq 1$, there is a G-homoclinicity class in X containing more than n points (Corollary 4.5). Furthermore, generalizing a result also previously obtained by Schmidt [28] for $G = \mathbb{Z}^d$, we prove that if A is a finite set, G a countable residually finite amenable group, and $\Sigma \subset A^G$ a subshift of finite type with zero topological entropy whose periodic configurations are dense, then every G-homoclinicity class in Σ is trivial. In Section 5, we prove the surjunctivity of expansive dynamical systems containing a dense set of periodic points. This last result is well known for subshifts and does not require amenability of the acting group. The final section contains a description of some examples of expansive dynamical systems that show the importance of the hypotheses in the above results.

2. Background and notation

In this section, we set up notation and collect basic facts that will be used in the sequel. Some proofs are given for convenience.

2.1. Dynamical systems. The cardinality of a set X is denoted $\mathrm{card}(X)$. A set X is *countable* if $\mathrm{card}(X) = \mathrm{card}(\mathbb{N})$. Here $\mathbb{N}$ denotes the set of non-negative integers. An action of a group G on a set X is a map $\alpha\colon G \times X \to X$ satisfying $\alpha(g_1, \alpha(g_2, x)) = \alpha(g_1 g_2, x)$ and $\alpha(1_G, x) = x$ for all $g_1, g_2 \in G$ and $x \in X$, where 1_G denotes the identity element of G. When the action α is clear from the context, we shall write gx instead of $\alpha(g, x)$. If a group G acts on two sets X and Y, a map $\varphi\colon X \to Y$ is called G-*equivariant* if one has $\varphi(gx) = g\varphi(x)$ for all $g \in G$ and $x \in X$. An action of a group G on a topological space X is said to be *continuous* if the permutation of X given by $x \mapsto gx$ is continuous for each $g \in G$.

Throughout this paper, by a *dynamical system*, we shall mean a triple (X, G, α), where X is a compact metrizable space, G is a countable group, and α is a continuous action of G on X. If there is no risk of confusion about the action, we shall write (X, G), or even sometimes simply X, instead of (X, G, α). We shall denote by d a metric on X that is compatible with the topology.

Given a dynamical system (X, G), the *orbit* of a point $x \in X$ is the set $\{gx : g \in G\} \subset X$. The point x is called *periodic* if its orbit is finite. The set $\mathrm{Per}(X, G) \subset X$ consisting of all periodic points of the dynamical system (X, G) satisfies

$$\mathrm{Per}(X, G) = \bigcup_H \mathrm{Fix}(H), \tag{2.1}$$

where H runs over all finite-index subgroups of G and $\mathrm{Fix}(H)$ is the closed subset of X consisting of all the points in X that are fixed by H.

A subset $Y \subset X$ is said to be *invariant* if the orbit of every point of Y is contained in Y. If $Y \subset X$ is an invariant subset then the action of G on X induces, by restriction, a continuous action of G on Y.

One says that the dynamical system (X, G) is *expansive* if there exists a constant $\delta > 0$ such that, for every pair of distinct points $x, y \in X$, there exists an element $g = g(x, y) \in G$ such that $d(gx, gy) \geq \delta$. Such a constant δ is called an *expansiveness constant* for (X, G, d). The fact that (X, G) is expansive or not does not depend on the choice of the metric d. Actually, the dynamical system (X, G) is expansive if and only if there is a neighborhood $W \subset X \times X$ of the diagonal such that, for every pair of distinct points $x, y \in X$, there exists an element $g = g(x, y) \in G$ such that $(gx, gy) \notin W$. Such a set W is then called an *expansiveness set* for (X, G).

One says that the dynamical system (X, G) is *topologically mixing* if, for any pair of non-empty open subsets U and V of X, there exists a finite subset $F \subset G$ such that $U \cap gV \neq \varnothing$ for all $g \in G \setminus F$.

Suppose that the group G acts continuously on two compact metrizable spaces X and $\widetilde{X}$.

One says that the dynamical systems (X, G) and $(\widetilde{X}, G)$ are *topologically conjugate* if there exists a G-equivariant homeomorphism $h \colon \widetilde{X} \to X$.

One says that the dynamical system (X, G) is a *factor* of the dynamical system $(\widetilde{X}, G)$ if there exists a G-equivariant continuous surjective map $\theta \colon \widetilde{X} \to X$. Such a map θ is then called a *factor map.* A factor map $\theta \colon \widetilde{X} \to X$ is said to be *finite-to-one* if the pre-image set $\theta^{-1}(x)$ is finite for each $x \in X$. A finite-to-one factor map is said to be *uniformly bounded-to-one* if there is an integer $K \geq 1$ such that $\operatorname{card}(\theta^{-1}(x)) \leq K$ for all $x \in X$.

2.2. Homoclinicity. Let X be a compact metrizable space equipped with a continuous action of a countable group G. Let d be a metric on X compatible with the topology. Two points $x, y \in X$ are called *homoclinic* with respect to the action of G, or more briefly *G-homoclinic*, if for every $\varepsilon > 0$, there is a finite subset $F \subset G$ such that $d(gx, gy) < \varepsilon$ for all $g \in G \setminus F$. Homoclinicity defines an equivalence relation on X. By compactness of X, this equivalence relation is independent of the choice of the metric d. Its equivalence classes are called the *G-homoclinicity classes* of X.

Proposition 2.1. *Let $\widetilde{X}$ and X be compact metrizable spaces, each equipped with a continuous action of a countable group G. Suppose that the dynamical system (X, G) is a factor of the dynamical system $(\widetilde{X}, G)$ and let $\theta \colon \widetilde{X} \to X$ be a factor map. Let $\tilde{x}$ and $\tilde{y}$ be points in $\widetilde{X}$ that are G-homoclinic. Then the points $x := \theta(\tilde{x})$ and $y := \theta(\tilde{y})$ are G-homoclinic.*

Proof. Let $\tilde{d}$ (resp. d) be a metric on $\widetilde{X}$ (resp. X) that is compatible with the topology.

Let $\varepsilon > 0$. By compactness, θ is uniformly continuous. Therefore, there exists $\eta > 0$ such that

$$\tilde{d}(\tilde{x}_1, \tilde{x}_2) < \eta \quad \Rightarrow \quad d\big(\theta(\tilde{x}_1), \theta(\tilde{x}_2)\big) < \varepsilon \tag{2.2}$$

for all $\tilde{x}_1, \tilde{x}_2 \in \widetilde{X}$. Since the points $\tilde{x}$ and $\tilde{y}$ are G-homoclinic, there is a finite subset $F \subset G$ such that $\tilde{d}(g\tilde{x}, g\tilde{y}) < \eta$ for all $g \in G \setminus F$. It follows that, for all $g \in G \setminus F$,

$$\begin{aligned} d(gx, gy) &= d\big(g\theta(\tilde{x}), g\theta(\tilde{y})\big) \\ &= d\big(\theta(g\tilde{x}), \theta(g\tilde{y})\big) \quad \text{(since } \theta \text{ is } G\text{-equivariant)} \\ &< \varepsilon \quad \text{(by (2.2))}. \end{aligned}$$

This shows that the points x and y are G-homoclinic. □

2.3. Amenability. A countable group G is called *amenable* if there exists a sequence $(F_n)_{n\in\mathbb{N}}$ of non-empty finite subsets of G satisfying, for all $g \in G$,

$$\lim_{n\to\infty} \frac{\operatorname{card}(F_n \setminus F_n g)}{\operatorname{card}(F_n)} = 0. \tag{2.3}$$

Such a sequence $(F_n)_{n\in\mathbb{N}}$ is called a *Følner sequence* for G.

There are many other equivalent definitions of amenability in the literature (see, e.g., the monographs [18], [26], [5] and the references therein).

All locally finite groups, all solvable groups (and therefore all Abelian groups), and all finitely generated groups of subexponential growth are amenable. The free group of rank 2 provides an example of a non-amenable group. As the class of amenable groups is closed under taking subgroups, it follows that if a group G contains a non-Abelian free subgroup then G is not amenable.

We shall use the following well known results about Følner sequences.

Proposition 2.2. *Let G be a countable amenable group and let $(F_n)_{n\in\mathbb{N}}$ be a Følner sequence for G. Let E be a non-empty finite subset of G. Then the following hold*:

(i) *one has*

$$\lim_{n\to\infty} \operatorname{card}(F_n) = \infty; \tag{2.4}$$

(ii) *the sequence $(EF_n)_{n\in\mathbb{N}}$ is a Følner sequence for G*;

(iii) *one has*

$$\lim_{n\to\infty} \frac{\operatorname{card}(F_n E \setminus F_n)}{\operatorname{card}(F_n)} = 0; \tag{2.5}$$

(iv) *one has*

$$\lim_{n\to\infty} \frac{\operatorname{card}(F_n E)}{\operatorname{card}(F_n)} = 1. \tag{2.6}$$

Proof. Let $K \in \mathbb{N}$. Since G is infinite, we can find a finite subset $R \subset G$ with $\operatorname{card}(R) = K^2$. It follows from (2.3) that, for each $g \in R$, there exists $N(g) \in \mathbb{N}$ such that $\operatorname{card}(F_n \setminus F_n g) < \operatorname{card}(F_n)$ for all $n \geq N(g)$. This implies that F_n meets $F_n g$ and hence $g \in F_n^{-1} F_n$ for all $n \geq N(g)$. If we take $N := \max_{g \in R} N(g)$, we then get $R \subset F_n^{-1} F_n$ for all $n \geq N$. We deduce that

$$K^2 = \operatorname{card}(R) \leq \operatorname{card}\big(F_n^{-1} F_n\big) \leq \operatorname{card}\big(F_n^{-1}\big) \operatorname{card}(F_n) = \big(\operatorname{card}(F_n)\big)^2$$

and hence $\operatorname{card}(F_n) \geq K$ for all $n \geq N$. This shows (i).

For all $g \in G$, we have the inclusion

$$EF_n \setminus EF_n g \subset E(F_n \setminus F_n g).$$

This implies

$$\operatorname{card}(EF_n \setminus EF_n g) \leq \operatorname{card}(E) \operatorname{card}(F_n \setminus F_n g).$$

As $\operatorname{card}(EF_n) \geq \operatorname{card}(F_n)$, this gives us

$$\frac{\operatorname{card}(EF_n \setminus EF_n g)}{\operatorname{card}(EF_n)} \leq \operatorname{card}(E) \frac{\operatorname{card}(F_n \setminus F_n g)}{\operatorname{card}(F_n)}$$

and hence

$$\lim_{n \to \infty} \frac{\operatorname{card}(EF_n \setminus EF_n g)}{\operatorname{card}(EF_n)} = 0$$

by using (2.3). This shows (ii).

To prove (iii), we first observe that

$$F_n E \setminus F_n = \bigcup_{e \in E} (F_n e \setminus F_n)$$

so that

$$\begin{aligned}
\operatorname{card}(F_n E \setminus F_n) &= \operatorname{card}\Big(\bigcup_{e \in E} (F_n e \setminus F_n)\Big) \\
&\leq \sum_{e \in E} \operatorname{card}(F_n e \setminus F_n) \\
&= \sum_{e \in E} \operatorname{card}(F_n \setminus F_n e) \quad \big(\text{since } \operatorname{card}(F_n) = \operatorname{card}(F_n e)\big)
\end{aligned}$$

and hence

$$\frac{\operatorname{card}(F_n E \setminus F_n)}{\operatorname{card}(F_n)} \leq \sum_{e \in E} \frac{\operatorname{card}(F_n \setminus F_n e)}{\operatorname{card}(F_n)}. \tag{2.7}$$

This shows (iii) since the right-hand side of (2.7) tends to 0 as $n \to \infty$ by (2.3).

To prove (iv), we observe that

$$F_n E \subset F_n \cup (F_n E \setminus F_n)$$

so that

$$\operatorname{card}(F_n E) \leq \operatorname{card}\big(F_n \cup (F_n E \setminus F_n)\big) \leq \operatorname{card}(F_n) + \operatorname{card}(F_n E \setminus F_n).$$

As $\operatorname{card}(F_n E) \geq \operatorname{card}(F_n)$, we deduce that

$$1 \leq \frac{\operatorname{card}(F_n E)}{\operatorname{card}(F_n)} \leq 1 + \frac{\operatorname{card}(F_n E \setminus F_n)}{\operatorname{card}(F_n)}.$$

This gives us (iv) by using (iii). □

2.4. Topological entropy. Consider a dynamical system (X, G), where X is a compact metrizable space and G is a countable amenable group acting continuously on X.

Let $\mathcal{U} = (U_i)_{i \in I}$ be an open cover of X. The *cardinality* of $\mathcal{U}$ is by definition the cardinality of its index set I. One says that an open cover $\mathcal{V} = (V_j)_{j \in J}$ is a *subcover* of $\mathcal{U}$ if $J \subset I$ and $V_j = U_j$ for all $j \in J$. Since X is compact, $\mathcal{U}$ admits a subcover with finite cardinality. We denote by $N(\mathcal{U})$ the smallest integer $n \geq 0$ such that $\mathcal{U}$ admits a subcover with cardinality n. For $g \in G$, we define the open cover $g\mathcal{U}$ by $g\mathcal{U} := (gU_i)_{i \in I}$.

Let $\mathcal{V} = (V_j)_{j \in J}$ be an open cover of X. The *join* of the open covers $\mathcal{U}$ and $\mathcal{V}$ is the open cover $\mathcal{U} \vee \mathcal{V}$ of x defined by $\mathcal{U} \vee \mathcal{V} := (U_i \cap V_j)_{(i,j) \in I \times J}$.

Given an open cover $\mathcal{U}$ of X and a non-empty finite subset $F \subset G$, we define the open cover $\mathcal{U}_F$ by

$$\mathcal{U}_F := \bigvee_{g \in F} g^{-1}\mathcal{U}.$$

Now let $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ be a Følner sequence for G. It follows from the Ornstein–Weiss lemma as stated in [22, Theorem 6.1] (see also [9] and the references therein) that the limit

$$h(\mathcal{U}, X, G) := \lim_{n \to \infty} \frac{\log N(\mathcal{U}_{F_n})}{\operatorname{card}(F_n)}$$

exists, is finite, and does not depend on the choice of the Følner sequence $\mathcal{F}$ for G.

The *topological entropy* of the dynamical system (X, G) is the quantity $0 \leq h_{\text{top}}(X, G) \leq \infty$ given by

$$h_{\text{top}}(X, G) := \sup_{\mathcal{U}} h(\mathcal{U}, X, G),$$

where $\mathcal{U}$ runs over all open covers of X.

The above definition of topological entropy was introduced by Adler, Konheim, and McAndrew [1] in the case when $G = \mathbb{Z}$. Let us now briefly review the metric approach to topological entropy that was developed independently by Bowen [4] and Dinaburg [13] for $G = \mathbb{Z}$.

Let d be a metric on X compatible with the topology.

Given a non-empty finite subset $F \subset G$, we define the metric d_F on X by

$$d_F(x, y) := \max_{g \in F} d(gx, gy) \quad \text{for all } x, y \in X. \tag{2.8}$$

The metric d_F is also compatible with the topology on X. Given a real number $\varepsilon > 0$, one says that a subset $Z \subset X$ is an $(F, \varepsilon, X, G, d)$-*spanning subset* of X if for every $x \in X$, there exists $z \in Z$ such that $d_F(x, z) < \varepsilon$. By compactness, X always contains a finite $(F, \varepsilon, X, G, d)$-spanning subset. Let $S(F, \varepsilon, X, G, d)$ denote the minimal cardinality of an $(F, \varepsilon, X, G, d)$-spanning subset $Z \subset X$.

Let $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ be a Følner sequence for G. We define $h(\mathcal{F}, \varepsilon, X, G, d)$ by

$$h(\mathcal{F}, \varepsilon, X, G, d) := \limsup_{n \to \infty} \frac{\log S(F_n, \varepsilon, X, G, d)}{\operatorname{card}(F_n)}. \tag{2.9}$$

Observe that the map $\varepsilon \mapsto h(\mathcal{F}, \varepsilon, X, G, d)$ is non-increasing. This implies that

$$\sup_{\varepsilon > 0} h(\mathcal{F}, \varepsilon, X, G, d) = \lim_{\varepsilon \to 0} h(\mathcal{F}, \varepsilon, X, G, d).$$

PROPOSITION 2.3. *With the notation above, one has*

$$h_{\text{top}}(X, G) = \sup_{\varepsilon > 0} h(\mathcal{F}, \varepsilon, X, G, d) = \lim_{\varepsilon \to 0} h(\mathcal{F}, \varepsilon, X, G, d). \tag{2.10}$$

Proof. To simplify notation, let us set

$$H = \sup_{\varepsilon > 0} h(\mathcal{F}, \varepsilon, X, G, d).$$

Let $\mathcal{U}$ be an open cover of X. Choose some Lebesgue number $\lambda > 0$ for $\mathcal{U}$ with respect to the metric d. We recall that this means that every open d-ball of X with radius λ is contained in some member of $\mathcal{U}$. Observe that, for every non-empty finite subset $F \subset G$, the open cover $\mathcal{U}_F$ admits λ as a Lebesgue number with respect to the metric d_F. We deduce that

$$N(\mathcal{U}_F) \leq S(F, \lambda, X, G, d).$$

It follows that

$$h(\mathcal{U}, X, G) \leq h(\mathcal{F}, \lambda, X, G, d) \leq H$$

and hence

$$h_{\text{top}}(X, G) = \sup_{\mathcal{U}} h(\mathcal{U}, X, G) \leq H.$$

It remains only to prove that $H \leq h_{\text{top}}(X, G)$. Let $\varepsilon > 0$ and consider the open cover $\mathcal{U}$ of X consisting of all open balls of radius ε. Then, for every non-empty finite subset $F \subset G$, we clearly have

$$S(F, \varepsilon, X, G, d) \leq N(\mathcal{U}_F).$$

This gives us

$$h(\mathcal{F}, \varepsilon, X, G, d) \leq h(\mathcal{U}, X, G) \leq h_{\text{top}}(X, G)$$

and hence

$$H = \sup_{\varepsilon > 0} h(\mathcal{F}, \varepsilon, X, G, d) \leq h_{\text{top}}(X, G).$$

This completes the proof of (2.10). $\square$

One of the key properties of topological entropy is that it never increases when taking factors. More precisely, we have the following.

Proposition 2.4. *Let G be a countable amenable group and let X (resp. $\widetilde{X}$) be a compact metrizable space equipped with a continuous action of G. Suppose that the dynamical system (X,G) is a factor of the dynamical system $(\widetilde{X},G)$. Then one has $h_{\text{top}}(X,G) \leq h_{\text{top}}(\widetilde{X},G)$.*

Proof. Let $\theta\colon \widetilde{X} \to X$ be a factor map and let $\mathcal{F} = (F_n)_{n\in\mathbb{N}}$ be a Følner sequence for G. Let $\varepsilon > 0$ and take $\tilde{d}$, d, and η as in the proof of Proposition 2.1.

Suppose that F is a non-empty finite subset of G. If $\widetilde{Z}$ is an $(F,\eta,\widetilde{X},G,\tilde{d})$-spanning subset of $\widetilde{X}$, then $Z = \theta(\widetilde{Z})$ is an (F,ε,X,G,d)-spanning subset of X. It follows that

$$S(F,\varepsilon,X,G,d) \leq S(F,\eta,\widetilde{X},G,\tilde{d}).$$

This gives us

$$S(\mathcal{F},\varepsilon,X,G,d) \leq S(\mathcal{F},\eta,\widetilde{X},G,\tilde{d}) \leq h_{\text{top}}(\widetilde{X},G)$$

and hence

$$h_{\text{top}}(X,G) = \sup_{\varepsilon>0} S(\mathcal{F},\varepsilon,X,G,d) \leq h_{\text{top}}(\widetilde{X},G). \qquad \square$$

2.5. Shifts and subshifts. Let G be a countable group and let A be a finite set. We equip A^G with its prodiscrete topology. This is the product topology obtained by taking the discrete topology on each factor A of A^G. The space A^G, being the product of countably many finite discrete spaces, is metrizable, compact, and totally disconnected. It is homeomorphic to the Cantor set as soon as A has more than one element.

Let us choose a non-decreasing sequence $(R_n)_{n\in\mathbb{N}}$ of finite subsets of G such that $R_0 = \varnothing$, $R_1 = \{1_G\}$, and $G = \bigcup_{n\in\mathbb{N}} R_n$. Then the metric ρ on A^G, defined by

$$\rho(u,v) := \frac{1}{k}, \tag{2.11}$$

where

$$k := \sup\bigl\{n \geq 1 \text{ such that } u(g) = v(g) \text{ for all } g \in R_{n-1}\bigr\}$$

(with the usual convention $1/\infty = 0$), is compatible with the topology.

The G-shift action on A^G is expansive and admits $\delta = 1$ as an expansiveness constant with respect to the metric ρ. Actually, the open set

$$W := \bigl\{(u,v) \in A^G \times A^G : u(1_G) \neq v(1_G)\bigr\} \subset A^G \times A^G$$

is an expansiveness set for (A^G,G).

Proposition 2.5. *Let G be a countable group and let A be a finite set. Then two configurations $u,v \in A^G$ are G-homoclinic if and only if they are almost equal.*

Proof. Suppose first that u and v are almost equal. Then there exists a finite subset $F \subset G$ such that $u(g) = v(g)$ for all $g \in G \setminus F$. Let $\varepsilon > 0$ and choose a positive integer k such that $1/(k+1) < \varepsilon$. If $g \in G$ is such that $g^{-1}F$ does not meet the set R_k, that is, $g \notin FR_k^{-1}$, then $u(g) = v(g)$. As the set FR_k^{-1} is finite, this shows that u and v are G-homoclinic.

Conversely, suppose that u and v are G-homoclinic. Then we can find a finite subset F of G such that $\rho(gu, gv) \leq 1/2$ for all $g \in G \setminus F$. This implies that u and v coincide outside of F^{-1}. Consequently, the configurations u and v are almost equal. □

A G-invariant closed subset $\Sigma \subset A^G$ is called a *subshift*.

Let Δ be a finite subset of G. Given subsets Ω_1 and Ω_2 of G, one says that Ω_1 is Δ-*apart* from Ω_2 if one has $\Omega_1 \Delta \cap \Omega_2 = \varnothing$, that is, there is no element $g \in \Delta$ such that the right-translate by g of Ω_1 meets Ω_2. A subshift $X \subset A^G$ is called Δ-*irreducible* if it satisfies the following condition: given any two finite subsets $\Omega_1, \Omega_2 \subset G$ such that Ω_1 is Δ-apart from Ω_2 and any two configurations $u_1, u_2 \in \Sigma$, there exists a configuration $u \in \Sigma$ which coincides with u_1 on Ω_1 and with u_2 on Ω_2. A subshift $\Sigma \subset A^G$ is called *strongly irreducible* if there exists a finite subset $\Delta \subset G$ such that Σ is Δ-irreducible.

We shall use the following result.

Proposition 2.6. *Let G be a countable group and let A be a finite set. Let Δ be a finite subset of G and let $\Sigma \subset A^G$ be a Δ-irreducible subshift. Suppose that Ω_1 and Ω_2 are (possibly infinite) subsets of G such that Ω_1 is Δ-apart from Ω_2. Then, given any two configurations u_1 and u_2 in Σ, there is a configuration u in Σ which coincides with u_1 on Ω_1 and with u_2 on Ω_2.*

Proof. This is a particular case of [6, Lemma 4.6]. □

A subshift $\Sigma \subset A^G$ is said to be of *finite type* if there exist a finite subset $\Omega \subset G$ and a subset $\mathcal{P} \subset A^\Omega$ such that

$$\Sigma = \left\{ u \in A^G : (gu)|_\Omega \in \mathcal{P} \text{ for all } g \in G \right\}. \tag{2.12}$$

A subshift $\Sigma \subset A^G$ is called *sofic* if there exist a finite set B and a subshift of finite type $\Psi \subset B^G$ such that Σ is a factor of Ψ.

2.6. Topological entropy of subshifts. Let G be a countable group and A a finite set. Given a subset $F \subset G$, let $\pi_F \colon A^G \to A^F$ denote the canonical projection map, i.e., the map defined by $\pi_F(u) = u|_F$ for all $u \in A^G$, where $u|_F \in A^F$ denotes the restriction of u to F.

Proposition 2.7. *Let G be a countable amenable group and A a finite set. Let $\Sigma \subset A^G$ be a subshift. Let $\mathcal{F} = (F_n)_{n \in \mathbb{N}}$ be a Følner sequence for G. Then the topological entropy of Σ satisfies*

$$h_{\text{top}}(\Sigma, G) = \lim_{n \to \infty} \frac{\log \operatorname{card}(\pi_{F_n^{-1}}(\Sigma))}{\operatorname{card}(F_n)}. \tag{2.13}$$

Proof. We first observe that the existence of the limit in the right-hand side of (2.13) is again an immediate consequence of the Ornstein–Weiss lemma that was mentioned above. Indeed, let $\mathcal{P}_{\text{fin}}(G)$ denote the set of all non-empty finite subsets of G. Then, one immediately checks that the map $\eta\colon \mathcal{P}_{\text{fin}}(G) \to \mathbb{N}$ defined by $\eta(F) := \text{card}(\pi_{F^{-1}}(\Sigma))$ is right-invariant (i.e., $\eta(Fg) = \eta(F)$ for all $F \in \mathcal{P}_{\text{fin}}(G)$ and $g \in G$) non-decreasing (i.e., $\eta(F) \leq \eta(F')$ for all $F, F' \in \mathcal{P}_{\text{fin}}(G)$ with $F \subset F'$), and submultiplicative in the sense that $\eta(F \cup F') \leq \eta(F)\eta(F')$ for all disjoint $F, F' \in \mathcal{P}_{\text{fin}}(G)$. Thus, the map $F \mapsto \log \eta(F)$ satisfies the hypotheses needed to apply the Ornstein–Weiss lemma [22, Theorem 6.1]. Consequently, the limit

$$\lim_{n\to\infty} \frac{\log \text{card}(\pi_{F_n^{-1}}(\Sigma))}{\text{card}(F_n)}$$

exists and does not depend on the choice of the Følner sequence $\mathcal{F}$.

Now let us assume that the Følner sequence $\mathcal{F}$ is *symmetric*, that is, satisfies $F_n = F_n^{-1}$ for all $n \in \mathbb{N}$ (the existence of such a Følner sequence follows from [25, Corollary 5.3]). Note that this implies in particular that

$$\lim_{n\to\infty} \frac{\log \text{card}(\pi_{F_n^{-1}}(\Sigma))}{\text{card}(F_n)} = \lim_{n\to\infty} \frac{\log \text{card}(\pi_{F_n}(\Sigma))}{\text{card}(F_n)}.$$

Equip Σ with the metric ρ defined by (2.11). Let F be a non-empty finite subset of G and fix an integer $k \geq 1$. It immediately follows from (2.11) and (2.8) that if two configurations $u, v \in \Sigma$ coincide on $F^{-1}R_k$ then $\rho(u,v) < 1/k$ while $\rho(u,v) \geq 1/k$ if u and v do not coincide on $F^{-1}R_k$. This gives us

$$S(F, 1/k, \Sigma, G, \rho) = \text{card}\big(\pi_{F^{-1}R_k}(\Sigma)\big)$$

and hence

$$\begin{aligned}
(2.14)\quad h(\mathcal{F}, 1/k, \Sigma, G, \rho) &= \limsup_{n\to\infty} \frac{\log \text{card}(\pi_{F_n^{-1}R_k}(\Sigma))}{\text{card}(F_n)} \quad \text{(by (2.9))}\\
&= \limsup_{n\to\infty} \frac{\log \text{card}(\pi_{F_nR_k}(\Sigma))}{\text{card}(F_n)}\\
&\qquad \text{(since } \mathcal{F} \text{ is symmetric).}
\end{aligned}$$

Now observe that $1_G \in R_k$ so that $F_n \subset F_nR_k$. We deduce that

$$\text{card}\big(\pi_{F_n}(\Sigma)\big) \leq \text{card}\big(\pi_{F_nR_k}(\Sigma)\big) \leq \text{card}\big(\pi_{F_n}(\Sigma)\big) \times \text{card}\big(A^{F_nR_k \setminus F_n}\big).$$

This yields

$$\begin{aligned}
\frac{\log \text{card}(\pi_{F_n}(\Sigma))}{\text{card}(F_n)} &\leq \frac{\log \text{card}(\pi_{F_nR_k}(\Sigma))}{\text{card}(F_n)}\\
&\leq \frac{\log \text{card}(\pi_{F_n}(\Sigma))}{\text{card}(F_n)} + \frac{\text{card}(F_nR_k \setminus F_n)}{\text{card}(F_n)}
\end{aligned}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$, this gives us

$$\lim_{n\to\infty} \frac{\log \operatorname{card}(\pi_{F_n R_k}(\Sigma))}{\operatorname{card}(F_n)} = \lim_{n\to\infty} \frac{\log \operatorname{card}(\pi_{F_n}(\Sigma))}{\operatorname{card}(F_n)} \tag{2.15}$$

since

$$\lim_{n\to\infty} \frac{\operatorname{card}(F_n R_k \setminus F_n)}{\operatorname{card}(F_n)} = 0$$

by (2.5).

By applying (2.14), we then get

$$h(\mathcal{F}, 1/k, \Sigma, G, \rho) = \lim_{n\to\infty} \frac{\log \operatorname{card}(\pi_{F_n}(\Sigma))}{\operatorname{card}(F_n)}$$

for all $k \geq 1$. As

$$h_{\text{top}}(\Sigma, G) = \lim_{k\to\infty} h(\mathcal{F}, 1/k, \Sigma, G, \rho)$$

by Proposition 2.3, this shows (2.13). □

Remark 2.8. If the Følner sequence $\mathcal{F}$ is symmetric, then formula (2.13) shows that $h_{\text{top}}(\Sigma, G)$ coincides with the entropy $\operatorname{ent}_{\mathcal{F}}(\Sigma)$ defined in [6].

We shall also need the following result about topological entropy of strongly irreducible subshifts.

Proposition 2.9. *Let G be a countable amenable group, A a finite set, and $\Sigma \subset A^G$ a strongly irreducible subshift. Suppose that $\Psi \subset A^G$ is a subshift that is strictly contained in Σ. Then one has $h_{\text{top}}(\Psi, G) < h_{\text{top}}(\Sigma, G)$.*

Proof. See [6, Proposition 4.2]. □

3. Proof of the main result

In this section, we present the proof of Theorem 1.1. Recall that we are given a countable amenable group G acting continuously on a compact metrizable space X, a finite set A, a strongly irreducible subshift $\Sigma \subset A^G$, a uniformly bounded-to-one factor map $\theta \colon \Sigma \to X$, and a G-equivariant continuous map $\tau \colon X \to X$. We want to show that if τ is pre-injective with respect to the action of G on X then τ is surjective. We shall proceed by contradiction. So assume that τ is not surjective. Then $Y := \tau(X)$ is a closed G-invariant proper subset of X. As θ is a factor map, it follows that $\Psi := \theta^{-1}(Y)$ is a proper subshift of Σ. Since Σ is strongly irreducible, we deduce from Proposition 2.9 that

$$h_{\text{top}}(\Psi, G) < h_{\text{top}}(\Sigma, G). \tag{3.1}$$

Let us choose a metric d on X that is compatible with the topology and let $\delta > 0$ be an expansiveness constant for (X, G, d). As Σ is compact, the composite map $\tau\circ\theta\colon\Sigma\to X$ is uniformly continuous. Consequently, there is a finite subset $L\subset G$ such that

$$v|_L = w|_L \quad\Rightarrow\quad d\big(\tau\big(\theta(v)\big),\tau\big(\theta(w)\big)\big)<\delta \tag{3.2}$$

for all $v,w\in\Sigma$.

Let $\mathcal{F}=(F_n)_{n\in\mathbb{N}}$ be a symmetric Følner sequence for G (as already mentioned in the proof of Proposition 2.7, the existence of such a Følner sequence follows from [25, Corollary 5.3]). Let Δ be a finite subset of G such that Σ is Δ-irreducible. Up to replacing Δ by $\Delta\cup\Delta^{-1}$, we can assume that Δ is symmetric.

Now let us fix some configuration $u\in\Sigma$ and consider, for each $n\in\mathbb{N}$, the subset $\Phi(n)\subset\Sigma$ consisting of all configurations in Σ that coincide with u outside of $F_n\Delta$, that is,

$$\Phi(n) := \{v\in\Sigma : v|_{G\setminus F_n\Delta}=u|_{G\setminus F_n\Delta}\}\subset\Sigma.$$

The set $\Omega_1:=F_n$ is Δ-apart from the set $\Omega_2:=G\setminus F_n\Delta$. Therefore, it follows from Proposition 2.6 that, given any configuration $w\in\Sigma$, there exists a configuration $v\in\Phi(n)$ such that v coincides with w on F_n. This implies that

$$\operatorname{card}\big(\Phi(n)\big)\geq\operatorname{card}\big(\pi_{F_n}(\Sigma)\big) \tag{3.3}$$

for all $n\in\mathbb{N}$.

As θ is uniformly bounded-to-one, there is an integer $K\geq 1$ such that every point in X has at most K pre-images under θ. By using (3.3), this gives us

$$\operatorname{card}\big(\theta\big(\Phi(n)\big)\big)\geq K^{-1}\operatorname{card}\big(\Phi(n)\big)\geq K^{-1}\operatorname{card}\big(\pi_{F_n}(\Sigma)\big) \tag{3.4}$$

for all $n\in\mathbb{N}$.

If $g\in G\setminus L\Delta F_n$, then

$$g^{-1}\in G\setminus F_n^{-1}\Delta^{-1}L^{-1}=G\setminus F_n\Delta L^{-1}$$

and hence $g^{-1}L\subset G\setminus F_n\Delta$. Thus, if $v,w\in\Phi(n)$ and $g\in G\setminus L\Delta F_n$, then $(gv)|_L=(gu)|_L=(gw)|_L$. By applying (3.2), we deduce that all $v,w\in\Phi(n)$ satisfy

$$d\big(\tau\big(\theta(gv)\big),\tau\big(\theta(gw)\big)\big)<\delta\quad\text{for all } g\in G\setminus L\Delta F_n. \tag{3.5}$$

Let now $x,y\in\theta(\Phi(n))$ and choose $v,w\in\Phi(n)$ such that $x=\theta(v)$ and $y=\theta(w)$. We have that

$$\begin{aligned}
d\big(g\tau(x),g\tau(y)\big) &= d\big(g\tau\big(\theta(v)\big),g\tau\big(\theta(w)\big)\big)\\
&= d\big(\tau\big(g\theta(v)\big),\tau\big(g\theta(w)\big)\big)\quad\text{(since τ is G-equivariant)}\\
&= d\big(\tau\big(\theta(gv)\big),\tau\big(\theta(gw)\big)\big)\quad\text{(since θ is G-equivariant).}
\end{aligned}$$

Therefore, by using (3.5), we get

$$d\big(g\tau(x),g\tau(y)\big)<\delta\quad\text{for all } g\in G\setminus L\Delta F_n. \tag{3.6}$$

We now observe that

$$\begin{aligned} h_{\mathrm{top}}(\Sigma, G) &= \lim_{n\to\infty} \frac{\log \operatorname{card}(\pi_{F_n^{-1}}(\Sigma))}{\operatorname{card}(F_n)} \quad \text{(by Proposition 2.7)} \\ &= \lim_{n\to\infty} \frac{\log \operatorname{card}(\pi_{F_n}(\Sigma))}{\operatorname{card}(F_n)} \quad \text{(since } \mathcal{F} \text{ is symmetric)} \\ &= \lim_{n\to\infty} \frac{\log(K^{-1} \operatorname{card}(\pi_{F_n}(\Sigma)))}{\operatorname{card}(F_n)}. \end{aligned}$$

On the other hand, the sequence $\mathcal{F}' = (L\Delta F_n)_{n\in\mathbb{N}}$ is a Følner sequence for G by Proposition 2.2(ii). Therefore, it follows from (2.10) that

$$\begin{aligned} h_{\mathrm{top}}(\Psi, G) &\geq h_{\mathrm{top}}(Y, G) \quad \text{(by Proposition 2.4)} \\ &\geq h\big(\mathcal{F}', \delta/2, Y, G, d\big) \quad \text{(by Proposition 2.3)} \\ &= \limsup_{n\to\infty} \frac{\log S(L\Delta F_n, \delta/2, Y, G, d)}{\operatorname{card}(L\Delta F_n)} \quad \text{(by (2.9))} \\ &= \limsup_{n\to\infty} \frac{\log S(L\Delta F_n, \delta/2, Y, G, d)}{\operatorname{card}(F_n^{-1}\Delta^{-1}L^{-1})} \\ &\quad \big(\text{since } \operatorname{card}(L\Delta F_n) = \operatorname{card}\big(F_n^{-1}\Delta^{-1}L^{-1}\big)\big) \\ &= \limsup_{n\to\infty} \frac{\log S(L\Delta F_n, \delta/2, Y, G, d)}{\operatorname{card}(F_n\Delta^{-1}L^{-1})} \quad \big(\text{since } F_n^{-1} = F_n\big) \\ &= \limsup_{n\to\infty} \frac{\log S(L\Delta F_n, \delta/2, Y, G, d)}{\operatorname{card}(F_n)} \quad \text{(by using (2.6))}. \end{aligned}$$

As $h_{\mathrm{top}}(\Sigma, G) > h_{\mathrm{top}}(\Psi, G)$ by (3.1), we deduce from the two estimations above that there exists $n \in \mathbb{N}$ such that

$$K^{-1} \operatorname{card}\big(\pi_{F_n}(\Sigma)\big) > S(L\Delta F_n, \delta/2, Y, G, d). \tag{3.7}$$

Let $Z \subset Y$ be an $(L\Delta F_n, \delta/2, Y, G, d)$-spanning subset with minimal cardinality. It follows from (3.4) and (3.7) that

$$\operatorname{card}\big(\theta\big(\Phi(n)\big)\big) > S(L\Delta F_n, \delta/2, Y, G, d) = \operatorname{card}(Z).$$

Therefore, by the pigeon-hole principle, there exist two distinct points $x, y \in \theta(\Phi(n))$ and a point $z \in Z$ such that

$$d_{L\Delta F_n}\big(\tau(x), z\big) < \delta/2 \quad \text{and} \quad d_{L\Delta F_n}\big(\tau(y), z\big) < \delta/2.$$

By applying the triangle inequality, this gives us

$$d_{L\Delta F_n}\big(\tau(x), \tau(y)\big) < \delta,$$

that is,

$$d\big(g\tau(x), g\tau(y)\big) < \delta \quad \text{for all } g \in L\Delta F_n. \tag{3.8}$$

Since δ is an expansiveness constant for (X, G, d), we deduce from (3.6) and (3.8) that $\tau(x) = \tau(y)$. The elements of $\Phi(n)$ are almost equal. Therefore they belong to the same homoclinicity class with respect to the G-shift by Proposition 2.5. As θ is a factor map, it follows from Proposition 2.1 that all points in $\theta(\Phi(n))$ are in the same homoclinicity class with respect to the action of G on X. Consequently, the points x and y are G-homoclinic. As x and y are distinct and have the same image under τ, this shows that τ is not pre-injective. This completes the proof of Theorem 1.1.

4. Existence of non-trivial homoclinicity classes

If all G-homoclinicity classes of a dynamical system (X,G) are trivial, that is, each reduced to a single point, then, by definition, every continuous G-equivariant map $\tau\colon X \to X$ is pre-injective. This implies in particular that (X,g) has the Moore property. In this section, we present some results about the triviality or non-triviality of homoclinicity classes in certain subshifts and factors of subshifts.

Proposition 4.1. *Let G be a countable group and let A be a finite set. Suppose that $\Sigma \subset A^G$ is an infinite strongly irreducible subshift. Then every G-homoclinicity class in Σ is infinite.*

Proof. Fix a configuration $u \in \Sigma$ and let $\Phi \subset \Sigma$ denote the G-homoclinicity class of u. By Proposition 2.5, the class Φ consists of all configurations $v \in \Sigma$ that are almost equal to u.

Let Δ be a finite subset of G such that Σ is Δ-irreducible. For every finite subset $F \subset G$, the set F is Δ-apart from the set $G \setminus F\Delta$. Therefore, it follows from Proposition 2.6 that, given any configuration $w \in \Sigma$, there exists a configuration $v \in \Sigma$ such that v coincides with w on F and with u on $G \setminus F\Delta$. Observe that such a configuration v is in Φ since the set $F\Delta$ is finite. This implies that

$$\operatorname{card}(\Phi) \geq \operatorname{card}\bigl(\pi_F(\Sigma)\bigr) \tag{4.1}$$

for every finite subset $F \subset G$ (cf. the proof of Theorem 1.1 in Section 3).

On the other hand, as the subshift Σ is infinite, for every $n \in \mathbb{N}$, there exists a finite subset $F \subset G$ such that $\operatorname{card}(\pi_F(\Sigma)) \geq n$. We then deduce from (4.1) that the set Φ is infinite. □

The corollary below applies in particular to any non-trivial dynamical system (X,G) satisfying the hypotheses of Theorem 1.1. Note that it requires neither expansiveness of the system nor amenability of the acting group.

Corollary 4.2. *Let X be an infinite compact metrizable space equipped with a continuous action of a countable group G. Suppose that there exist a finite set A, a strongly irreducible subshift $\Sigma \subset A^G$, and a finite-to-one factor map $\theta\colon \Sigma \to X$. Then every G-homoclinicity class in X is infinite.*

Proof. Every G-homoclinicity class in Σ is infinite by Proposition 4.1. As the factor map θ is finite-to-one, we conclude that each G-homoclinicity class in X is infinite by applying Proposition 2.1. □

In [28, Proposition 2.1], Schmidt proved that if Σ is a subshift of finite type over $\mathbb{Z}^d$ with positive topological entropy, then the $\mathbb{Z}^d$-homoclinicity relation on Σ is non-trivial, i.e., there exist two distinct configurations in Σ that are $\mathbb{Z}^d$-homoclinic. The following statement extends Schmidt's result in two directions. First, it applies to a subshift of finite type with positive topological entropy Σ over an arbitrary countable amenable group G. Second, it says that, for any integer $n \geq 2$, one can find n distinct configurations in Σ that are G-homoclinic.

PROPOSITION 4.3. *Let G be a countable amenable group and let A be a finite set. Suppose that $\Sigma \subset A^G$ is a subshift of finite type with $h_{\text{top}}(\Sigma, G) > 0$. Then, for every integer $K \geq 1$, there exists a G-homoclinicity class in Σ containing more than K configurations.*

Proof. We shall proceed by contradiction. So let us assume that there is an integer $K \geq 1$ such that each G-homoclinicity class in Σ contains at most K configurations.

As the subshift Σ is of finite type, we can find a finite subset $\Omega \subset G$ and $\mathcal{P} \subset A^\Omega$ such that Σ satisfies (2.12). Up to enlarging Ω if necessary, we can assume that $1_G \in \Omega$ and $\Omega = \Omega^{-1}$.

Let $F \subset G$ be a finite subset. Observe that

$$F \subset F\Omega \subset F\Omega^2 \tag{4.2}$$

since $1_G \in \Omega$. Consider the finite subset $\partial F \subset G$ defined by

$$\partial F := F\Omega^2 \setminus F$$

and suppose that u and v are two configurations in Σ such that

$$u|_{\partial F} = v|_{\partial F}. \tag{4.3}$$

We claim that the configuration $w \in A^G$ defined by

$$w(g) = \begin{cases} v(g) & \text{if } g \in F\Omega, \\ u(g) & \text{if } g \in G \setminus F\Omega \end{cases}$$

also belongs to Σ.

To see this, we first observe that it follows from (4.2) and (4.3) that w coincides with u on $G \setminus F$ and with v on $F\Omega^2$. Now let $g \in G$.

If $g \in F\Omega$, then $g\Omega \subset F\Omega^2$ so that

$$(g^{-1}w)|_\Omega = (g^{-1}v)|_\Omega \in \mathcal{P}$$

since $v \in \Sigma$.

On the other hand, if $g \in G \setminus F\Omega$, then $g\Omega \subset G \setminus F$ so that

$$\big(g^{-1}w\big)|_\Omega = \big(g^{-1}u\big)|_\Omega \in \mathcal{P}$$

since $u \in \Sigma$.

Thus, we have that $(g^{-1}w)|_\Omega \in \mathcal{P}$ for all $g \in G$. This shows that $w \in \Sigma$.

The configurations u and w are G-homoclinic since they coincide outside of F. On the other hand, w coincides with v on F. As the G-homoclinicity class of u contains at most K configurations by our hypothesis, we deduce that

$$\operatorname{card}\big(\pi_F(\Sigma)\big) \le K \operatorname{card}\big(\pi_{\partial F}(\Sigma)\big) \tag{4.4}$$

for every finite subset $F \subset G$.

Now let $(F_n)_{n\in\mathbb{N}}$ be a symmetric Følner sequence for G. We then have

$$h_{\mathrm{top}}(\Sigma, G) = \lim_{n\to\infty} \frac{\log \operatorname{card}(\pi_{F_n}(\Sigma))}{\operatorname{card}(F_n)}$$

by Proposition 2.7. Since

$$\begin{aligned}
\frac{\log \operatorname{card}(\pi_{F_n}(\Sigma))}{\operatorname{card}(F_n)} &\le \frac{\log(K \operatorname{card}(\pi_{\partial F_n}(\Sigma)))}{\operatorname{card}(F_n)} \quad \text{(by (4.4))}\\
&= \frac{\log K}{\operatorname{card}(F_n)} + \frac{\log \operatorname{card}(\pi_{\partial F_n}(\Sigma))}{\operatorname{card}(F_n)}\\
&\le \frac{\log K}{\operatorname{card}(F_n)} + \frac{\log \operatorname{card}(A^{\partial F_n})}{\operatorname{card}(F_n)}\\
&= \frac{\log K}{\operatorname{card}(F_n)} + \frac{\operatorname{card}(\partial F_n)}{\operatorname{card}(F_n)} \log \operatorname{card}(A)
\end{aligned}$$

and

$$\lim_{n\to\infty} \frac{\operatorname{card}(\partial F_n)}{\operatorname{card}(F_n)} = \lim_{n\to\infty} \frac{\operatorname{card}(F_n\Omega^2 \setminus F_n)}{\operatorname{card}(F_n)} = 0$$

by applying Proposition 2.2(iii), we deduce that $h_{\mathrm{top}}(\sigma, G) = 0$. □

The following example shows that a subshift $\Sigma \subset A^G$ that satisfies the hypotheses of Proposition 4.3 may contain a trivial G-homoclinicity class.

Example 4.4. Let G be a finitely generated amenable group (e.g., $G = \mathbb{Z}^d$) and let A be a finite set with more than two elements. Fix an element $a_0 \in A$ and consider the subshift $\Sigma \subset A^G$ consisting of all configurations $u \in A^G$ such that either $u(g) = a_0$ for all $g \in G$ or $u(g) \neq a_0$ for all $g \in G$. This is a subshift of finite type. Indeed, if S is a finite generating subset of G, then Σ satisfies (2.12) for $\Omega = \{1_G\} \cup S$ by taking as $\mathcal{P}$ the set consisting of all maps $p \colon \Omega \to A$ such that either $p(g) = a_0$ for all $g \in \Omega$ or $p(g) \neq a_0$ for all $g \in G$. We clearly have $h_{\mathrm{top}}(\Sigma, G) = \log(\operatorname{card}(A) - 1) > 0$. On the other hand, the G-homoclinicity class of the constant configuration $u_0 \in \Sigma$, given by $u_0(g) = a_0$ for all $g \in G$, is reduced to u_0.

As an immediate consequence of Proposition 4.3, we get the following result. Note that the expansiveness of the dynamical system (X,G) is not required in the hypotheses.

COROLLARY 4.5. *Let X be an infinite compact metrizable space equipped with a continuous action of a countable amenable group G that satisfies $h_{\text{top}}(X,G) > 0$. Suppose that there exist a finite set A, a subshift of finite type $\Sigma \subset A^G$, and a uniformly bounded-to-one factor map $\theta\colon \Sigma \to X$. Then, for every integer $n \geq 1$, there exists a G-homoclinicity class in X containing more than n points.*

Proof. As θ is uniformly bounded-to-one, there is an integer $K \geq 1$ such that $\text{card}(\theta^{-1}(x)) \leq K$ for all $x \in X$. We have $h_{\text{top}}(\Sigma, G) > 0$ by Proposition 2.4. Therefore, it follows from Proposition 4.3 that, given an integer $n \geq 1$, there is a G-homoclinicity class $\Phi \subset \Sigma$ that contains more than Kn configurations. Then its image $\theta(\Phi) \subset X$ has more than n points and is contained in a G-homoclinicity class since θ preserves homoclinicity by Proposition 2.1. $\square$

REMARK 4.6. In [11, Question 1.1], Chung and Li asked whether every expansive system (X,G), where X is a compact metrizable space, G is a countable amenable group, and $h_{\text{top}}(x,G) > 0$, must contain two distinct points that are G-homoclinic. This question is known to have an affirmative answer for algebraic systems when $G = \mathbb{Z}^d$ [21, Theorem 4.1] and, more generally, when G is polycyclic-by-finite [11, Theorem 1.1]. We recall that the system (X,G) is said to be *algebraic* if X is a compact Abelian group and G acts by automorphisms on X. However, according to [11], the answer to this question in the general case is unknown even for $G = \mathbb{Z}$.

REMARK 4.7. In [21, Example 3.4], Lind and Schmidt constructed examples of expansive algebraic actions of $\mathbb{Z}$ on the n-torus $\mathbb{T}^n$ with positive topological entropy for which every $\mathbb{Z}$-homoclinicity class is trivial.

Recall that a group G is *residually finite* provided that for every finite subset $F \subset G$ there exists a finite index subgroup $H \subset G$ such that $\text{card}(H \cap F) \leq 1$. All finitely generated Abelian groups (e.g., $\mathbb{Z}^d$, for $d \geq 1$) and, more generally, all finitely generated virtually nilpotent groups are residually finite.

The following result yields a weak converse to Proposition 4.3 above. In the particular case $G = \mathbb{Z}^d$, it reduces to the second statement of Proposition 2.1 in [28].

PROPOSITION 4.8. *Let A be a finite set, G a countable amenable group, and $\Sigma \subset A^G$ a subshift of finite type. Suppose that G is residually finite and that the set $\text{Per}(\Sigma, G)$ is dense in Σ. If $h_{\text{top}}(\Sigma, G) = 0$, then every G-homoclinicity class in Σ is reduced to a single configuration.*

Proof. We shall proceed by contradiction. So, let us suppose that u_0 and u_1 are two distinct G-homoclinic configurations in Σ. Then we can find a finite subset $E \subset G$ such that $u_0|_{G \setminus E} = u_1|_{G \setminus E}$. Since Σ is of finite type, we can find, as in the proof of Proposition 4.3, a finite symmetric subset $\Omega \subset G$, with $1_G \in \Omega$, and $\mathcal{P} \subset A^\Omega$ such that Σ satisfies (2.12).

Since $\mathrm{Per}(\Sigma, G)$ is dense in Σ, there is a periodic configuration $v \in \Sigma$ such that $v|_{E\Omega^2} = u_0|_{E\Omega^2}$. Let H_v denote the stabilizer of v in G and observe that $[G : H_v] = \mathrm{card}(Gv) < \infty$.

We claim that we can find a finite index subgroup $H \subset H_v$ such that the left-translates $hE\Omega^2$, $h \in H$, are all pairwise disjoint. Indeed, since G is residually finite, there exists a finite index subgroup $H' \subset G$ such that $H' \cap E\Omega^2(E\Omega^2)^{-1} = \{1_G\}$. Then $H := H_v \cap H'$ is a finite index subgroup of G. Moreover, if $h_1, h_2 \in H$ are distinct, then $h_1 h_2^{-1} \neq 1_G$, so that $h_1 h_2^{-1} \notin E\Omega^2(E\Omega^2)^{-1}$ and hence $h_1 E\Omega^2$ does not meet $h_2 E\Omega^2$. Thus, the subgroup H has the required properties.

Let $R \subset G$ be a complete set of representatives of the right cosets of H in G. Then H satisfies $\bigcup_{h \in H} hR = G$. As the sets $hE\Omega^2$, $h \in H$, are pairwise disjoint, we deduce that H is an $(E\Omega^2, R)$-tiling of G in the sense of [5, Section 5.6].

Consider now, for each $z \in \{0,1\}^H$, the configuration $w_z \in A^G$ defined by

$$w_z(g) = \begin{cases} u_{z(h)}(h^{-1}g) & \text{if } g \in hE\Omega^2 \text{ for some (necessarily unique) } h \in H, \\ v(g) & \text{otherwise} \end{cases}$$

for all $g \in G$. Observe that, since $u_0|_{E\Omega^2 \setminus E} = u_1|_{E\Omega^2 \setminus E}$, we have that

$$w_z|_{G \setminus \bigcup_{h \in H} hE} = v|_{G \setminus \bigcup_{h \in H} hE}. \tag{4.5}$$

We claim that $w_z \in \Sigma$ for every $z \in \{0,1\}^H$. Indeed, let $g \in G$. We distinguish two cases.

First case: $g \in hE\Omega$ for some $h \in H$. Then $g\Omega \subset hE\Omega^2$ so that

$$\left(g^{-1} w_z\right)|_\Omega = \left(g^{-1} h u_{z(h)}\right)|_\Omega \in \mathcal{P},$$

since $u_{z(h)} \in \Sigma$.

Second case: $g \in G \setminus \bigcup_{h \in H} hE\Omega$. Then $g\Omega \subset G \setminus \bigcup_{h \in H} hE$ (here we use the fact that Ω is symmetric), so that, by virtue of (4.5),

$$\left(g^{-1} w_z\right)|_\Omega = \left(g^{-1} v\right)|_\Omega \in \mathcal{P}$$

since $v \in \Sigma$.

Thus, we have that $(g^{-1} w_z)|_\Omega \in \mathcal{P}$ for all $g \in G$. This shows that $w_z \in \Sigma$ for all $z \in \{0,1\}^H$.

We are now in position to prove that $h_{\mathrm{top}}(\Sigma, G) > 0$. Let $(F_n)_{n \in \mathbb{N}}$ be a symmetric Følner sequence for G. For each $n \in \mathbb{N}$, let H_n be the finite subset of G defined by

$$H_n := \{h \in H : hE \subset F_n\}.$$

By [5, Proposition 5.6.4] applied to the $(E\Omega^2, R)$-tiling H, there exist a constant $\alpha > 0$ and $n_0 \in \mathbb{N}$ such that

$$\operatorname{card}(H_n) \geq \alpha \operatorname{card}(F_n) \tag{4.6}$$

for all $n \geq n_0$.

As $u_0|_F \neq u_1|_F$, it follows from the definition of w_z and H_n that

$$\operatorname{card}\big(\pi_{F_n}\big(\{w_z : z \in \{0,1\}^H\}\big)\big) \geq 2^{\operatorname{card}(H_n)}.$$

Since $w_z \in \Sigma$ for all $z \in \{0,1\}^H$, it follows from (4.6) that

$$\operatorname{card}\big(\pi_{F_n}(\Sigma)\big) \geq 2^{\operatorname{card}(H_n)} \geq 2^{\alpha \operatorname{card}(F_n)}$$

for all $n \geq n_0$. We then conclude that

$$h_{\mathrm{top}}(\Sigma, G) = \lim_{n \to \infty} \frac{\log(\operatorname{card}(\pi_{F_n}(\Sigma)))}{\operatorname{card}(F_n)} \geq \alpha \log 2 > 0$$

by applying Proposition 2.7. □

5. Surjunctivity and density of periodic points

The following surjunctivity result does not require amenability of the acting group but only expansiveness of the dynamical system and density of periodic points. It is well known for subshifts (cf. [17], [20, Section 4], [15, Theorem 8.2], [7, Proposition 2.1]) and was previously obtained by the authors in [8, Proposition 4.1] for $G = \mathbb{Z}$.

PROPOSITION 5.1. *Let X be a compact metrizable space equipped with a continuous action of a countable group G. Suppose that the dynamical system (X, G) is expansive and that the set $\operatorname{Per}(X, G)$ of periodic points of (X, G) is dense in X. Then the dynamical system (X, G) is surjunctive.*

Proof. Suppose that $\tau \colon X \to X$ is an injective G-equivariant continuous map. Let d be a metric on X that is compatible with the topology and let $\delta > 0$ be an expansiveness constant for (X, G, d). Fix a finite-index subgroup H of G and let $R \subset G$ be a complete set of representatives of the left-cosets of H in G, so that every element $g \in G$ can be uniquely written in the form $g = rh$ with $r \in R$ and $h \in H$. Since the set R is finite, for $\varepsilon > 0$ small enough, all points $x, y \in X$ such that $d(x, y) < \varepsilon$ satisfy $d(rx, ry) < \delta$ for all $r \in R$. By expansivity, this implies that all distinct points in $\operatorname{Fix}(H)$ are at least ε-apart. As X is compact, we deduce that $\operatorname{Fix}(H)$ is finite. On the other hand, the G-equivariance of τ implies that $\tau(\operatorname{Fix}(H)) \subset \operatorname{Fix}(H)$. Since τ is injective, it follows that $\tau(\operatorname{Fix}(H)) = \operatorname{Fix}(H)$. We deduce that $\tau(\operatorname{Per}(X, G)) = \operatorname{Per}(X, G)$ by applying (2.1). As $\operatorname{Per}(X, G)$ is dense in X, we conclude that $\tau(X) = X$. This shows that τ is surjective. □

Corollary 5.2. *Let X be a compact metrizable space equipped with a continuous action of a countable residually finite group G. Suppose that the dynamical system (X,G) is expansive and that there exist a finite set A, a strongly irreducible subshift of finite type $\Sigma \subset A^G$ admitting a periodic configuration, such that (X,G) is a factor of (Σ,G). Then the dynamical system (X,G) is surjunctive.*

Proof. By [7, Theorem 1.1], the set of periodic configurations is dense in Σ. On the other hand, since θ is G-equivariant, every periodic configuration in Σ is mapped by θ to a periodic point of X. As θ is continuous and onto, we deduce that $\mathrm{Per}(X,G)$ is dense in X. □

6. Some examples of expansive dynamical systems

In this section, we describe expansive dynamical systems that may be used as counterexamples for showing the necessity of some of the hypotheses in our results.

Example 6.1. Let G be a countable group. Consider a dynamical system (X,G), where X is a finite discrete space with $k \geq 2$ points and G fixes each point of X. Each G-homoclinicity class of X is reduced to one point and any map $\tau\colon X \to X$ is continuous and G-equivariant. As X has more than one point, we deduce that (X,G) is surjunctive but does not have the Myhill property. The dynamical system (X,G) satisfies the hypotheses of Proposition 5.1 without having the Myhill property. Observe that (X,G) is topologically conjugate to the subshift $\Sigma \subset \{0,1,\dots,k-1\}^G$ consisting of the k constant configurations. Note also that Σ is of finite type if the group G is finitely generated. Indeed, if S is a finite generating subset of G, then Σ satisfies (2.12) for $\Omega = \{1_G\} \cup S$ by taking as $\mathcal{P}$ the set of constant maps from Ω to $\{0,1,\dots,k-1\}$.

Example 6.2. Let G be a countable group equipped with the discrete topology and consider its one-point compactification $X = G \cup \{\infty\}$. Observe that X is compact and metrizable. Actually, X is homeomorphic to the subset of $\mathbb{R}$ consisting of 0 and all the inverses of the positive integers. The action of G on itself by left-multiplication continuously extends to an action of G on X that fixes ∞. This action is clearly expansive but not topologically mixing. The points of X are all in the same G-homoclinicity class. On the other hand, a map $\tau\colon X \to X$ is continuous and G-equivariant if and only if either τ sends every point of X to ∞ or there exists $g_0 \in G$ such that $\tau(\infty) = \infty$ and $\tau(g) = gg_0$ for all $g \in G$. We deduce that the dynamical system (X,G) has the Myhill property and is therefore surjunctive. However, when G is amenable, (X,G) does not satisfy the hypotheses of Theorem 1.1. Indeed, every strongly irreducible subshift is topologically mixing (see, e.g., [6, Proposition 3.3]) and it is clear that any factor of a topologically mixing dynamical system is itself topologically mixing. Clearly, (X,G) is topologically conjugate to the subshift $\Sigma \subset \{0,1\}^G$ consisting of all configurations $u \in \{0,1\}^G$ such that there is at most one element $g \in G$ such that $u(g) = 1$. Consequently, the dynamical system (X,G) is finitely presented if and only if the subshift Σ is sofic. It follows from [12, Corollary 4.4] that (X,G) is finitely presented if G is polyhyperbolic (e.g., polycyclic). On the other hand, it is shown in [2, Proposition 2.3 and Theorem 2.11] that if G is a finitely generated and recursively presented group with undecidable word problem then (X,G) is not finitely presented.

EXAMPLE 6.3. Fix an integer $n \geq 2$ and let G denote the solvable (and hence amenable) Baumslag–Solitar group given by the presentation $G = \langle a, b : bab^{-1} = a^n \rangle$. Consider the dynamical system (X,G) where X is the real projective line $S^1 = \mathbb{R} \cup \{\infty\}$ and the action of G on X is the projective action defined by $ax = x + 1$ and $bx = nx$ for all $x \in X = \mathbb{R} \cup \{\infty\}$. Observe that ∞ is the only point of X fixed by G and that the orbit of any point $x \in X \setminus \{\infty\}$ is dense in X. Note also that the action of G on X is expansive but not topologically mixing and that every G-homoclinicity class is reduced to one single point. On the other hand, it is easy to check that a map $\tau \colon X \to X$ is continuous and G-equivariant if and only if either $\tau(x) = \infty$ for all $x \in X$ or τ is the identity map on X. We deduce that (X,G) is surjunctive but does not have the Myhill property. Finally, let us note that (X,G) does not satisfy the conclusion of Proposition 4.3.

EXAMPLE 6.4. Let A be a finite set and G a countable, amenable, residually finite group. Suppose that $\Sigma \subset A^G$ is a subshift of finite type with $h_{\text{top}}(\Sigma, G) = 0$ and $\text{Per}(\Sigma, G)$ dense in Σ. Suppose also that Σ contains a constant configuration u_0 but is not reduced to u_0. Then all G-homoclinicity classes in Σ are trivial by Proposition 4.8. Thus, the map $\tau \colon \Sigma \to \Sigma$ defined by $\tau(u) = u_0$ for all $u \in \Sigma$ is pre-injective. As τ is continuous and G-equivariant but not surjective, this shows that Σ does not have the Myhill property. Note however that Σ is surjunctive by Proposition 5.1.

An example of a subshift satisfying all these conditions is provided by the Ledrappier subshift (cf. [27, Section 3]). Recall that the *Ledrappier subshift* is the subshift $\Sigma \subset A^G$, for $G = \mathbb{Z}^2$ and $A = \mathbb{Z}/2\mathbb{Z}$, consisting of all $x \colon \mathbb{Z}^2 \to \mathbb{Z}/2\mathbb{Z}$ satisfying

$$x(m,n) + x(m+1,n) + x(m,n+1) = 0$$

for all $(m,n) \in \mathbb{Z}^2$. As the Ledrappier subshift is topologically mixing, we deduce that Theorem 1.1 becomes false if the hypothesis saying that the subshift Σ is strongly irreducible is replaced by the condition that Σ is topologically mixing and of finite type. Note that Σ is a compact Abelian group and that G acts on Σ by group automorphisms, so that (Σ, G) is an expansive algebraic dynamical system (cf. [29]).

Example 6.5. Weiss [30, p. 358] described a topologically mixing subshift of finite type containing a constant configuration over $\mathbb{Z}^2$ that is not surjunctive and hence does not have the Myhill property. This shows that both Corollary 1.2 and Corollary 5.2 become false if the hypothesis saying that the subshift Σ is strongly irreducible is replaced by the condition that Σ is topologically mixing and of finite type.

Example 6.6. Let G be a group containing a non-Abelian free subgroup. It is known that there is a finite set A such that A^G does not have the Myhill property (cf. [5, Proposition 5.11.2]). This shows in particular that amenability of the group G cannot be removed from the hypotheses of Theorem 1.1. However, there are non-amenable groups with no non-Abelian free subgroups and we do not know if Theorem 1.1 becomes also false when applied to such groups. An affirmative answer to this question would give a new characterization of amenablity for groups.

References

[1] R. L. Adler, A. G. Konheim and M. H. McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. **114** (1965), 309–319. MR 0175106

[2] N. Aubrun, S. Barbieri and M. Sablik, *A notion of effectiveness for subshifts on finitely generated groups*, available at `arXiv:1412.2582`.

[3] L. Bartholdi, *Gardens of Eden and amenability on cellular automata*, J. Eur. Math. Soc. (JEMS) **12** (2010), 241–248. MR 2578610

[4] R. Bowen, *Periodic points and measures for Axiom A diffeomorphisms*, Trans. Amer. Math. Soc. **154** (1971), 377–397. MR 0282372

[5] T. Ceccherini-Silberstein and M. Coornaert, *Cellular automata and groups*, Springer Monographs in Mathematics, Springer, Berlin, 2010. MR 2683112

[6] T. Ceccherini-Silberstein and M. Coornaert, *The Myhill property for strongly irreducible subshifts over amenable groups*, Monatsh. Math. **165** (2012), 155–172. MR 2886128

[7] T. Ceccherini-Silberstein and M. Coornaert, *On the density of periodic configurations in strongly irreducible subshifts*, Nonlinearity **25** (2012), 2119–2131. MR 2947938

[8] T. Ceccherini-Silberstein and M. Coornaert, *The Myhill property for hyperbolic homeomorphisms*, available at `arXiv:1506.06945v3`.

[9] T. Ceccherini-Silberstein, M. Coornaert and F. Krieger, *An analogue of Fekete's lemma for subadditive functions on cancellative amenable semigroups*, J. Anal. Math. **124** (2014), 59–81. MR 3286049

[10] T. Ceccherini-Silberstein, A. Machì and F. Scarabotti, *Amenable groups and cellular automata*, Ann. Inst. Fourier (Grenoble) **49** (1999), 673–685. MR 1697376

[11] N.-P. Chung and H. Li, *Homoclinic groups, IE groups, and expansive algebraic actions*, Invent. Math. **199** (2015), 805–858. MR 3314515

[12] F. Dahmani and A. Yaman, *Symbolic dynamics and relatively hyperbolic groups*, Groups Geom. Dyn. **2** (2008), 165–184. MR 2393178

[13] E. I. Dinaburg, *A correlation between topological entropy and metric entropy*, Dokl. Akad. Nauk SSSR **190** (1970), 19–22. MR 0255765

[14] F. Fiorenzi, *The Garden of Eden theorem for sofic shifts*, Pure Math. Appl. (PU.M.A.) **11** (2000), 471–484. MR 1839563

[15] F. Fiorenzi, *Periodic configurations of subshifts on groups*, Internat. J. Algebra Comput. **19** (2009), 315–335. MR 2522838

[16] D. Fried, *Finitely presented dynamical systems*, Ergodic Theory Dynam. Systems **7** (1987), 489–507. MR 0922362

[17] W. Gottschalk, *Some general dynamical notions*, Recent advances in topological dynamics, Proc. conf. topological dynamics (Yale Univ., New Haven, Conn., 1972), Lecture Notes in Math., vol. 318, Springer, Berlin, 1973, pp. 120–125. In honor of Gustav Arnold Hedlund. MR 0407821

[18] F. P. Greenleaf, *Invariant means on topological groups and their applications*, Van Nostrand Mathematical Studies, vol. 16, Van Nostrand-Reinhold, New York, 1969. MR 0251549

[19] M. Gromov, *Hyperbolic manifolds, groups and actions*, Riemann surfaces and related topics, Proceedings of the 1978 Stony Brook conference (State Univ. New York, Stony Brook, NY, 1978), Ann. of Math. Stud., vol. 97, Princeton University Press, Princeton, NJ, 1981, pp. 183–213. MR 0624814

[20] M. Gromov, *Endomorphisms of symbolic algebraic varieties*, J. Eur. Math. Soc. (JEMS) **1** (1999), 109–197. MR 1694588

[21] D. Lind and K. Schmidt, *Homoclinic points of algebraic* $\mathbf{Z}^d$*-actions*, J. Amer. Math. Soc. **12** (1999), 953–980. MR 1678035

[22] E. Lindenstrauss and B. Weiss, *Mean topological dimension*, Israel J. Math. **115** (2000), 1–24. MR 1749670

[23] E. F. Moore, *Machine models of self-reproduction*, Proc. symp. appl. math., vol. 14, Am. Math. Soc., Providence, RI, 1963, pp. 17–34.

[24] J. Myhill, *The converse of Moore's Garden-of-Eden theorem*, Proc. Amer. Math. Soc. **14** (1963), 685–686. MR 0155764

[25] I. Namioka, *Følner's conditions for amenable semi-groups*, Math. Scand. **15** (1964), 18–28. MR 0180832

[26] A. L. T. Paterson, *Amenability*, Mathematical Surveys and Monographs, vol. 29, Am. Math. Soc., Providence, RI, 1988. MR 0961261

[27] A. N. Quas and P. B. Trow, *Subshifts of multi-dimensional shifts of finite type*, Ergodic Theory Dynam. Systems **20** (2000), 859–874. MR 1764932

[28] K. Schmidt, *The cohomology of higher-dimensional shifts of finite type*, Pacific J. Math. **170** (1995), 237–269. MR 1359979

[29] K. Schmidt, *Dynamical systems of algebraic origin*, Progress in Mathematics, vol. 128, Birkhäuser, Basel, 1995. MR 3024809

[30] B. Weiss, *Sofic groups and dynamical systems*, Sankhya, Ser. A **62** (2000), 350–359. Ergodic theory and harmonic analysis (Mumbai, 1999). MR 1803462

Tullio Ceccherini-Silberstein, Dipartimento di Ingegneria, Università del Sannio, C.so Garibaldi 107, 82100 Benevento, Italy

E-mail address: tceccher@mat.uniroma3.it

Michel Coornaert, Institut de Recherche Mathématique Avancée, UMR 7501, Université de Strasbourg et CNRS, 7 rue René-Descartes, 67000 Strasbourg, France

E-mail address: coornaert@math.unistra.fr