Expansive actions of countable amenable groups, homoclinic pairs, and the Myhill property

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This is joint work with Tullio Ceccherini-Silberstein.

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Our motivation came from the following phrase of Gromov [Gro-1999, p. 195]:

> *"...the Garden of Eden theorem can be generalized to a suitable class of hyperbolic actions ..."*

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If $f\colon X \to X$ is a homeomorphism, the d.s. $(X, \mathbb{Z})$, where

$$nx := f^n(x) \quad \forall n \in \mathbb{Z}, \forall x \in X,$$

is also denoted (X, f).

Examples of Dynamical systems

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Example (Arnold's cat)

This is the d.s. $(\mathbb{T}^2, f)$, where f is the homeomorphism of the 2-torus $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ given by

$$\begin{aligned} f \colon \mathbb{T}^2 &\to \mathbb{T}^2 \\ (x_1, x_2) &\mapsto (x_2, x_1 + x_2). \end{aligned}$$

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We take a discrete finite space A, called the alphabet or the set of states, and a countable group G. The associated shift is the d.s. (A^G, G), where

$$A^G = \{x\colon G \to A\}$$

is equipped with the product topology and G acts on A^G by

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Examples of Dynamical systems (continued)

Example (The Ledrappier subshift)

The Ledrappier subshift is the subshift $(X, \mathbb{Z}^2)$ over the alphabet $A := \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$ consisting of all $x\colon \mathbb{Z}^2 \to A$ such that

$$x(g) = x(g + e_1) + x(g + e_2) \quad \forall g \in \mathbb{Z}^2,$$

where $e_1 = (1, 0)$ and $e_2 = (0, 1)$.

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Two points $x, y \in X$ are caled homoclinic if

$$\lim_{g \to \infty} d(gx, gy) = 0,$$

i.e., for every $\varepsilon > 0$, there exists a finite subset $F \subset G$ such that

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The homoclinicity class of a point $x \in \mathbb{T}^2$ is $D \cap D'$, where D is the line passing through x whose slope is the golden mean $\phi := (1 + \sqrt{5})/2$ and D' is the line passing through x and orthogonal to D'.

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Therefore, the homoclinicity relations is trivial: the homoclinicity class of every configuration $x \in X$ is reduced to x.

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Of course

$$\boxed{\tau \text{ injective} \implies \tau \text{ pre-injective}}$$

but the converse implication is false in general.

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Example (The Ledrappier subshift)

The constant map that sends each configuration $x \in X$ to the 0-configuration is an endomorphism of the Ledrappier subshift $(X, \mathbb{Z}^2)$ that is pre-injective but neither injective nor surjective.

Amenable groups

Let G be a countable group.

Definition

The group G is called amenable if there exists a sequence $(F_n)_{n\geq 1}$ of non-empty finite subsets of G such that

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The proof consists in showing that

$$\tau \text{ surjective} \iff h_{top}(\tau(A^G), G) = h_{top}(A^G, G) \iff \tau \text{ pre-injective},$$

where $h_{top}(X, G)$ denotes the topological entropy of the d.s. (X, G).

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A d.s. has the Moore-Myhill property if it has both the Moore and the Myill property.

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Example

Arnold's cat $(\mathbb{T}^2, f)$ has the Moore-Myhill property. Indeed, it is easy to show that any endomorphism τ of the cat is of the form $\tau = m\,\mathrm{Id} + nf$, for some $m, n \in \mathbb{Z}$. Thus, with the exception of the 0-endomorphism, every endomorphism of the cat is both surjective and pre-injective.

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Arnold's cat $(\mathbb{T}^2, f)$ has the Moore-Myhill property. Indeed, it is easy to show that any endomorphism τ of the cat is of the form $\tau = m\,\mathrm{Id} + nf$, for some $m, n \in \mathbb{Z}$. Thus, with the exception of the 0-endomorphism, every endomorphism of the cat is both surjective and pre-injective.

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The GOE theorem says that the shift A^G has the Moore-Myhill property whenever A is finite and G is amenable . Bartholdi [Bar-2010] proved that if G is non-amenable then there is a finite set A such that A^G does not have the Moore property. It is known that if G contains a nonabelian free subgroup then there is a finite set A such that A^G does not have the Myhill property.

Example

The Ledrappier subshift $(X, \mathbb{Z}^2)$ has the Moore property (since every endomorphism is pre-injective) but does not have the Myhill property (since the 0-endomorphism is pre-injective but not surjective).

The Moore and the Myhill property (continued)

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Remark

The Moore property is a finiteness condition (i.e., every d.s. (X, G) with X finite has the Moore property) whereas the Myhill property is not (consider a finite discrete space X with more than one point and a group G fixing each point of X).

Subshifts of finite type and strongly irreducible subshifts

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Definition

A subshift $X \subset A^G$ is said to be of *finite type* if there exist a finite subset $\Omega \subset G$ and a subset $\mathcal{P} \subset A^\Omega$ such that

$$X = \{x \in A^G : (gx)|_\Omega \in \mathcal{P} \text{ for all } g \in G\}.$$

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A subshift $X \subset A^G$ is said to be strongly irreducible if there exists a finite subset $\Delta \subset G$ with the following property:
if Ω_1 and Ω_2 are finite subsets of G such that there is no element $g \in \Delta$ such that the set $\Omega_1 g$ meets Ω_2 (i.e., $\Omega_1\Delta \cap \Omega_2 = \varnothing$) then, given any two configurations $x_1, x_2 \in X$, there exists a configuration $x \in X$ such that $x|_{\Omega_1} = x_1|_{\Omega_1}$ and $x|_{\Omega_2} = x_2|_{\Omega_2}$.

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Example

The hard sphere model is the subshift $X \subset \{0,1\}^{\mathbb{Z}^d}$ consisting of all $x \colon \mathbb{Z}^d \to \{0,1\}$ with no two 1s appearing at Euclidean distance 1 on $\mathbb{Z}^d$.

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Remark

For $d = 1$, the hard sphere model is also called the golden mean subshift because its topological entropy is equal to the golden mean.

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Expansive dynamical systems

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The d.s. (X, G) is expansive if there is a constant $\varepsilon > 0$ such that, for all distinct points $x, y \in X$, there exists $g \in G$ such that

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Example

All shifts and subshifts are expansive.

Theorem (CC-2015b)

Let X be a compact metrizable space equipped with a continuous action of a countable amenable group G.
Suppose that the d.s. (X, G) is expansive and that there exist a finite set A, a strongly irreducible subshift $\Sigma \subset A^G$, and a continuous, surjective, G-equivariant and uniformly finite-to-one map $\pi \colon \Sigma \to X$.
Then the dynamical system (X, G) has the Myhill property.

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A closed f-invariant subset $\Lambda \subset M$ is hyperbolic if the restriction to Λ of the tangent bundle of M splits as a direct sum of two invariant subbundles E_s and E_u such that, with respect to some (or equivalently any) Riemannian metric on M, the differential df is uniformly contracting on E_s and uniformly expanding on E_u.

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A point $x \in M$ is called *non-wandering* if for every neighborhood U of x, there is an integer $n \geq 1$ such that $f^n(U)$ meets U. The set $\Omega(f)$ consisting of all non-wandering points of f is a closed invariant subset of M.

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If f is Axiom A, then $\Omega(f)$ can be uniquely written as a disjoint union of closed invariant subsets $\Omega(f) = X_1 \cup \cdots \cup X_k$, such that the restriction of f to each X_i is topologically transitive (spectral decomposition theorem).

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Axiom A diffeomorphisms (continued)

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A dynamical system (X, G) is topologically mixing if, given any two non-empty open subsets $U, V \subset X$, one has $U \cap gV \neq \varnothing$ for all but finitely many $g \in G$.

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Corollary (CC-2015a)

Let f be an Axiom A diffeomorphism of a smooth compact manifold M. Suppose that X is a topologically mixing basic set of (M, f). Then the dynamical system $(X, f|_X)$ has the Myhill property.

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Corollary (CC-2015a)

Let f be an Axiom A diffeomorphism of a smooth compact manifold M. Suppose that X is a topologically mixing basic set of (M, f). Then the dynamical system $(X, f|_X)$ has the Myhill property.

Proof.

The fact that the dynamical system $(X, f|_X)$ satisfies the hypotheses of the theorem follows from results obtained by Rufus Bowen in the 1970s. □

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Example (Hyperbolic toral automorphisms)

Consider a matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ with no eigenvalue of modulus 1. Then A induces a topologically mixing Anosov diffeomorphism f_A of the n-torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. One says that f_A is a hyperbolic toral automorphism.

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