Expansive actions of countable amenable groups, homoclinic pairs, and the Myhill property

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This is joint work with Tullio Ceccherini-Silberstein.
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Our motivation came from the following phrase of Gromov [Gro-1999, p. 195]:

"...the Garden of Eden theorem can be generalized to a suitable class of hyperbolic actions..."
Dynamical systems

A dynamical system is a pair \((X, G)\), where \(X\) is a compact metrizable topological space, \(G\) is a countable group acting continuously on \(X\). The space \(X\) is called the phase space.

If \(f: X \to X\) is a homeomorphism, the d.s. \((X, f)\), where \(nx := f^n(x)\) for all \(n \in \mathbb{Z}\), \(x \in X\), is also denoted \((X, f)\).
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\[ nx := f^n(x) \quad \forall n \in \mathbb{Z}, \forall x \in X, \]

is also denoted \((X, f)\).
Examples of Dynamical systems

Example (Arnold’s cat)
This is the d.s. \((T^2, f)\), where \(f\) is the homeomorphism of the 2-torus \(T^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}\) given by \(f: T^2 \to T^2 (x_1, x_2) \mapsto (x_2, x_1 + x_2)\).

Example (Shifts and subshifts)
We take a discrete finite space \(A\), called the alphabet or the set of states, and a countable group \(G\). The associated shift is the d.s. \((A^G, G)\), where \(A^G = \{x: G \to A\}\) is equipped with the product topology and \(G\) acts on \(A^G\) by \((gx)(h) := x(g^{-1}h)\) for all \(g, h \in G\), \(x \in A^G\).

An element of \(A^G\) is called a configuration. A subsystem of the shift (i.e., a pair \((X, G)\), where \(X \subset A^G\) is a closed \(G\)-invariant subspace) is called a subshift.
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Examples of Dynamical systems (continued)

Example (The Ledrappier subshift)

The Ledrappier subshift is the subshift \((X, \mathbb{Z}^2)\) over the alphabet

\[ A := \{0, 1\} = \mathbb{Z}/2\mathbb{Z} \]

consisting of all \(x : \mathbb{Z}^2 \to A\) such that

\[ x(g) = x(g + e_1) + x(g + e_2) \quad \forall g \in \mathbb{Z}^2, \]

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Homoclinicity

Let $(X, G)$ be a dynamical system. Let $d$ be a metric on $X$ that is compatible with the topology.

**Definition**

Two points $x, y \in X$ are called homoclinic if

$$\lim_{g \to \infty} d(gx, gy) = 0,$$

i.e., for every $\varepsilon > 0$, there exists a finite subset $F \subset G$ such that $d(gx, gy) < \varepsilon \quad \forall g \in G \setminus F$.

Homoclinicity is an equivalence relation on $X$. This relation is $G$-invariant and does not depend on the choice of $d$. 

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Example
Consider Arnold’s cat (\(T^2, f\)). Equip \(T^2 = \mathbb{R}^2/\mathbb{Z}^2\) with its Euclidean structure. The homoclinicity class of a point \(x \in T^2\) is \(D \cap D'\), where \(D\) is the line passing through \(x\) whose slope is the golden mean \(\phi := \frac{1 + \sqrt{5}}{2}\) and \(D'\) is the line passing through \(x\) and orthogonal to \(D'\). Each homoclinicity class is countably-infinite.

Example
Consider the full shift \((A^G, G)\) over a finite alphabet \(A\) and a countable group \(G\). Two configurations \(x, y \in A^G\) are homoclinic if and only if they coincide outside of a finite subset of \(G\). Thus, each homoclinicity class is countably-infinite as soon as \(A\) has more than one element and \(G\) is infinite.

Example
Consider the Ledrappier subshift \((X, Z_2)\). Observe that if two configurations \(x, y \in X\) coincide on the horizontal line \(Z \times \{n\} \subset Z^2\), then they coincide on \(Z \times \{n + 1\}\). Therefore, the homoclinicity relations is trivial: the homoclinicity class of every configuration \(x \in X\) is reduced to \(x\).
Example

Consider Arnold’s cat \((\mathbb{T}^2, f)\).

Equip \(\mathbb{T}^2 = \mathbb{R}^2 / \mathbb{Z}^2\) with its Euclidean structure.

The homoclinicity class of a point \(x \in \mathbb{T}^2\) is \(D \cap D'\), where \(D\) is the line passing through \(x\) whose slope is the golden mean \(\phi := (1 + \sqrt{5})/2\) and \(D'\) is the line passing through \(x\) and orthogonal to \(D'\).

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Consider the full shift \((\mathcal{A}G, G)\) over a finite alphabet \(\mathcal{A}\) and a countable group \(G\).

Two configurations \(x, y \in \mathcal{A}G\) are homoclinic if and only if they coincide outside of a finite subset of \(G\).

Thus, each homoclinicity class is countably-infinite as soon as \(\mathcal{A}\) has more than one element and \(G\) is infinite.

Example

Consider the Ledrappier subshift \((X, \mathbb{Z}_2^n)\).

Observe that if two configurations \(x, y \in X\) coincide on the horizontal line \(\mathbb{Z} \times \{n\} \subset \mathbb{Z}^2\), then they coincide on \(\mathbb{Z} \times \{n + 1\}\).

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Pre-injective endomorphisms

Let \((X, G)\) be a dynamical system.

**Definition**
A continuous map \(\tau: X \to X\) is an endomorphism of the d.s. \((X, G)\) if it is \(G\)-equivariant, i.e.,
\[
\tau(gx) = g \tau(x) \quad \forall g \in G, x \in X.
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**Remark**
An endomorphism of a shift (or subshift) is also called a cellular automaton.

**Definition**
An endomorphism \(\tau: X \to X\) of the d.s. \((X, G)\) is called pre-injective if its restriction to each homoclinicity class is injective.

Of course \(\tau\) injective \(\Rightarrow\) \(\tau\) pre-injective but the converse implication is false in general.
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Examples of pre-injective but not injective endomorphisms

Example (Arnold's cat)

The group endomorphism $\tau : \mathbb{T}^2 \to \mathbb{T}^2$ given by $x \mapsto 2x$ is an endomorphism of Arnold's cat ($\mathbb{T}^2, f$).

The kernel of $\tau$ is $\text{Ker}(\tau) = \{(0,0), (1/2,0), (0,1/2), (1/2,1/2)\}$.

The endomorphism $\tau$ is surjective and pre-injective but not injective.

Example

The endomorphism $\tau$ of the full shift ($\mathbb{A}_Z, Z$) on the alphabet $\mathbb{A} = \mathbb{Z}/2\mathbb{Z}$ defined by $\tau(x)_n := x_{n+1} + x_n$ $\forall x \in \{0,1\}^\mathbb{Z}, \forall n \in \mathbb{Z}$ is surjective and pre-injective but not injective.

Example (The Ledrappier subshift)

The constant map that sends each configuration $x \in X$ to the 0-configuration is an endomorphism of the Ledrappier subshift ($X, Z_2$) that is pre-injective but neither injective nor surjective.
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Amenable groups

Let $G$ be a countable group. Definition: The group $G$ is called amenable if there exists a sequence $(F_n)_{n \geq 1}$ of non-empty finite subsets of $G$ such that
\[
\lim_{n \to \infty} |F_n \setminus gF_n| / |F_n| = 0 \quad \forall g \in G.
\]
Such a sequence is called a Følner sequence for $G$.

- Every locally finite group is amenable.
- Every abelian group and, more generally, every solvable group is amenable.
- Every finitely generated group with subexponential growth is amenable.
- An example of a non-amenable group is provided by the free group on 2 generators. More generally, every group containing a non-abelian free subgroup is non-amenable.
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The group $G$ is called **amenable** if there exists a sequence $(F_n)_{n \geq 1}$ of non-empty finite subsets of $G$ such that

$$\lim_{n \to \infty} \frac{|F_n \setminus F_ng|}{|F_n|} = 0 \quad \forall g \in G.$$ 

Such a sequence is called a **Følner sequence** for $G$. 

• Every locally finite group is amenable.
• Every abelian group and, more generally, every solvable group is amenable.
• Every finitely generated group with subexponential growth is amenable.
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The Garden of Eden theorem

The following result is known as the Garden of Eden theorem:

Theorem (CMS-1999)

Let $G$ be a countable amenable group and $A$ a finite set. Then every endomorphism $\tau$ of the shift $(A^G, G)$ satisfies

$\tau$ surjective $\iff \tau$ pre-injective.

Moore [Moo-1963] proved $\Rightarrow$ for $G = \mathbb{Z}^d$,
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Expansive actions of countable amenable groups
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The Moore and the Myhill property

Let \((X, G)\) be a dynamical system.

Definition

The d.s. \((X, G)\) has the Moore property if every surjective endomorphism of \((X, G)\) is pre-injective.

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The d.s. \((X, G)\) has the Myhill property if every pre-injective endomorphism of \((X, G)\) is surjective.

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Example
Arnold’s cat \((T^2, f)\) has the Moore-Myhill property. Indeed, it is easy to show that any endomorphism \(\tau\) of the cat is of the form \(\tau = m \text{Id} + nf\), for some \(m, n \in \mathbb{Z}\). Thus, with the exception of the 0-endomorphism, every endomorphism of the cat is both surjective and pre-injective.

Example
The GOE theorem says that the shift \(A_G\) has the Moore-Myhill property whenever \(A\) is finite and \(G\) is amenable. Bartholdi [Bar-2010] proved that if \(G\) is non-amenable then there is a finite set \(A\) such that \(A_G\) does not have the Moore property. It is known that if \(G\) contains a nonabelian free subgroup then there is a finite set \(A\) such that \(A_G\) does not have the Myhill property.

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The Ledrappier subshift \((X, Z_2)\) has the Moore property (since every endomorphism is pre-injective) but does not have the Myhill property (since the 0-endomorphism is pre-injective but not surjective).
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The Moore property is a finiteness condition (i.e., every d.s. \((X, G)\) with \(X\) finite has the Moore property) whereas the Myhill property is not (consider a finite discrete space \(X\) with more than one point and a group \(G\) fixing each point of \(X\)).
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Subshifts of finite type and strongly irreducible subshifts

Definition
A subshift $X \subset \mathcal{A}^G$ is said to be of finite type if there exist a finite subset $\Omega \subset \mathcal{G}$ and a subset $P \subset \mathcal{A}^\Omega$ such that $X = \{x \in \mathcal{A}^G : (gx)_{\Omega} \in P \text{ for all } g \in \mathcal{G}\}$.

Definition
A subshift $X \subset \mathcal{A}^G$ is said to be strongly irreducible if there exists a finite subset $\Delta \subset \mathcal{G}$ with the following property: if $\Omega_1$ and $\Omega_2$ are finite subsets of $\mathcal{G}$ such that there is no element $g \in \Delta$ such that the set $\Omega_1 g$ meets $\Omega_2$ (i.e., $\Omega_1 \Delta \cap \Omega_2 = \emptyset$) then, given any two configurations $x_1, x_2 \in X$, there exists a configuration $x \in X$ such that $x|_{\Omega_1} = x_1|_{\Omega_1}$ and $x|_{\Omega_2} = x_2|_{\Omega_2}$.
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Fiorenzi extended the Garden of Eden theorem in the following way:

**Theorem (Fio-2003)**

Let $G$ be a countable amenable group and $A$ a finite set. Then every strongly irreducible subshift of finite type $X \subset A^G$ has the Moore-Myhill property.

**Example**

The hard sphere model is the subshift $X \subset \{0, 1\}^{\mathbb{Z}^d}$ consisting of all $x: \mathbb{Z}^d \to \{0, 1\}$ with no two 1s appearing at Euclidean distance 1 on $\mathbb{Z}^d$.

The hard sphere model is strongly irreducible and of finite type. Thus, it has the Moore-Myhill property.

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For $d = 1$, the hard sphere model is also called the golden mean subshift because its topological entropy is equal to the golden mean.
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Note that the even subshift is not of finite type. Actually, Fiorenzi [Fio-2000] proved that the even subshift does not have the Moore property: it admits endomorphisms that are surjective but not pre-injective.
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Let \((X, G)\) be a dynamical system and let \(d\) be a metric on \(X\) that is compatible with the topology.

**Definition**
The d.s. \((X, G)\) is expansive if there is a constant \(\varepsilon > 0\) such that, for all distinct points \(x, y \in X\), there exists \(g \in G\) such that 

\[d(gx, gy) \geq \varepsilon.\]

This definition does not depend on the choice of \(d\).

**Example**
Arnold's cat is expansive.

**Example**
All shifts and subshifts are expansive.
Expansive dynamical systems

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The Myhill property for a class of expansive dynamical systems

Theorem (CC-2015b)

Let $X$ be a compact metrizable space equipped with a continuous action of a countable amenable group $G$.

Suppose that the d.s. $(X, G)$ is expansive and that there exist a finite set $A$, a strongly irreducible subshift $\Sigma \subset A^G$, and a continuous, surjective, $G$-equivariant and uniformly finite-to-one map $\pi : \Sigma \to X$.

Then the dynamical system $(X, G)$ has the Myhill property.
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Smale’s Axiom A diffeomorphisms

Let \( f : M \to M \) be a diffeomorphism of a smooth compact manifold \( M \).

A closed \( f \)-invariant subset \( \Lambda \subset M \) is hyperbolic if the restriction to \( \Lambda \) of the tangent bundle of \( M \) splits as a direct sum of two invariant subbundles \( E^s \) and \( E^u \) such that, with respect to some (or equivalently any) Riemannian metric on \( M \), the differential \( df \) is uniformly contracting on \( E^s \) and uniformly expanding on \( E^u \).

A point \( x \in M \) is called non-wandering if for every neighborhood \( U \) of \( x \), there is an integer \( n \geq 1 \) such that \( f^n(U) \) meets \( U \). The set \( \Omega(f) \) consisting of all non-wandering points of \( f \) is a closed invariant subset of \( M \).

If \( \text{Per}(f) \) denotes the set of periodic points of \( f \), one always has \( \text{Per}(f) \subset \Omega(f) \).

Definition

One says that \( f \) is Axiom A if \( \Omega(f) \) is hyperbolic, and \( \text{Per}(f) \) is dense in \( \Omega(f) \).

If \( f \) is Axiom A, then \( \Omega(f) \) can be uniquely written as a disjoint union of closed invariant subsets \( \Omega(f) = \bigcup_{i=1}^{k} X_i \), such that the restriction of \( f \) to each \( X_i \) is topologically transitive (spectral decomposition theorem).

These subsets \( X_i \) are called the basic sets of \((M, f)\).
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Let $f : M \to M$ be a diffeomorphism of a smooth compact manifold $M$. A closed $f$-invariant subset $\Lambda \subset M$ is **hyperbolic** if the restriction to $\Lambda$ of the tangent bundle of $M$ splits as a direct sum of two invariant subbundles $E_s$ and $E_u$ such that, with respect to some (or equivalently any) Riemannian metric on $M$, the differential $df$ is uniformly contracting on $E_s$ and uniformly expanding on $E_u$. 

A point $x \in M$ is called **non-wandering** if for every neighborhood $U$ of $x$, there is an integer $n \geq 1$ such that $f^n(U)$ meets $U$. The set $\Omega(f) = \{x \in M : x \text{ is non-wandering} \}$ is a closed invariant subset of $M$. If $\text{Per}(f)$ denotes the set of periodic points of $f$, one always has $\text{Per}(f) \subset \Omega(f)$.

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A dynamical system \((X, G)\) is topologically mixing if, given any two non-empty open subsets \(U, V \subset X\), one has \(U \cap gV \neq \emptyset\) for all but finitely many \(g \in G\).

Corollary (CC-2015a)

Let \(f\) be an Axiom A diffeomorphism of a smooth compact manifold \(M\). Suppose that \(X\) is a topologically mixing basic set of \((M, f)\). Then the dynamical system \((X, f|_X)\) has the Myhill property.

Proof.
The fact that the dynamical system \((X, f|_X)\) satisfies the hypotheses of the theorem follows from results obtained by Rufus Bowen in the 1970s.
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Let $f: M \to M$ be a diffeomorphism of a smooth compact manifold $M$. One says that $f$ is Anosov if the whole manifold $M$ is hyperbolic for $f$.

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Example (Hyperbolic toral automorphisms) Consider a matrix $A \in \text{GL}_n(\mathbb{Z})$ with no eigenvalue of modulus 1. Then $A$ induces a topologically mixing Anosov diffeomorphism $f_A$ of the $n$-torus $T^n = \mathbb{R}^n / \mathbb{Z}^n$. One says that $f_A$ is a hyperbolic toral automorphism.

Arnold’s cat is the hyperbolic toral automorphism associated with the matrix $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$.

Every Anosov diffeomorphisms of $T^n$ is topologically conjugate to a hyperbolic toral automorphism. In particular, every Anosov diffeomorphism of $T^n$ is topologically mixing.

Theorem (CC-2015a) Let $f$ be an Anosov diffeomorphism of the $n$-torus $T^n$. Then the d.s. $(T^n, f)$ has the Moore-Myhill property.
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