Some Remarks on Sofic Monoids

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“Groups acting on rooted trees”, 24–28 February 2014, Institut Henri Poincaré, Paris, France

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We introduce a notion of soficity for monoids.

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Remark

Every monoid M embeds as a submonoid of $\mathrm{Map}(M)$ via its Cayley map

$$\begin{aligned} M &\hookrightarrow \mathrm{Map}(M) \\ s &\mapsto (t \mapsto st). \end{aligned}$$

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The Hamming metric on $\mathrm{Map}(X)$ is the metric defined by

$$d_X^{\mathrm{Ham}}(f,g) := \frac{1}{|X|}|\{x \in X : f(x) \neq g(x)\}| \quad \forall f,g \in \mathrm{Map}(X)$$

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Thus the Hamming distance between f and g is the proportion of elements of X at which f and g take different values.

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$$d(\varphi(k_1 k_2), \varphi(k_1)\varphi(k_2)) \leq \varepsilon \quad \forall k_1, k_2 \in K$$

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Definition

A monoid M is called sofic if for every finite subset $K \subset M$ and every $\varepsilon > 0$, there exist a finite set $X \neq \varnothing$ and a map

$$\varphi \colon M \to \mathrm{Map}(X)$$

that is a $(K, 1-\varepsilon)$-injective (K, ε)-morphism with respect to the Hamming metric on $\mathrm{Map}(X)$.

Proposition

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Proof.

If N is a submonoid of a monoid M, $K \subset N$, and $\varphi \colon M \to \mathrm{Map}(X)$ is a $(K, 1-\varepsilon)$-injective (K, ε)-morphism, so is the restriction of φ to N. $\square$

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As every monoid embeds in its symmetric monoid, it suffices to prove that $\mathrm{Map}(X)$ is sofic for every finite set $X \neq \varnothing$.

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$$\Delta(f)(x_1, \dots, x_n) := (f(x_1), \dots, f(x_n))$$

for all $f \in \mathrm{Map}(X)$ and $(x_1, \dots, x_n) \in X^n$.

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$$\Delta(f)(x_1, \dots, x_n) = \Delta(g)(x_1, \dots, x_n) \iff (f(x_i) = g(x_i) \text{ for all } 1 \leq i \leq n,$$

and hence

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If $k := |X|$ and $f \neq g$, this implies that

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We deduce that the monoid morphism δ is $(\mathrm{Map}(X), 1 - \varepsilon)$-injective for n large enough. $\square$

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Every cancellative one-sided amenable monoid is sofic.

Proof.

All amenable groups are sofic [Wei-2000] and it is known [WW-1967] that every cancellative one-sided amenable semigroup can be embedded in an amenable group. □

Examples of Sofic Monoids (continued)

Let $\mathcal{C}$ be a class of monoids. One says that a monoid M is locally embeddable in $\mathcal{C}$ if, for every finite subset $K \subset M$, there exists a monoid $N \in \mathcal{C}$ and a map $\varphi \colon M \to N$ satisfying the following properties:

- the restriction of φ to K is injective,
- $\varphi(k_1 k_2) = \varphi(k_1)\varphi(k_2) \quad \forall k_1, k_2 \in K$,
- $\varphi(1_M) = 1_N$.

(note that φ is not required to be globally injective nor to be a monoid morphism).

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Every monoid that is locally embeddable in the class of sofic monoids is itself sofic. $\square$

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Let $\mathcal{P}$ be a property of monoids. One says that a monoid M is locally $\mathcal{P}$ if every finitely generated submonoid of M satisfies $\mathcal{P}$. One says that a monoid M is residually $\mathcal{P}$ if, given any pair of distinct elements $m_1, m_2 \in M$, there exist a monoid N satisfying $\mathcal{P}$ and a monoid morphism $\varphi \colon M \to N$ such that $\varphi(m_1) \neq \varphi(m_2)$.

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Every linear monoid is sofic. □

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Proposition

If a monoid M is obtained by adjoining to a semigroup S an identity element $1_M \notin S$, then M is sofic. $\square$

Remark

The hypothesis on M in the previous statement amounts to saying that M has no non-trivial one-sided invertible element, i.e., it satisfies

$$xy = 1_M \Longrightarrow x = y = 1_M.$$

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$$p(n) = \begin{cases} n - 1 & \text{if } n \geq 1 \\ 0 & \text{if } n = 0 \end{cases} \quad \text{and} \quad q(n) = n + 1 \quad \forall n \in \mathbb{N}.$$

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Proposition

The bicyclic monoid B is not sofic.

Non-Soficity of the Bicyclic Monoid

Lemma

Let X be a non-empty finite set. Then one has

$$d_X^{Ham}(fg, \mathrm{Id}_X) = d_X^{Ham}(gf, \mathrm{Id}_X), \quad \forall f, g \in \mathrm{Map}(X).$$

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$$d_X^{\mathrm{Ham}}(fg, \mathrm{Id}_X) = 1 - \frac{\mathrm{Tr}(\Phi(fg))}{|X|} = 1 - \frac{\mathrm{Tr}(\Phi(f)\Phi(g))}{|X|}$$

for all $f, g \in \mathrm{Map}(X)$, and hence $d_X^{\mathrm{Ham}}(fg, \mathrm{Id}_X) = d_X^{\mathrm{Ham}}(gf, \mathrm{Id}_X)$ since $\mathrm{Tr}(\Phi(f)\Phi(g)) = \mathrm{Tr}(\Phi(g)\Phi(f))$. □

Proof that B is not sofic

Take $K := \{1_B, p, q, qp\}$ and $0 < \varepsilon < \dfrac{1}{5}$. Suppose that X is a non-empty finite set and that $\varphi \colon B \to \mathrm{Map}(X)$ is a $(K, 1-\varepsilon)$-injective (K,ε)-morphism. Let $f := \varphi(p)$, $g := \varphi(q) \in \mathrm{Map}(X)$. We then have

$$\begin{aligned}
d_X^{\mathrm{Ham}}(fg, \mathrm{Id}_X) &= d_X^{\mathrm{Ham}}(\varphi(p)\varphi(q), \mathrm{Id}_X) \\
&\leq d_X^{\mathrm{Ham}}(\varphi(pq), \mathrm{Id}_X) + d_X^{\mathrm{Ham}}(\varphi(pq), \varphi(p)\varphi(q)) \quad \text{(by the triangle inequality)} \\
&= d_X^{\mathrm{Ham}}(\varphi(1_B), \mathrm{Id}_X) + d_X^{\mathrm{Ham}}(\varphi(pq), \varphi(p)\varphi(q)) \quad \text{(since } pq = 1_B\text{)} \\
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By applying the preceding lemma, we obtain

$$d_X^{\mathrm{Ham}}(gf, \mathrm{Id}_X) \leq 2\varepsilon. \tag{1}$$

Proof that B is not sofic

Take $K := \{1_B, p, q, qp\}$ and $0 < \varepsilon < \dfrac{1}{5}$. Suppose that X is a non-empty finite set and that $\varphi\colon B \to \mathrm{Map}(X)$ is a $(K, 1-\varepsilon)$-injective (K,ε)-morphism. Let $f := \varphi(p)$, $g := \varphi(q) \in \mathrm{Map}(X)$. We then have

$$\begin{aligned} d_X^{\mathrm{Ham}}(fg, \mathrm{Id}_X) &= d_X^{\mathrm{Ham}}(\varphi(p)\varphi(q), \mathrm{Id}_X) \\ &\leq d_X^{\mathrm{Ham}}(\varphi(pq), \mathrm{Id}_X) + d_X^{\mathrm{Ham}}(\varphi(pq), \varphi(p)\varphi(q)) \quad \text{(by the triangle inequality)} \\ &= d_X^{\mathrm{Ham}}(\varphi(1_B), \mathrm{Id}_X) + d_X^{\mathrm{Ham}}(\varphi(pq), \varphi(p)\varphi(q)) \quad (\text{since } pq = 1_B) \\ &\leq 2\varepsilon. \end{aligned}$$

By applying the preceding lemma, we obtain

$$d_X^{\mathrm{Ham}}(gf, \mathrm{Id}_X) \leq 2\varepsilon. \tag{1}$$

Finally, using again the triangle inequality, we get

$$\begin{aligned} d_X^{\mathrm{Ham}}(\varphi(qp), \varphi(1_B)) &\leq d_X^{\mathrm{Ham}}(\varphi(qp), gf) + d_X^{\mathrm{Ham}}(gf, \mathrm{Id}_X) + d_X^{\mathrm{Ham}}(\varphi(1_B), \mathrm{Id}_X) \\ &\leq d_X^{\mathrm{Ham}}(\varphi(qp), \varphi(q)\varphi(p)) + 2\varepsilon + d_X^{\mathrm{Ham}}(\varphi(1_B), \mathrm{Id}_X) \quad \text{(by (1))} \\ &\leq 4\varepsilon \qquad (\text{since } \varphi \text{ is a } (K,\varepsilon)\text{-morphism}). \end{aligned}$$

This contradicts the fact that φ is $(K, 1-\varepsilon)$-injective since qp and 1_B are distinct elements of K and $4\varepsilon < 1 - \varepsilon$. Consequently, the monoid B is not sofic.

Geometric Characterization of Sofic Monoids

Let M be a finitely generated monoid and $\Sigma \subset M$ a finite generating subset.

Let M be a finitely generated monoid and $\Sigma \subset M$ a finite generating subset.
The Cayley graph of (M, Σ) is the Σ-labelled graph $\mathcal{C}(M, \Sigma)$ with vertex set $V := M$ and edge set

$$E := \{(s, \sigma, s\sigma) : s \in M, \sigma \in \Sigma\} \subset V \times \Sigma \times V.$$

This means that t there is an oriented edge labelled σ from s to $s\sigma$ for all $s \in M$ and $\sigma \in \Sigma$.

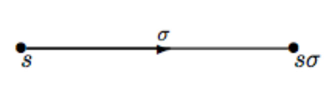

Figure 1: An edge in the Cayley graph

Geometric Characterization of Sofic Monoids (continued)

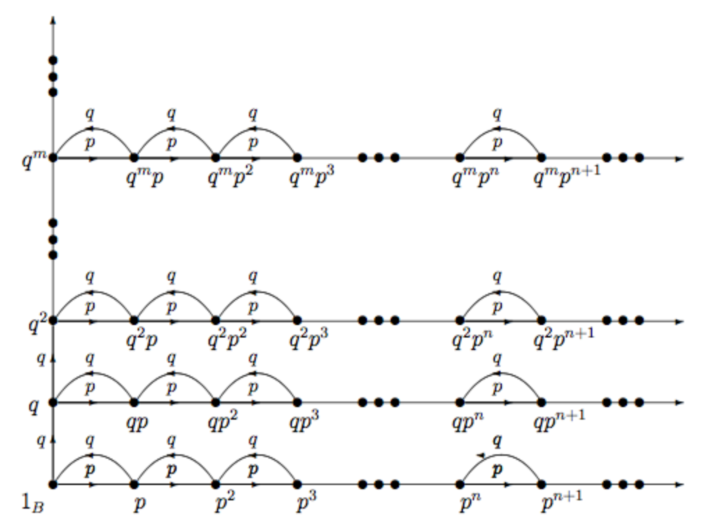

Figure 2: The cayley graph of the bicyclic monoid B

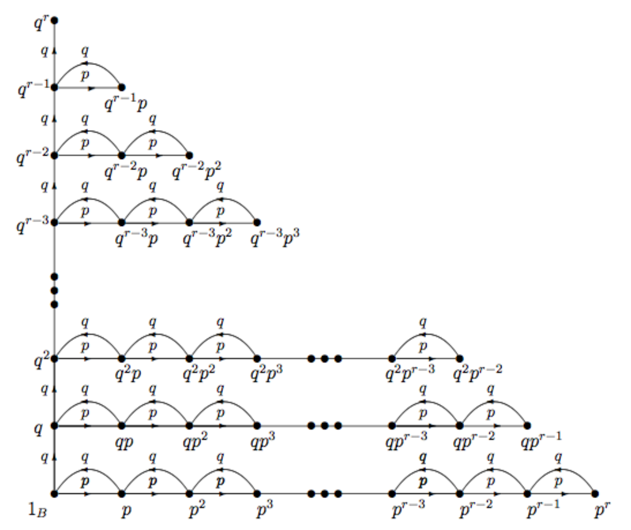

Figure 3: The ball $B_r(1_B)$ in the Cayley graph of the bicyclic monoid B

Theorem

Let M be a finitely generated left-cancellative monoid and $\Sigma \subset M$ a finite generating subset of M. Then the following conditions are equivalent:

(a) *the monoid M is sofic;*

(b) *for every $r \in \mathbb{N}$ and every $\varepsilon > 0$, there exists a finite Σ-labeled graph $\mathcal{G} = (V, E)$ with the following property: the subset $V(r) \subset V$, consisting of all the vertices $v \in V$ such that the ball of radius r centered at v in $\mathcal{G}$ is isomorphic, as a pointed Σ-labeled graph, to the ball of radius r centered at 1_M in the Cayley graph $\mathcal{C}(M, \Sigma)$, satisfies*

$$|V(r)| \geq (1 - \varepsilon)|V|.$$

□

Geometric Characterization of Sofic Monoids (continued)

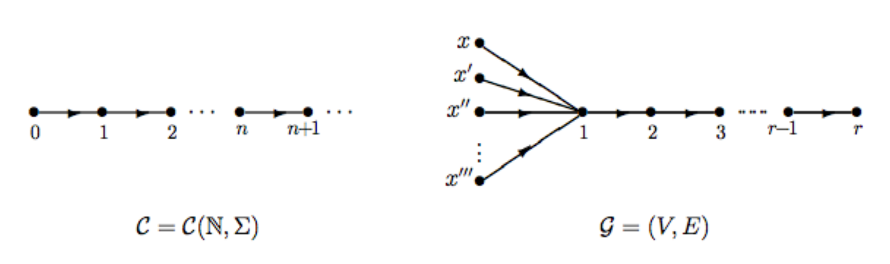

Figure 4: Graph-theoretic proof that the monoid $\mathbb{N}$ is sofic

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