Algebraic Cellular Automata and Groups

Michel Coornaert

IRMA, Strasbourg, France

International Conference on Geometry and Analysis, Kyoto, Japan

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Ideas and results are mostly taken from:

[Gr-1999] M. Gromov, *Endomorphisms of symbolic algebraic varieties*, J. Eur. Math. Soc. (JEMS) **1** (1999), 109–197.

The space of configurations

Take:

• a group G,

• a set A (called the alphabet or the set of symbols).

The set

$$A^G = \{x \colon G \to A\}$$

is endowed with its prodiscrete topology, i.e., the product topology obtained by taking the discrete topology on each factor A of A^G.

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Thus, a base of open neighborhoods of $x \in A^G$ is provided by the sets

$$V(x, \Omega) := \{y \in A^G : x|_\Omega = y|_\Omega\},$$

where Ω runs over all finite subsets of G (we denote by $x|_\Omega \in A^\Omega$ the restriction of $x \in A^G$ to Ω).

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Example

If G is countably infinite, A is finite of cardinality $|A| \geq 2$, then A^G is homeomorphic to the Cantor set. This is the case for $G = \mathbb{Z}$ and $A = \{0,1\}$, where A^G is the space of bi-infinite sequences of 0's and 1's.

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The space A^G is called the space of configurations over the group G and the alphabet A.

The shift action

There is a natural continuous left action of G on A^G given by

$$\begin{aligned} G \times A^G &\to A^G \\ (g, x) &\mapsto gx \end{aligned}$$

where

$$gx(h) = x(g^{-1}h) \quad \forall h \in G.$$

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The $\mathbb{Z}$-shift on $\{0,1\}^{\mathbb{Z}}$:

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The study of the shift action on A^G is the central theme in symbolic dynamics.

Cellular automata

Definition

A cellular automaton over the group G and the alphabet A is a map

$$\tau \colon A^G \to A^G$$

satisfying the following condition:
there exist a finite subset $M \subset G$ and a map $\mu_M \colon A^M \to A$ such that:

$$(\tau(x))(g) = \mu_M((g^{-1}x)|_M) \quad \forall x \in A^G, \forall g \in G,$$

where $(g^{-1}x)|_M$ denotes the restriction of the configuration $g^{-1}x$ to M.

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• Every cellular automaton $\tau \colon A^G \to A^G$ admits a minimal memory set M_0. It is characterized by the fact that a finite subset $M \subset G$ is a memory set for τ if and only if $M_0 \subset M$. Moreover, one then has

$$\mu_M = \mu_{M_0} \circ \pi,$$

where $\pi \colon A^M \to A^{M_0}$ denotes the projection map.

Example: Conway's Game of Life

Life was introduced by J. H. Conway in the 1970's.
Take $G = \mathbb{Z}^2$ and $A = \{0,1\}$.
Life is the cellular automaton

$$\tau\colon \{0,1\}^{\mathbb{Z}^2} \to \{0,1\}^{\mathbb{Z}^2}$$

with memory set $M = \{-1,0,1\}^2 \subset \mathbb{Z}^2$ and local defining map $\mu\colon A^M \to A$ given by

$$\mu_M(y) = \begin{cases} 1 & \text{if} \begin{cases} \displaystyle\sum_{m \in M} y(m) = 3 \\ \text{or} \displaystyle\sum_{m \in M} y(m) = 4 \text{ and } y((0,0)) = 1 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

$\forall y \in A^M$.

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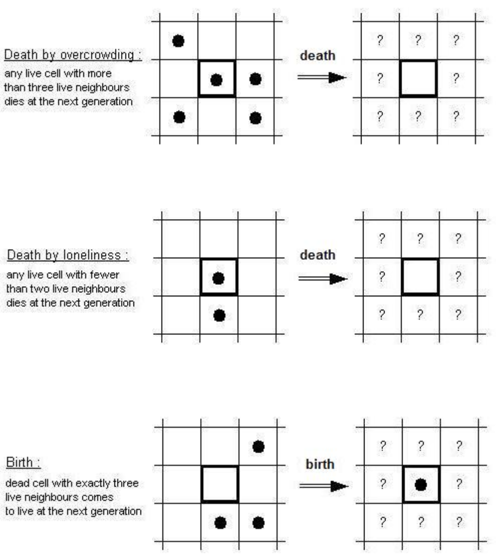

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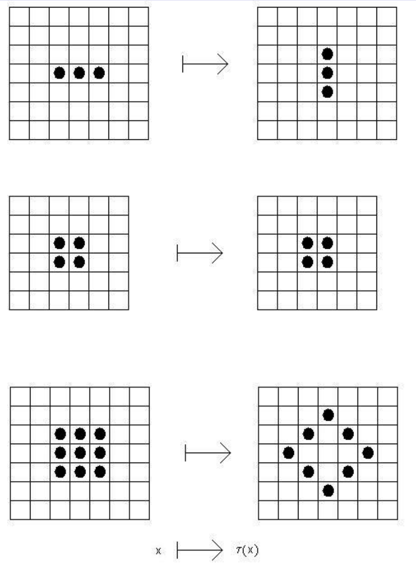

The Curtis-Hedlund theorem

From the definition, it easily follows that:

• Every cellular automaton $\tau \colon A^G \to A^G$ is G-equivariant, i.e.,

$$\tau(gx) = g\tau(x) \quad \forall x \in A^G, \forall g \in G.$$

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Conversely, one has the Curtis-Hedlund theorem:

Theorem (He-1969)

Let G be a group and let A be a finite set. Let $\tau \colon A^G \to A^G$ be a map. Then the following conditions are equivalent:

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Example (CC-2008)

For $G = A = \mathbb{Z}$, the map $\tau\colon A^G \to A^G$, defined by $\tau(x)(n) = x(x(n) + n)$, is G-equivariant and continuous, but τ is not a cellular automaton.

Uniform spaces

Let X be a set.
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Definition

A uniform structure on X is a non–empty set $\mathcal{U}$ of subsets of $X \times X$ called entourages satisfying the following conditions:

(UN-1) if $V \in \mathcal{U}$, then $\Delta_X \subset V$;

(UN-2) if $V \in \mathcal{U}$ and $V \subset V' \subset X \times X$, then $V' \in \mathcal{U}$;

(UN-3) if $V \in \mathcal{U}$ and $W \in \mathcal{U}$, then $V \cap W \in \mathcal{U}$;

(UN-4) if $V \in \mathcal{U}$, then $\overset{-1}{V} := \{(x,y) : (y,x) \in V\} \in \mathcal{U}$;

(UN-5) if $V \in \mathcal{U}$, then there exists $W \in \mathcal{U}$ such that
$W \circ W := \{(x,y) : \exists z \in X \text{ s. t. } (x,z),(z,y) \in W\} \subset V$.

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A set equipped with a uniform structure is called a uniform space.
The discrete uniform structure on X is the one for which every subset of $X \times X$ containing the diagonal is an entourage.
A map $f : X \to Y$ between uniform spaces is said to be uniformly continuous if

$$\forall W \text{ entourage of } Y, \exists V \text{ entourage of } X \text{ s. t.}$$
$$(f \times f)(V) \subset W$$

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A base of entourages for the prodiscrete uniform structure on A^G is provided by the sets:

$$N(\Omega) = \{(x, y) \in A^G \times A^G : x|_\Omega = y|_\Omega\} \subset A^G \times A^G,$$

where Ω runs over all finite subsets of G.

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Algebraic subsets

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Definition

A subset $A \subset K^m$ is called an algebraic subset if there exists a subset $S \subset K[t_1, \ldots, t_m]$ such that A is the set of common zeroes of the polynomials in S, i.e.,

$$A = \mathrm{Z}(S) = \{a = (a_1, \ldots, a_m) \in K^m : P(a) = 0 \quad \forall P \in S\}.$$

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A map $P \colon K^m \to K^n$ is called polynomial if there exist polynomials $P_1, \ldots, P_n \in K[t_1, \ldots, t_n]$ such that

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Definition

Let $A \subset K^m$ and $B \subset K^n$ be algebraic subsets.
A map $f \colon A \to B$ is called regular if f is the restriction of some polynomial map $P \colon K^m \to K^n$.

The category of affine algebraic sets

The identity map on any algebraic subset is regular. The composite of two regular maps is regular.
Thus, the algebraic subsets of K^m, $m = 0, 1, \ldots,$ are the objects of a category whose morphisms are the regular maps.

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Thus, the algebraic subsets of K^m, $m = 0, 1, \ldots$, are the objects of a category whose morphisms are the regular maps.
This category is the category of affine algebraic sets over K.
This category admits finite direct products. Indeed, if $A \subset K^m$ and $B \subset K^n$ are algebraic subsets then

$$A \times B \subset K^m \times K^n = K^{m+n}$$

is also an algebraic subset. It is the direct product of A and B in the category of algebraic sets over K.

Definition

Let G be a group and let K be a field. One says that a cellular automaton $\tau\colon A^G \to A^G$ is an algebraic cellular automaton over K if:

- A is an affine algebraic set over K;
- for some (or, equivalently, any) memory set M, the associated local defining map $\mu_M\colon A^M \to A$ is regular.

Examples of algebraic cellular automata

1) The map $\tau\colon K^{\mathbb{Z}} \to K^{\mathbb{Z}}$ defined by

$$\tau(x)(n) = x(n+1) - x(n)^2 \quad \forall x \in K^{\mathbb{Z}}, \forall n \in \mathbb{Z},$$

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2) Let G be a group, A an affine algebraic set, $f \colon A \to A$ a regular map, and $g_0 \in G$. Then the map $\tau \colon A^G \to A^G$, defined by

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3) Let A be an affine algebraic group (e.g. $A = \mathrm{SL}_n(K)$). Then the map $\tau\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$, defined by

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Remark

Every cellular automaton with finite alphabet A may be regarded as an algebraic cellular automaton (embed A in some field K).

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If X is compact and Y Hausdorff, then every continuous map $f\colon X \to Y$ has the CIP. In particular, if A is a finite set, then every cellular automaton $\tau\colon A^G \to A^G$ has the CIP.

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Remark

When A is infinite and the group G is non-periodic, one can always construct a cellular automaton $\tau\colon A^G \to A^G$ which does not satisfy the closed image property [CC-2011].

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Theorem (Gr-1999, CC-2010a)

Let G be a group, K an uncountable algebraically closed field, and A an affine algebraic set over K. Then every algebraic cellular automaton $\tau\colon A^G \to A^G$ over K has the CIP with respect to the prodiscrete topology on A^G.

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- More generally, by a result of Malcev, any finitely generated linear group is residually finite. Recall that a group is called linear if one can find a field K such that G embeds into $\mathrm{GL}_n(K)$ for n large enough.

A group G is called residually finite if the intersection of its finite-index subgroups is reduced to the identity element.

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Corollary

Let G be a residually finite group (e.g., $G = \mathbb{Z}^d$), and K an uncountable algebraically closed field. Then every injective algebraic cellular automaton $\tau \colon A^G \to A^G$ over K is surjective and hence bijective.

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Remark

The polynomial map $f: \mathbb{Q} \to \mathbb{Q}$ given by $f(t) = t^3$ is injective but not surjective.

Proof of the corollary

A configuration $x \in A^G$ is periodic if its orbit under the G-shift is finite.

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If τ is injective, then f is injective and hence surjective by the Ax-Grothendieck theorem. Thus $\tau(\mathrm{Fix}(H)) = \mathrm{Fix}(H)$. As $x \in \mathrm{Fix}(H)$, this implies that every periodic configuration is in the image of τ.

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Thus $\tau(\mathrm{Fix}(H)) = \mathrm{Fix}(H)$. As $x \in \mathrm{Fix}(H)$, this implies that every periodic configuration is in the image of τ.
By density of periodic configurations and the CIP theorem, this implies that $\tau(A^G) = A^G$.

First ingredient in the proof of the CIP theorem

Let K be a field. If A is an affine algebraic set over K, the algebraic subsets of A are the closed subsets of a topology.

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Theorem (Chevalley)

Let K be an algebraically closed field. Let A and B be affine algebraic sets over K, and let $f : A \to B$ be a regular map. Then every constructible subset $C \subset A$ has a constructible image $f(C) \subset B$. In particular, $f(A)$ is a constructible subset of B.

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Remark

The image of the polynomial map $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(t) = t^2$ is $[0, \infty)$ which is not constructible in $\mathbb{R}$ for the Zariski topology (the only constructible subsets of $\mathbb{R}$ for the Zariski topology are the finite subsets of $\mathbb{R}$ and their complements).

Second ingredient in the proof of the CIP theorem

Lemma 1

Let K be an uncountable algebraically closed field and let A be an affine algebraic set over K. Suppose that $C_0, C_1, C_2, \ldots$ is a sequence of nonempty constructible subsets of A such that

$$C_0 \supset C_1 \supset C_2 \supset \ldots$$

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Then one has $\bigcap_{n \geq 0} C_n \neq \varnothing$.

Remark

The preceding lemma becomes false if the field K is countable, e.g., $K = \overline{\mathbb{Q}}$ or $K = \overline{F_p}$.

A real counterexample to the CIP

Here we take $G = \mathbb{Z}$ and $A = \mathbb{R}$.
Consider the algebraic cellular automaton $\tau \colon \mathbb{R}^{\mathbb{Z}} \to \mathbb{R}^{\mathbb{Z}}$ defined by

$$\tau(x)(n) = x(n+1) - x(n)^2 \quad \forall x \in \mathbb{R}^{\mathbb{Z}}, \forall n \in \mathbb{Z}.$$

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Thus $x(n)$ would have a finite limit as $n \to -\infty$.
This is impossible since $\alpha - \alpha^2 = 1$ has no real roots.

Reversibility

Definition

Let G be a group and let A be a set. A cellular automaton $\tau\colon A^G \to A^G$ is called *reversible* if τ is bijective and its inverse map $\tau^{-1}\colon A^G \to A^G$ is also a cellular automaton.

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Let $\tau \colon A^G \to A^G$ be a bijective cellular automaton. As τ is continuous and G-equivariant, its inverse map τ^{-1} is also G-equivariant and continuous by compactness of A^G. We deduce that τ^{-1} is a cellular automaton by the Curtis-Hedlund theorem. □

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Remark

When A is infinite and the group G is non-periodic, one can always construct a bijective cellular automaton $\tau \colon A^G \to A^G$ which is not reversible [CC-2011].

Reversibility of algebraic cellular automata

Theorem (CC-2010a)

Let G be a group, and K an uncountable algebraically closed field. Then every bijective algebraic cellular automaton $\tau \colon A^G \to A^G$ over K is reversible.

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Example

Let K be an uncountable algebraically closed field of characteristic $p > 0$ and consider the Frobenius automorphism $f\colon K \to K$ given by $\lambda \mapsto \lambda^p$. Then the map $\tau\colon K^G \to K^G$, defined by

$$\tau(x)(g) = f(x(g)) \quad \forall x \in K^G, \forall g \in G,$$

is a bijective algebraic cellular automaton with memory set $\{1_G\}$ and local defining map f. The inverse cellular automaton $\tau^{-1}\colon K^G \to K^G$ is given by

$$\tau^{-1}(x)(g) = f^{-1}(x(g)) \quad \forall x \in K^G, \forall g \in G,$$

Therefore τ^{-1} is not algebraic since f^{-1} is not polynomial.

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(Q3) — For $K = \overline{\mathbb{Q}}$ or $K = \overline{F_p}$, does there exist an injective algebraic cellular automaton $\tau\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ over K which is not surjective ?

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(Q4) — For $K = \overline{\mathbb{Q}}$ or $K = \overline{F_p}$, does there exist an algebraic cellular automaton $\tau\colon A^{\mathbb{Z}} \to A^{\mathbb{Z}}$ over K which does not satisfy the closed image property ?

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