The Garden of Eden theorem: old and new

Michel Coornaert

IRMA, Université de Strasbourg

“Groups and Computation”
Conference dedicated to the 80th birthday of Paul Schupp
Stevens Institute of Technology
June 26–30, 2017
This is joint work with Tullio Ceccherini-Silberstein.
Configurations and Shifts

Take a group $G$ (called the universe), a finite set $A$ (called the alphabet). The set $A^G = \{ x : G \to A \}$ is called the set of configurations. The shift on $A^G$ is the left action of $G$ on $A^G$ given by $G \times A^G \to A^G \ (g, x) \mapsto gx$ where $gx(h) = x(g^{-1}h) \ \forall \ h \in G$. 

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The Garden of Eden theorem

June 26, 2017
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Cellular Automata

Definition

A cellular automaton over the group $G$ and the alphabet $A$ is a map $\tau : A^G \to A^G$ satisfying the following condition: there exist a finite subset $M \subset G$ and a map $\mu : A^M \to A$ such that $\tau(x)(g) = \mu((g^{-1}x)|M) \forall x \in A^G, \forall g \in G$, where $(g^{-1}x)|M$ denotes the restriction of the configuration $g^{-1}x$ to $M$. Such a set $M$ is called a memory set and $\mu$ is called a local defining map for $\tau$. 

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Example: Conway’s Game of Life
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Death by overcrowding:
any live cell with more than three live neighbours dies at the next generation

Death by loneliness:
any live cell with fewer than two live neighbours dies at the next generation

Birth:
dead cell with exactly three live neighbours comes to live at the next generation
Example: Conway’s Game of Life (continued)
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Here \( G = \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z} \) and \( A = \{0, 1\} \).
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$$\tau : \{0, 1\}^{\mathbb{Z}^2} \rightarrow \{0, 1\}^{\mathbb{Z}^2}$$
Example: Conway’s Game of Life (continued)

Here $G = \mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ and $A = \{0, 1\}$. Life is described by the cellular automaton

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with memory set $M = \{-1, 0, 1\}^2 \subset \mathbb{Z}^2$ and local defining map $\mu: A^M \to A$ given by

$$\mu(y) = \begin{cases} 
1 & \text{if } \sum_{m \in M} y(m) = 3 \\
0 & \text{if } \sum_{m \in M} y(m) = 4 \text{ and } y((0, 0)) = 1 \\
0 & \text{otherwise}
\end{cases}$$

$\forall y \in A^M$. 
Let $\tau: A^G \rightarrow A^G$ be a cellular automaton.

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Two configurations $x_1, x_2 \in A^G$ are almost equal if they coincide outside of a finite subset of $G$.

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Two configurations $x_1, x_2 \in A^G$ form a diamond for $\tau$ if $x_1 \neq x_2$; $x_1$ and $x_2$ are almost equal; $\tau(x_1) = \tau(x_2)$.

**Definition**
One says that $\tau$ is pre-injective if it has no diamonds.
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Example

Take \( G = \mathbb{Z}_2 \) and \( A = \{0, 1\} \).

Conway's Game of Life \( \tau \): \( A^G \rightarrow A^G \) is not pre-injective.

The configurations \( x_1, x_2 \in A^G \) defined by

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x_1(g) = 0 \quad \forall g \in G
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and

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x_2(0) = 1 \quad \text{and} \quad x_2(g) = 0 \quad \forall g \in G \setminus \{0\}
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Diamonds and Pre-injectivity (continued)

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form a diamond.
Injectivity vs Pre-injectivity

Note that $\tau$ injective $\Rightarrow$ $\tau$ pre-injective. The converse is false.

Example: Take $G = \mathbb{Z}$, $A = \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$, and $\tau: A \times G \rightarrow A \times G$ given by $\tau(x)(g) = x(g) + x(g + 1)$ for all $x \in A \times G$, $g \in G$.

$\tau$ is a cellular automaton admitting $M = \{0, 1\} \subset G$ as a memory set.

$\tau$ is pre-injective.

$\tau$ is not injective (the two constant configurations have the same image).
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The GOE Theorem for $\mathbb{Z}^d$

The following theorem is due to Moore [Mo-1963] and Myhill [My-1963].

Theorem (GOE theorem)

Let $G = \mathbb{Z}^d$ and $A$ a finite set. Let $\tau: A^G \rightarrow A^G$ be a cellular automaton. Then

\[ \tau \text{ surjective} \iff \tau \text{ pre-injective.} \]

Moore's implication

Moore: $\Rightarrow$

Myhill: $\Leftarrow$

Example ($G = \mathbb{Z}^2$)

Conway's Game of Life is not pre-injective. Therefore it is not surjective by Moore's implication.
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The GOE theorem for Groups of Subexponential Growth

Schupp [S-1988] asked the following.

**Question**
Is the analogue of the Moore-Myhill theorem true exactly for virtually nilpotent groups?

**Definition**
A group $G$ with finite generating set $S$ has subexponential growth if
$$\lim_{n \to \infty} \frac{\log |B_n|}{n} = 0,$$
where $B_n$ is a ball of radius $n$ in the Cayley graph of $(G, S)$ and $|\cdot|$ denotes cardinality.

Machı and Mignosi [MM-1993] proved that the GOE theorem remains valid when $G$ is a f.g. group with subexponential growth.

Every f.g. virtually nilpotent group has subexponential growth but there are f.g. groups of subexponential growth that are not virtually nilpotent.
The first examples of such groups were given by Grigorchuk [Gri-1984].
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The GOE Theorem for Amenable Groups

Definition

A group $G$ is amenable if there exists a finitely-additive invariant probability measure defined on the set of all subsets of $G$. All f.g. groups of subexponential growth, all solvable groups, all locally finite groups are amenable.

Ceccherini-Silberstein, Machì and Scarabotti [CMS-1999] proved that the GOE theorem remains valid for amenable groups.

Bartholdi [B-2010] proved that if $G$ is a non-amenable group then $G$ does not satisfy Moore’s implication, i.e., there exist a finite set $A$ and a cellular automaton $\tau: A \rightarrow G$ that is surjective but not pre-injective.

Bartholdi and Kielak [BK-2016] proved that if $G$ is a non-amenable group then $G$ does not satisfy Myhill’s implication either, i.e., there exist a finite set $A$ and a cellular automaton $\tau: A \rightarrow G$ that is pre-injective but not surjective.
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What Gromov Said

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A dynamical system is a pair \((X,G)\), where \(X\) is a compact metrizable topological space, \(G\) is a countable group acting continuously on \(X\). The space \(X\) is called the phase space. If \(f: X \rightarrow X\) is a homeomorphism, the d.s. generated by \(f\) is the d.s. \((X,Z)\), where \(Z\) acts on \(X\) by \((n,x) \mapsto f^n(x)\) for all \(n \in Z\), \(x \in X\). This d.s. is also denoted \((X,f)\).
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Examples of Dynamical Systems

Example

Let $A$ be a finite set and $G$ a countable group. Equip $A$ with its discrete topology and $A^G$ with the product topology. Then the shift $(A^G, G)$ is a d.s.

Example (Arnold's cat)

This is the d.s. $(\mathbb{T}^2, f)$, where $f$ is the automorphism of the 2-torus $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ given by $f(x) = (x_2 x_1 + x_2) \forall x = (x_1, x_2) \in \mathbb{T}^2$. Thus we have $f(x) = Ax$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is the cat matrix.
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Thus we have $f(x) = Ax$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is the cat matrix.
Let $(X, G)$ be a dynamical system. Let $d$ be a metric on $X$ that is compatible with the topology.

**Definition**

Two points $x, y \in X$ are called homoclinic if

$$\lim_{g \to \infty} d(gx, gy) = 0,$$

i.e., for every $\varepsilon > 0$, there exists a finite subset $F \subset G$ such that

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i.e., for every \(\varepsilon > 0\), there exists a finite subset \(F \subset G\) such that

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Homoclinicity is an equivalence relation on \(X\).
Let \((X, G)\) be a dynamical system. Let \(d\) be a metric on \(X\) that is compatible with the topology.

**Definition**

Two points \(x, y \in X\) are called **homoclinic** if

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Homoclinicity is an equivalence relation on \(X\). This relation does not depend on the choice of \(d\).
Homoclinicity (continued)

Example

Let $A$ be a finite set and $G$ a countable group. Consider the shift $(A, G)$. Two configurations $x, y \in A^G$ are homoclinic if and only if they are almost equal.

Example

Consider Arnold's cat $(T^2, f)$. Equip $T^2 = \mathbb{R}^2 / \mathbb{Z}^2$ with its Euclidean structure. The homoclinicity class of a point $x \in T^2$ is $D \cap D'$, where $D$ is the line passing through $x$ whose slope is the golden mean $1 + \sqrt{5}/2 = 1.618...$ and $D'$ is the line passing through $x$ and orthogonal to $D'$. The slopes of $D$ and $D'$ are the eigenvalues of the cat matrix. Each homoclinicity class is countably-infinite and dense in $T^2$.
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Let \((X, G)\) be a dynamical system.

**Definition**

An endomorphism of the d.s. \((X, G)\) is a continuous map \(\tau: X \to X\) such that \(\tau\) commutes with the action of \(G\), that is, such that \(\tau(gx) = g\tau(x)\) for all \(g \in G\), \(x \in X\).

**Example**

Let \(A\) be a finite set and \(G\) a countable group. Then the endomorphisms of the shift \((A^G, G)\) are precisely the cellular automata \(\tau: A^G \to A^G\) (Curtis-Hedlund-Lyndon theorem).

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The Garden of Eden theorem

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Endomorphisms of Dynamical Systems

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An endomorphism $\tau: X \to X$ of the d.s. $(X, G)$ is called pre-injective if its restriction to each homoclinicity class is injective.

**Example**
For shift systems $(A^G, G)$, the two definitions of pre-injectivity are equivalent.

**Example**
The group endomorphism $\tau: T^2 \to T^2$, given by $\tau(x) := 2x$ for all $x \in T^2$, is an endomorphism of Arnold’s cat $(T^2, f)$.

The kernel of $\tau$ consists of four points: $\text{Ker}(\tau) = \{ (0,0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2}) \}$.

The endomorphism $\tau$ is pre-injective but not injective.
Pre-injective Endomorphisms

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Let \((X, G)\) be a dynamical system.

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One says that the d.s. \((X, G)\) satisfies the Garden of Eden theorem if every endomorphism \(\tau: X \to X\) of \((X, G)\) satisfies \(\tau\) surjective \(\iff\) \(\tau\) pre-injective.

**Example**

Arnold's cat \((T^2, f)\) satisfies the GOE theorem. Indeed, it is easy to show, using spectral analysis, that any endomorphism \(\tau\) of the cat is of the form \(\tau = m\text{Id} + nf\), for some \(m, n \in \mathbb{Z}\). With the exception of the 0-endomorphism, every endomorphism of the cat is both surjective and pre-injective.
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Anosov Diffeomorphisms

Let $f: M \to M$ be a diffeomorphism of a smooth compact manifold $M$. One says that $f$ is Anosov if the tangent bundle $TM$ of $M$ continuously splits as a direct sum $TM = E_s \oplus E_u$ of two $df$-invariant subbundles $E_s$ and $E_u$ such that, with respect to some (or equivalently any) Riemannian metric on $M$, the differential $df$ is exponentially contracting on $E_s$ and exponentially expanding on $E_u$, i.e., there are constants $C > 0$ and $0 < \lambda < 1$ such that

$$\|df^n(v)\| \leq C \lambda^n \|v\|,$$

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for all $x \in M$, $v \in E_s(x)$, $w \in E_u(x)$, and $n \geq 0$.

Example
Arnold's cat is Anosov. If we identify the tangent space at $x \in T^2$ with $\mathbb{R}^2$, the two eigenlines of the cat matrix yield $E_u(x)$ and $E_s(x)$.Michel Coornaert (IRMA, Université de Strasbourg)
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**Example**

Arnold’s cat is Anosov. If we identify the tangent space at $x \in \mathbb{T}^2$ with $\mathbb{R}^2$, the two eigenlines of the cat matrix yield $E_u(x)$ and $E_s(x)$. 

Arnold’s cat can be generalized as follows. Consider a matrix $A \in \text{GL}_n(\mathbb{Z})$ with no eigenvalue of modulus 1. Then the map $f : T^n \to T^n$ defined by $x \mapsto Ax$ is an Anosov diffeomorphism of the $n$-dimensional torus $T^n := \mathbb{R}^n / \mathbb{Z}^n$. One says that $f$ is the hyperbolic toral automorphism associated with $A$. 

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Hyperbolic toral automorphisms
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Theorem (CC-2016)

Let $f$ be an Anosov diffeomorphism of the $n$-dimensional torus $T^n$. Then the d.s. $(T^n, f)$ satisfies the GOE theorem.

The proof uses two classical results:

Result 1 (Franks [Fra-1970], Manning [Man-1974]) Every Anosov diffeomorphisms of $T^n$ is topologically conjugate to a hyperbolic toral automorphism.

Result 2 (Walters [Wal-1968]) Every endomorphism of a hyperbolic toral automorphism on $T^n$ is affine, i.e., of the form $x \mapsto Bx + c$, where $B$ is an integral $n \times n$ matrix and $c \in T^n$. 

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A GOE Theorem for Anosov Diffeomorphisms on Tori

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Let $f$ be an Anosov diffeomorphism of a smooth compact manifold $M$. Does the dynamical system $(M, f)$ satisfy the GOE theorem?

A homeomorphism $f$ of a topological space $X$ is topologically mixing if, given any two non-empty open subsets $U, V \subset X$, one has $U \cap f^n(V) \neq \emptyset$ for all but finitely many $n \in \mathbb{Z}$.

Theorem (CC-2015) Let $f$ be a topologically mixing Anosov diffeomorphism of a smooth compact manifold $M$. Then $(M, f)$ has the Myhill property, i.e., every pre-injective continuous map $\tau : M \to M$ commuting with $f$ is surjective.

Remark All known examples of Anosov diffeomorphisms are topologically mixing.
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Algebraic Dynamical Systems

Definition
An algebraic dynamical system is a d.s. \((X, G)\), where \(X\) is a compact metrizable abelian topological group and \(G\) is a countable group acting on \(X\) by continuous group automorphisms.

Example
Let \(G\) be a countable group and \(A\) a c.m.a.t. group. Then \(AG\) is a c.m.a.t. group. The shift system \((AG, G)\) is an a.d.s.

Example
Arnold’s cat \((T^2, \mathbb{Z})\) is an a.d.s.
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Arnold’s cat \((\mathbb{T}^2, \mathbb{Z})\) is an a.d.s.
Let $G$ be a countable group and denote by $\mathbb{Z}[G]$ its integral group ring. If $M$ is a countable left $\mathbb{Z}[G]$-module, then its Pontryagin dual $\hat{M}$ (the character group of the additive group $M$) is a c.m.a.t. group. $G$ acts on $M$ and hence (by dualizing) on $\hat{M}$ by continuous group automorphisms. $(\hat{M}, G)$ is an a.d.s. Every a.d.s. can be obtained in this way (see [Sch-1995]).

In the case $M = \mathbb{Z}[G]/\mathbb{Z}[G]f$, where $f \in \mathbb{Z}[G]$, one writes $X_f := \hat{M}$ and one says that $(X_f, G)$ is the principal a.d.s. associated with $f$. 
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A GOE Theorem for Principal Algebraic Dynamical Systems

Let $G$ be a countable abelian group (e.g. $G = \mathbb{Z}^d$). Let $f \in \mathbb{Z}[G]$ such that $f$ is invertible in $\ell_1(G)$ and $Xf$ is connected. Then the p.a.d.s. $(Xf, G)$ satisfies the GOE theorem.

The fact that $f \in \mathbb{Z}[G]$ is invertible in $\ell_1(G)$ is equivalent to the expansiveness of $(Xf, G)$.

A sufficient condition for $f \in \mathbb{Z}[G]$ to be invertible in $\ell_1(G)$ is that $f$ is lopsided, i.e., there exists $g_0 \in G$ such that $|f(g_0)| \geq \sum_{g \neq g_0} |f(g)|$. 

Michel Coornaert (IRMA, Université de Strasbourg)
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References


References (continued)


