The Garden of Eden theorem: old and new

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"Groups and Computation"
Conference dedicated to the 80th birthday of Paul Schupp
Stevens Institute of Technology
June 26–30, 2017

This is joint work with Tullio Ceccherini-Silberstein.

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The set

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is called the set of configurations.
The shift on A^G is the left action of G on A^G given by

$$\begin{aligned} G \times A^G &\to A^G \\ (g, x) &\mapsto gx \end{aligned}$$

where

$$gx(h) = x(g^{-1}h) \quad \forall h \in G.$$

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satisfying the following condition:
there exist a finite subset $M \subset G$ and a map $\mu \colon A^M \to A$ such that

$$(\tau(x))(g) = \mu((g^{-1}x)|_M) \quad \forall x \in A^G, \forall g \in G,$$

where $(g^{-1}x)|_M$ denotes the restriction of the configuration $g^{-1}x$ to M.

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Such a set M is called a memory set and μ is called a local defining map for τ.

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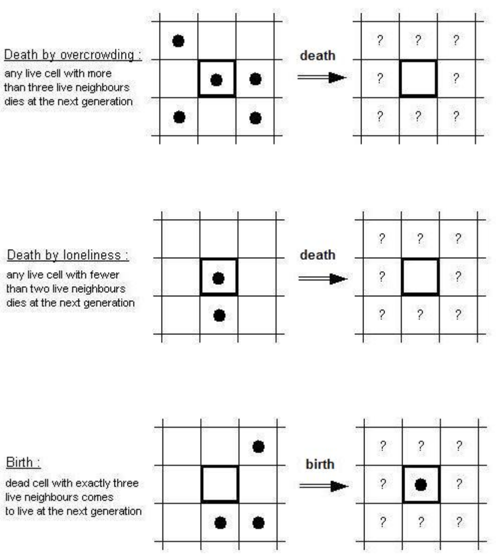

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with memory set $M = \{-1, 0, 1\}^2 \subset \mathbb{Z}^2$
and local defining map $\mu \colon A^M \to A$ given by

$$\mu(y) = \begin{cases} 1 & \text{if} \begin{cases} \displaystyle\sum_{m \in M} y(m) = 3 \\ \text{or} \displaystyle\sum_{m \in M} y(m) = 4 \text{ and } y((0,0)) = 1 \end{cases} \\ 0 & \text{otherwise} \end{cases}$$

$\forall y \in A^M$.

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Definition

One says that τ is pre-injective if it has no diamonds.

Diamonds and Pre-injectivity (continued)

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Example

Take $G = \mathbb{Z}^2$ and $A = \{0, 1\}$.
Conway's Game of Life $\tau \colon A^G \to A^G$ is not pre-injective.
The configurations $x_1, x_2 \in A^G$ defined by

$$x_1(g) = 0 \quad \forall g \in G$$

and

$$x_2(0_G) = 1 \quad \text{and} \quad x_2(g) = 0 \ \forall g \in G \setminus \{0_G\}$$

form a diamond.

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Example

Take $G = \mathbb{Z}$, $A = \{0,1\} = \mathbb{Z}/2\mathbb{Z}$, and $\tau \colon A^G \to A^G$ given by

$$\tau(x)(g) = x(g) + x(g+1) \quad \forall x \in A^G, g \in G.$$

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τ is pre-injective.
τ is not injective (the two constant configurations have the same image).

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Theorem (GOE theorem)

Let $G = \mathbb{Z}^d$ and A a finite set. Let $\tau \colon A^G \to A^G$ be a cellular automaton. Then

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Example ($G = \mathbb{Z}^2$)

Conway's Game of Life is not pre-injective.
Therefore it is not surjective by Moore's implication.

The GOE theorem for Groups of Subexponential Growth

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A group G with finite generating set S has subexponential growth if

$$\lim_{n\to\infty} \frac{\log |B_n|}{n} = 0,$$

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Every f.g. virtually nilpotent group has subexponential growth but there are f.g. groups of subexponential growth that are not virtually nilpotent. The first examples of such groups were given by Grigorchuk [Gri-1984].

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Bartholdi [B-2010] proved that if G is a non-amenable group then G does not satisfy Moore's implication, i.e., there exist a finite set A and a cellular automaton $\tau\colon A^G \to A^G$ that is surjective but not pre-injective.

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Bartholdi and Kielak [BK-2016] proved that if G is a non-amenable group then G does not satisfy Myhill's implication either, i.e., there exist a finite set A and a cellular automaton $\tau \colon A^G \to A^G$ that is pre-injective but not surjective.

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> *"...the Garden of Eden theorem can be generalized to a suitable class of hyperbolic actions ..."*

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If $f : X \to X$ is a homeomorphism, the d.s. generated by f is the d.s. $(X, \mathbb{Z})$, where $\mathbb{Z}$ acts on X by

$$(n, x) \mapsto f^n(x) \quad \forall n \in \mathbb{Z}, x \in X.$$

This d.s. is also denoted (X, f).

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Example (Arnold's cat)

This is the d.s. $(\mathbb{T}^2, f)$, where f is the automorphism of the 2-torus $\mathbb{T}^2 = \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$ given by

$$f(x) = \begin{pmatrix} x_2 \\ x_1 + x_2 \end{pmatrix} \quad \forall x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{T}^2.$$

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Thus we have $f(x) = Ax$, where $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is the cat matrix.

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Homoclinicity is an equivalence relation on X. This relation does not depend on the choice of d.

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Equip $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with its Euclidean structure.
The homoclinicity class of a point $x \in \mathbb{T}^2$ is $D \cap D'$, where D is the line passing through x whose slope is the golden mean $\dfrac{1+\sqrt{5}}{2} = 1.618\ldots$ and D' is the line passing through x and orthogonal to D'.

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Each homoclinicity class is countably-infinite and dense in $\mathbb{T}^2$.

Endomorphisms of Dynamical Systems

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An endomorphism of the d.s. (X, G) is a continuous map $\tau\colon X \to X$ such that τ commutes with the action of G, that is, such that

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The kernel of τ consists of four points:

$$\mathrm{Ker}(\tau) = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \end{pmatrix}, \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} \right\}.$$

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The endomorphism τ is pre-injective but not injective.

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Arnold's cat $(\mathbb{T}^2, f)$ satisfies the GOE theorem. Indeed, it is easy to show, using spectral analysis, that any endomorphism τ of the cat is of the form $\tau = m \operatorname{Id} + nf$, for some $m, n \in \mathbb{Z}$. With the exception of the 0-endomorphism, every endomorphism of the cat is both surjective and pre-injective.

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One says that f is Anosov if the tangent bundle TM of M continuously splits as a direct sum $TM = E_s \oplus E_u$ of two df-invariant subbundles E_s and E_u such that, with respect to some (or equivalently any) Riemannian metric on M, the differential df is exponentially contracting on E_s and exponentially expanding on E_u, i. e., there are constants $C > 0$ and $0 < \lambda < 1$ such that

- $\|df^n(v)\| \leq C\lambda^n\|v\|$,
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Example

Arnold's cat is Anosov.
If we identify the tangent space at $x \in \mathbb{T}^2$ with $\mathbb{R}^2$, the two eigenlines of the cat matrix yield $E_u(x)$ and $E_s(x)$.

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Consider a matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ with no eigenvalue of modulus 1. Then the map

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One says that f is the hyperbolic toral automorphism associated with A.

A GOE Theorem for Anosov Diffeomorphisms on Tori

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Result 2 (Walters [Wal-1968]) Every endomorphism of a hyperbolic toral automorphism on $\mathbb{T}^n$ is affine, i. e., of the form $x \mapsto Bx + c$, where B is an integral $n \times n$ matrix and $c \in \mathbb{T}^n$.

General Anosov Diffoemorphisms

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A homeomorphism f of a topological space X is topologically mixing if, given any two non-empty open subsets $U, V \subset X$, one has $U \cap f^n(V) \neq \varnothing$ for all but finitely many $n \in \mathbb{Z}$.

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Theorem (CC-2015)

Let f be a topologically mixing Anosov diffeomorphism of a smooth compact manifold M. Then (M, f) has the Myhill property, i.e., every pre-injective continuous map $\tau \colon M \to M$ commuting with f is surjective.

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Remark

All known examples of Anosov diffeomorphisms are topologically mixing.

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Every a.d.s. can be obtained in this way (see [Sch-1995]).

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Every a.d.s. can be obtained in this way (see [Sch-1995]).
In the case $M = \mathbb{Z}[G]/\mathbb{Z}[G]f$, where $f \in \mathbb{Z}[G]$, one writes $X_f := \widehat{M}$ and one says that (X_f, G) is the principal a.d.s. associated with f.

A GOE Theorem for Principal Algebraic Dynamical Systems

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The fact that $f \in \mathbb{Z}[G]$ is invertible in $\ell^1(G)$ is equivalent to the expansiveness of (X_f, G).

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The fact that $f \in \mathbb{Z}[G]$ is invertible in $\ell^1(G)$ is equivalent to the expansiveness of (X_f, G).
A sufficient condition for $f \in \mathbb{Z}[G]$ to be invertible in $\ell^1(G)$ is that f is lopsided, i.e., there exists $g_0 \in G$ such that

$$|f(g_0)| \geq \sum_{g \neq g_0} |f(g)|.$$

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