High speed excited multi-solitons in nonlinear Schrödinger equations

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Abstract

We consider the nonlinear Schrödinger equation in $\mathbb{R}^d$

$$i\partial_t u + \Delta u + f(u) = 0.$$

For $d \geqslant 2$, this equation admits travelling wave solutions of the form $e^{i\omega t}\Phi(x)$ (up to a Galilean transformation), where Φ is a fixed profile, solution to $-\Delta\Phi + \omega\Phi = f(\Phi)$, but *not the ground state.* This kind of profiles are called excited states. In this paper, we construct solutions to NLS behaving like a sum of N excited states which spread up quickly as time grows (which we call multi-solitons). We also show that if the flow around one of these excited states is linearly unstable, then the multi-soliton is not unique, and is unstable.

Résumé

On considère l'équation de Schrödinger non-linéaire dans $\mathbb{R}^d$

$$i\partial_t u + \Delta u + f(u) = 0.$$

Pour $d \geqslant 2$, cette équation admet des ondes progressives de la forme $e^{i\omega t}\Phi(x)$ (à une transformation galiléenne près), où Φ est un profil fixe, solution de $-\Delta\Phi + \omega\Phi = f(\Phi)$, mais *pas un état fondamental.* Ces profils sont appelés états excités. Dans cet article, nous construisons des solutions de NLS se comportant comme une somme d'états excités qui se séparent rapidement au cours du temps (nous les appelons multi-solitons). Nous montrons aussi que si le flot autour d'un des états excités est linéairement instable, alors le multi-soliton n'est pas unique et est instable.

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1. Introduction

Setting of the problem

We consider the nonlinear Schrödinger equation

$$iu_t + \Delta u + f(u) = 0 \tag{NLS}$$

where $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ and $f : \mathbb{C} \to \mathbb{C}$ is defined for any $z \in \mathbb{C}$ by $f(z) = g(|z|^2)z$ with $g \in \mathcal{C}^0([0, +\infty), \mathbb{R}) \cap \mathcal{C}^1((0, +\infty), \mathbb{R})$.

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Equation (NLS) admits special travelling wave solutions called solitons: given a frequency $\omega_0 > 0$, an initial phase $\gamma_0 \in \mathbb{R}$, initial position and speed $x_0, v_0 \in \mathbb{R}^d$ and a solution $\Phi_0 \in H^1(\mathbb{R}^d)$ of

$$-\Delta\Phi_0 + \omega_0\Phi_0 - f(\Phi_0) = 0, \tag{1}$$

a *soliton* solution of (NLS) travelling on the line $x = x_0 + v_0 t$ is given by

$$R_{\Phi_0,\omega_0,\gamma_0,v_0,x_0}(t,x) := \Phi_0(x - v_0 t - x_0)e^{i(\frac{1}{2}v_0\cdot x - \frac{1}{4}|v_0|^2 t + \omega_0 t + \gamma_0)}. \tag{2}$$

Among solutions of (1), it is common to distinguish between *ground states*, and *excited states.* A *ground state* (or *least energy solution*) minimizes among all solutions of (1) the action S_0, defined for $v \in H^1(\mathbb{R}^d)$ by

$$S_0(v) := \frac{1}{2}\|\nabla v\|^2_{L^2(\mathbb{R}^d)} + \frac{\omega_0}{2}\|v\|^2_{L^2(\mathbb{R}^d)} - \int_{\mathbb{R}^d} F(v)dx,$$

where $F(z) := \int_0^{|z|} g(s^2)sds$ for all $z \in \mathbb{C}$. An *excited state* is a solution to (1) which is not a ground state. In general, we shall refer to any solution of (1) as *bound state.* We also mention the existence of a particular type of excited states, the *vortices.* A vortex is a special solution of (1) which is non-trivially complex-valued, i.e. with a non-zero angular momentum. Vortices can be constructed following the ansatz described by Lions in [28]. We shall sometimes abuse terminology and call ground state (resp. excited state) a soliton build with a ground state (resp. an excited state).

A multi-soliton is a solution of (NLS) built with solitons. More precisely, let $N \in \mathbb{N}\backslash\{0,1\}$, $\omega_1, ..., \omega_N > 0$, $\gamma_1, ..., \gamma_N \in \mathbb{R}$, $v_1, ..., v_N \in \mathbb{R}^d$, $x_1, ..., x_N \in \mathbb{R}^d$ and $\Phi_1, ..., \Phi_N \in H^1(\mathbb{R}^d)$ solutions of (1) (with ω_0 replaced by $\omega_1, ..., \omega_N$). Set

$$R_j(t,x) := R_{\Phi_j,\omega_j,\gamma_j,v_j,x_j}(t,x), \quad R(t,x) := \sum_{j=1}^N R_j(t,x). \tag{3}$$

Due to the non-linearity, the function R is not a solution of (NLS) anymore. What we call *multi-soliton* is a solution u of (NLS) defined on $[T_0, +\infty)$ for some $T_0 \in \mathbb{R}$ and such that

$$\lim_{t\to+\infty} \|u(t) - R(t)\|_{H^1(\mathbb{R}^d)} = 0.$$

In this paper, we are concerned with existence, non-uniqueness and instability of multi-solitons build on excited states, which we will refer to as excited multi-solitons.

History and known results

Solitons and multi-solitons play a crucial role in understanding the dynamics of nonlinear dispersive evolution equations such as Korteweg-de Vries equations or nonlinear Schrödinger equations (see e.g. [37] for a general overview).

To fix ideas, consider the pure-power nonlinearity $f(u) = |u|^{p-1}u$. Equation (NLS) is L^2-critical (resp. subcritical, resp. supercritical) if $p = 1 + \frac{4}{d}$ (resp. $p < 1 + \frac{4}{d}$, $p > 1 + \frac{4}{d}$). The *soliton resolution conjecture* states that, at least in the L^2-subcritical case, a generic solution will eventually decompose into a sum of *ground state* solitons and a small radiative term, in some sense we will not try to make precise. However, this conjecture remains widely open, except when the equation is completely integrable (like the classical Korteweg-de Vries equation $u_t + u_{xxx} + uu_x = 0$) and explicit solutions are known [26, 36].

Nevertheless, multi-solitons *based on ground states* are supposed to be generic objects for large time; in contrast *excited* multi-solitons are believed to be singular objects of the flow of (NLS). However, their existence shows that a global approach of the large time dynamics must take care of them.

The first existence result of multi-solitons in a non-integrable setting was obtained by Merle [32] for multi-solitons composed of ground states or excited states for the L^2-critical nonlinear Schrödinger equation. For multi-solitons composed only of ground states, the L^2-subcritical case was treated by Martel and Merle [30] (see also Martel [29] for the generalized Korteweg-de Vries equation) and the L^2-supercritical case by Côte, Martel and Merle [13]. No excited multi-solitons were ever constructed except in the L^2-critical case and our result (Theorem 1) is the first in that direction: we construct excited multi-solitons based on excited states which move fast away from one another.

Study of the dynamics around *ground-states* solitons and multi-solitons, in particular stability properties, has attracted a lot of attention since the beginning of the 80's (see e.g. [2, 8, 21, 22, 38, 39, 40]). The main result states that ground-states solitons are orbitally stable only in the L^2-subcritical case.

So far, little is known about the stability of excited state solitons. All excited states are conjectured to be unstable, regardless of any assumption on the nonlinearity. For results on instability with a supercritical nonlinearity, see Grillakis [19] and Jones [24] in the case of real and radial excited states and Mizumachi for vortices [33, 34]. Partial results in the L^2-subcritical case are available in the works of Chang, Gustafson, Nakanishi and Tsai [10], Grillakis [20] and Mizumachi [35].

Here we show that under a very natural assumption of instability of the linearized flow around one excited state, the excited multi-soliton is not unique, and unstable in a strong sense.

Statement of the results

We make the following assumptions on the nonlinearity (recall that $f(z) = g(|z|^2)z$ for $z \in \mathbb{C}$).

(A1) $g \in \mathcal{C}^0([0,+\infty),\mathbb{R}) \cap \mathcal{C}^1((0,+\infty),\mathbb{R})$, $g(0) = 0$ and $\lim_{s\to 0} sg'(s) = 0$.

(A2) There exist $C > 0$ and $1 < p < 1 + \frac{4}{d-2}$ if $d \geqslant 3$, $1 < p < +\infty$ if $d = 1, 2$ such that $|s^2 g'(s^2)| \leqslant Cs^{p-1}$ for $s \geqslant 1$.

(A3) There exists $s_0 > 0$ such that $F(s_0) > \frac{s_0^2}{2}$.

Remark 1. *A typical example of a non-linearity satisfying (A1)-(A3) is given by the power type non-linearity* $f(z) = |z|^{p-1}z$ *with* $1 < p < 1 + \frac{4}{d-2}$ *if* $d \geqslant 3$, $1 < p < +\infty$ *if* $d = 1, 2$.

Assumptions (A1)-(A3) guarantee that, except in dimension $d = 1$ where all bound states are ground states, there exist ground states and infinitely many excited states (see e.g. [3, 4, 5, 20, 25]). In particular, excited states can have arbitrarily large energy and $L^\infty(\mathbb{R}^d)$-norm. Note that every solution of (1) is exponentially decaying (see e.g. [6]). More precisely, for all Φ_0 solution to (1) we have $e^{\sqrt{\omega}|x|}(|\Phi_0| + |\nabla\Phi_0|) \in L^\infty(\mathbb{R}^d)$ for all $\omega < \omega_0$.

Assumptions (A1)-(A2) ensure well-posedness in $H^1(\mathbb{R}^d)$ of (NLS), see e.g. [7] (the equation is then H^1-subcritical). In particular, for any $u_0 \in H^1(\mathbb{R}^d)$ there exists a unique maximal solution u such that energy, mass and momentum are conserved. Recall that energy, mass and momentum are defined in the following way.

$$E(u) := \frac{1}{2}\|\nabla u\|^2_{L^2(\mathbb{R}^d)} - \int_{\mathbb{R}^d} F(u)dx,$$
$$M(u) := \|u\|^2_{L^2(\mathbb{R}^d)},$$
$$P(u) := \mathscr{Im}\int_{\mathbb{R}^d} \bar{u}\nabla u dx.$$

Notice that (A3) makes the equation focusing.

Our first result is the existence of multi-solitons composed of excited states as soon as the relative speeds $v_j - v_k$ of the solitons are sufficiently large.

Theorem 1. *Assume (A1)-(A3). Let* $N \in \mathbb{N} \setminus \{0, 1\}$, *and for* $j = 1, ..., N$ *take* $\omega_j > 0$, $\gamma_j \in \mathbb{R}$, $v_j \in \mathbb{R}^d$, $x_j \in \mathbb{R}^d$ *and* $\Phi_j \in H^1(\mathbb{R}^d)$ *a solution of* (1) *(with* ω_0 *replaced by* ω_j*). Set*

$$R_j(t,x) = R_{\Phi_j,\omega_j,\gamma_j,v_j,x_j}(t,x) := \Phi_j(x - v_j t - x_j)e^{i(\frac{1}{2}v_j\cdot x - \frac{1}{4}|v_j|^2 t + \omega_j t + \gamma_j)}.$$

Let $\omega_\star$ *and* $v_\star$ *be given by*

$$\omega_\star := \frac{1}{2}\min\{\omega_j, j = 1, ..., N\}, \quad v_\star := \frac{1}{9}\min\{|v_j - v_k|; j, k = 1, ..., N, j \neq k\}.$$

Also introduce $\alpha := \sin\left(\frac{\sqrt{\pi}\Gamma(\frac{d-1}{2})}{N^2\Gamma(\frac{d}{2})}\right)$ *(this constant appears naturally in Claim 13).*
There exists $v_\sharp := v_\sharp(\Phi_1, ..., \Phi_N) > 0$ *such that if* $v_\star > \alpha^{-1}v_\sharp$ *then the following holds.*
There exist $T_0 \in \mathbb{R}$ *and a solution of* (NLS) $u \in \mathcal{C}([T_0, +\infty), H^1(\mathbb{R}^d))$ *such that for all* $t \in [T_0, +\infty)$ *we have*

$$\|u(t) - \sum_{j=1}^{N} R_j(t)\|_{H^1(\mathbb{R}^d)} \leqslant e^{-\alpha\omega_\star^{\frac{1}{2}} v_\star t}.$$

We now turn to the non-uniqueness and instability of a multi-soliton.

Assume that the flow around one of the R_j is linearly unstable, i.e. has an eigenvalue off the imaginary axis. As the R_j all play the same role, we can assume it is R_1.

(A4) $L = -i\Delta + i\omega_1 - idf(\Phi_1)$ has an eigenvalue $\lambda \in \mathbb{C}$ with $\rho := \mathscr{R}e(\lambda) > 0$.

This assumption is very natural if one expects R_1 to be unstable. Actually, (A4) holds for any real radial bound state in the L^2-supercritical case (see [19]). For excited states, (A4) is believed to hold for a wide class of non-linearities.

Under assumption (A4), we are able to construct a one parameter family of solutions to (NLS) that converge to the soliton R_1 as time goes to infinity, as described in the following Theorem.

Theorem 2. *Take* $\omega_1 > 0$, $\gamma_1 \in \mathbb{R}$, $v_1 \in \mathbb{R}^d$, $x_1 \in \mathbb{R}^d$ *and* $\Phi_1 \in H^1(\mathbb{R}^d)$ *a solution of* (1) *(with* ω_0 *replaced by* ω_1*). Set*

$$R_1(t,x) = R_{\Phi_1,\omega_1,\gamma_1,v_1,x_1}(t,x) := \Phi_1(x - v_1 t - x_1)e^{i(\frac{1}{2}v_1\cdot x - \frac{1}{4}|v_1|^2 t + \omega_1 t + \gamma_1)}.$$

Assume g is $\mathcal{C}^\infty$ *and (A1)-(A4) are satisfied.*

There exists a function $Y(t)$ *such that* $\|Y(t)\|_{H^1(\mathbb{R}^d)} \leqslant Ce^{-\rho t}$ *and* $e^{\rho t}\|Y(t)\|_{H^1}$ *is non-zero and periodic (here* ρ *is given by (A4) and* $Y(t)$ *is actually a solution to the linearized flow around* R_1*, see* (26), (27)*). For all* $a \in \mathbb{R}$*, there exist* $T_0 \in \mathbb{R}$ *large enough, a solution* u_a *to* (NLS) *defined on* $[T_0, +\infty)$*, and a constant* $C > 0$ *such that*

$$\forall t \geqslant T_0, \quad \|u_a(t) - R_1(t) - aY(t)\|_{H^1(\mathbb{R}^d)} \leqslant Ce^{-2\rho t}.$$

In particular, Theorem 2 implies that the soliton R_1 is orbitally unstable, as precised in the following corollary.

Corollary 2. *Under the hypotheses of Theorem 2,* R_1 *is orbitally unstable in the following sense. Let* $\sigma \geqslant 0$*. There exist* $\varepsilon > 0$, $(T_n) \subset (0, +\infty)$, $(u_{0,n}) \subset H^1(\mathbb{R}^d)$ *and solutions* (u_n) *of* (NLS) *defined on* $[0, T_n]$ *with* $u_n(0) = u_{0,n}$ *such that*

$$\lim_{n\to+\infty} \|u_{0,n} - R_1(0)\|_{H^\sigma(\mathbb{R}^d)} = 0 \text{ and } \inf_{y\in\mathbb{R}^d,\vartheta\in\mathbb{R}} \|u_n(T_n) - e^{i\vartheta}\Phi_1(\cdot - y)\|_{L^2(\mathbb{R}^d)} \geqslant \varepsilon \text{ for all } n \in \mathbb{N}.$$

From Theorem 2 we infer the existence of a one parameter family of multi-solitons. As a corollary, we obtain non-uniqueness and instability for high relative speeds multi-solitons.

Theorem 3. *Let* $N \in \mathbb{N} \setminus \{0,1\}$*, and for* $j = 1, ..., N$ *take* $\omega_j > 0$, $\gamma_j \in \mathbb{R}$, $v_j \in \mathbb{R}^d$, $x_j \in \mathbb{R}^d$ *and* $\Phi_j \in H^1(\mathbb{R}^d)$ *a solution of* (1) *(with* ω_0 *replaced by* ω_j*). Set*

$$R_j(t,x) = R_{\Phi_j,\omega_j,\gamma_j,v_j,x_j}(t,x) := \Phi_j(x - v_j t - x_j)e^{i(\frac{1}{2}v_j\cdot x - \frac{1}{4}|v_j|^2 t + \omega_j t + \gamma_j)}.$$

Let $v_\star := \frac{1}{9}\min\{|v_j - v_k|; j,k = 1, ..., N, j \neq k\}$*. Assume g is* $\mathcal{C}^\infty$ *and (A1)-(A4) are satisfied.*

There exists $v_\natural := v_\natural(\Phi_1, \ldots, \Phi_N) > 0$ *such that if* $v_\star > v_\natural$ *then the following holds.*
There exists a function $Y(t)$ *such that* $\|Y(t)\|_{H^1(\mathbb{R}^d)} \leqslant Ce^{-\rho t}$ *and* $e^{\rho t}\|Y(t)\|_{H^1}$ *is non-zero and periodic (here* ρ *is given by (A4) and* $Y(t)$ *is actually a solution to the linearized flow around* R_1*, see* (26), (27)*). For all* $a \in \mathbb{R}$*, there exist* $T_0 \in \mathbb{R}$ *large enough, a solution* u_a *to* (NLS) *defined on* $[T_0, +\infty)$*, and a constant* $C > 0$ *such that*

$$\forall t \geqslant T_0, \quad \|u_a(t) - \sum_{j=1}^{N} R_j(t) - aY(t)\|_{H^1(\mathbb{R}^d)} \leqslant Ce^{-2\rho t}.$$

Remark 3. *Notice that, in Theorem 3, if for $a, b \in \mathbb{R}$ we have $a \neq b$, then $u_a \not\equiv u_b$. Indeed, for t large enough we have*

$$\|u_a(t) - u_b(t)\|_{H^1(\mathbb{R}^d)} \geqslant |a-b| \|Y(t)\|_{H^1(\mathbb{R}^d)} - 2Ce^{-2\rho t}.$$

Since $e^{\rho t}\|Y(t)\|_{H^1}$ is non-zero and periodic, this implies that $u_a \not\equiv u_b$ if $a \neq b$.

Corollary 4. *Under the hypotheses of Theorem 3, the following instability property holds. Let $\sigma \geqslant 0$, there exists $\varepsilon > 0$, such that for all $n \in \mathbb{N} \setminus \{0\}$ and for all $T \in \mathbb{R}$ the following holds. There exists $I_n, J_n \in \mathbb{R}$, $T \leqslant I_n < J_n$ and a solution $w_n \in \mathscr{C}([I_n, J_n], H^1(\mathbb{R}^d))$ to* (NLS) *such that*

$$\lim_{n\to+\infty} \|w_n(I_n) - R(I_n)\|_{H^\sigma(\mathbb{R}^d)} = 0, \quad \text{and} \quad \inf_{\substack{y_j \in \mathbb{R}^d, \vartheta_j \in \mathbb{R},\\ j=1,\dots,N}} \|w_n(J_n) - \sum_{i=1}^N \Phi_j(x-y_j) e^{i(\frac{1}{2}v_j\cdot x + \vartheta_j)}\|_{L^2(\mathbb{R}^d)} \geqslant \varepsilon.$$

Remark 5. *The fact that instability holds backward in time (i.e. with $J_n < I_n$) is an easy consequence of Theorem 3. Hence the difficulty in Corollary 4 is to prove instability forward in time.*

Remark 6. *The classification of multi-solitons is now complete for the generalized Korteweg-de Vries equations (see [12, 29] and the references therein). In particular, uniqueness holds in the subcritical and critical cases, whereas in the supercritical case the set of multi-solitons consists in a N-parameters family. To the authors knowledge, no uniqueness nor classification result is available yet for multi-solitons of nonlinear Schrödinger equations.*

Scheme of proofs and comments

Our strategy for the proof of the existence result (Theorem 1) is inspired from the works [13, 30, 32]: we take a sequence of time $T_n \to +\infty$ and a set of final data $u_n(T_n) = R(T_n)$. Our goal is to prove that the solutions u_n to (NLS) backwards in time (which approximate a multi-soliton) exist up to some time T_0 independent of n, and enjoy uniform $H^1(\mathbb{R}^d)$ decay estimates on $[T_0, T_n]$. A compactness argument then shows that (u_n) converges to a multi-soliton solution to (NLS) defined on $[T_0, +\infty)$.

As in [13, 30], the uniform backward $H^1(\mathbb{R}^d)$-estimates rely on slow variation of localized conservation laws as well as coercivity of the Hessian of the action around each component of the multi-soliton. However, this Hessian has negative "bad directions" on which it is not coercive. When dealing with ground states, these were ruled out either by modulation and conservation of the mass (as in [30]) or with the help of explicit knowledge of eigenfunctions of the operator corresponding to the linearization of (NLS) around a soliton (as in [13]). In both cases, this could be done only because of the knowledge of precise spectral properties for ground states; this does no longer hold when dealing with the more general case of excited states.

Our remark is that the Hessian fails to be $H^1(\mathbb{R}^d)$-coercive only up to a $L^2(\mathbb{R}^d)$-scalar product with the bad directions. Hence the first step in our analysis is to find uniform $L^2(\mathbb{R}^d)$-backward estimates without the help of the Hessian. This rules out the "bad directions" and we can now take advantage of the coercivity of the Hessian to obtain the $H^1(\mathbb{R}^d)$-estimates. The main drawback of our approach is that the bootstrap of the $L^2(\mathbb{R}^d)$-estimates requires that the soliton components are well-separated. Thus we have to work with high-speed solitons.

To obtain the one parameter family of Theorem 2, we rely on a fixed point argument for smooth functions exponentially convergent (in time). This is possible because we now assume smoothness on the non-linearity. The main difficulty is to construct a very good approximate solution to the multi-soliton. Actually we build such a profile at arbitrary exponential order. This method is inspired by [15, 16, 17, 18] in the case of a single ground state, for the nonlinear wave or Schrödinger equations. It was also recently developed by Combet [11, 12] for multi-solitons in the context of the L^2-supercritical generalized Korteweg-de Vries equation.

However, an important difference in our case is that we consider excited states, and the linearized flow around them is much less understood than that around a ground state soliton. For example, to our knowledge, the exponential decay of eigenfunctions was not known in general (see [23] for a partial result). We prove it in Appendix A, see Proposition 25. Also, the unstable eigenvalue has no reason to be real, and this will make the construction of the profile much more intricate than in the ground state soliton case. This is the purpose of Proposition 22. Once the approximation profile is derived, the proofs of Theorem 2 and 3 follow from a fixed point argument around the profile.

The paper is organized as follows. In Section 2 we prove Theorem 1. Section 3 is devoted to the proofs of Theorems 2 and 3. In Appendix A we prove the exponential decay of eigenfunctions for matrix Schrödinger operators and in Appendix B we prove Corollaries 2 and 4.

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2. Existence

In this section, we assume (A1)-(A3) and suppose we are given $N \in \mathbb{N} \setminus \{0,1\}$, and for $j = 1, ..., N$, $\omega_j > 0$, $\gamma_j \in \mathbb{R}$, $v_j \in \mathbb{R}^d$, $x_j \in \mathbb{R}^d$ and $\Phi_j \in H^1(\mathbb{R}^d)$ a solution of (1) (with ω_0 replaced by ω_j). Recall that

$$R_j(t,x) = \Phi_j(x - v_j t - x_j) e^{i(\frac{1}{2}v_j \cdot x - \frac{1}{4}|v_j|^2 t + \omega_j t + \gamma_j)},$$

$$\omega_\star = \frac{1}{2} \min\{\omega_j, j = 1, ..., N\}, \quad v_\star = \frac{1}{9} \min\{|v_j - v_k|; j, k = 1, ..., N, j \neq k\},$$

and $\alpha := \sin\left(\frac{\sqrt{\pi}\Gamma(\frac{d-1}{2})}{N^2\Gamma(\frac{d}{2})}\right)$.

2.1. Approximate solutions and convergence toward a multi-soliton

Let $(T_n)_{n \geqslant 1} \subset \mathbb{R}$ be an increasing sequence of time such that $T_n \to +\infty$ and (u_n) be solutions to (NLS) such that $u_n(T_n) = R(T_n)$. We call u_n an *approximate multi-soliton.*

The proof of Theorem 1 relies on the following proposition.

Proposition 7 (Uniform estimates)**.** *There exists $v_\sharp := v_\sharp(\Phi_1, ..., \Phi_N) > 0$ such that if $v_\star > \alpha^{-1} v_\sharp$ then the following holds. There exist $n_0 \in \mathbb{N}$, $T_0 > 0$ such that for all $n \geqslant n_0$ every approximate multi-soliton u_n is defined on $[T_0, T_n]$ and for all $t \in [T_0, T_n]$ we have*

$$\|u_n(t) - R(t)\|_{H^1(\mathbb{R}^d)} \leqslant e^{-\alpha \omega_\star^{\frac{1}{2}} v_\star t}. \tag{4}$$

In this section, assuming Proposition 7, we prove Theorem 1 by establishing the convergence of the approximate multi-solitons u_n to a multi-soliton u existing on $[T_0, +\infty)$. Our proof follows the same line as in [13, 30].

From now on and in the rest of section 2.1 we assume that $v_\star > \alpha^{-1} v_\sharp$, where $v_\sharp$ is given by Proposition 7.

Since the approximate multi-solitons u_n are constructed by solving (NLS) backward in time, to prove Theorem 1 we first need to make sure that the initial data $u_n(T_0)$ converge to some initial datum u_0.

Lemma 8. *There exists $u_0 \in H^1(\mathbb{R}^d)$ such that, possibly for a subsequence only, $u_n(T_0) \to u_0$ strongly in $H^s(\mathbb{R}^d)$ as $n \to +\infty$ for any $s \in [0, 1)$.*

Lemma 8 is a consequence of the following claim.

Claim 9 ($L^2(\mathbb{R}^d)$-compactness)**.** *Take $\delta > 0$. There exists $r_\delta > 0$ such that for all n large enough we have*

$$\int_{|x| > r_\delta} |u_n(T_0)|^2 dx \leqslant \delta. \tag{5}$$

Proof. Let n be large enough so that the conclusions of Proposition 7 hold. Let T_δ be such that $e^{-\alpha \omega_\star^{\frac{1}{2}} v_\star T_\delta} \leqslant \sqrt{\frac{\delta}{4}}$. Then, by Proposition 7, we have

$$\|u_n(T_\delta) - R(T_\delta)\|_{H^1(\mathbb{R}^d)} \leqslant \sqrt{\frac{\delta}{4}}. \tag{6}$$

Let ρ_δ be such that

$$\int_{|x|>\rho_\delta} |R(T_\delta)|^2 dx < \frac{\delta}{4}. \tag{7}$$

From (6)-(7) we infer

$$\int_{|x|>\rho_\delta} |u_n(T_\delta)|^2 dx < \frac{\delta}{2}. \tag{8}$$

We define a $\mathcal{C}^1$ cut-off function $\tau : \mathbb{R} \to \mathbb{R}$ such that $\tau(s) = 0$ if $s \leqslant 0$, $\tau(s) = 1$ if $s \geqslant 1$, $\tau(s) \in [0,1]$ and $|\tau'(s)| \leqslant 2$ if $s \in [0,1]$. Let κ_δ to be determined later and consider

$$\Upsilon(t) := \int_{\mathbb{R}^d} |u_n(t)|^2 \tau\left(\frac{|x| - \rho_\delta}{\kappa_\delta}\right) dx.$$

To obtain (5) from (8) we need to establish a link between $\Upsilon(T_0)$ and $\Upsilon(T_\delta)$. Differentiating in time, we obtain after simple calculations (see e.g. [30, Claim 2])

$$\Upsilon'(t) = \frac{2}{\kappa_\delta} \mathscr{Im} \int_{\mathbb{R}^d} \bar{u}_n \nabla u_n \cdot \frac{x}{|x|} \tau'\left(\frac{|x| - \rho_\delta}{\kappa_\delta}\right) dx.$$

Since $\|u_n(t)\|_{H^1(\mathbb{R}^d)}$ is bounded independently of n and t, there exists

$$C_0 := \sup_{n\in\mathbb{N}} \sup_{t\in[T_0,T_n]} \|u_n(t)\|^2_{H^1(\mathbb{R}^d)} > 0$$

such that

$$|\Upsilon'(t)| \leqslant \frac{2C_0}{\kappa_\delta}.$$

Choose κ_δ such that $\frac{2C_0}{\kappa_\delta} T_\delta < \frac{\delta}{2}$. Then, by integrating between T_0 and T_δ we obtain

$$\Upsilon(T_0) - \Upsilon(T_\delta) \leqslant \frac{\delta}{2}. \tag{9}$$

From (8) we infer that

$$\Upsilon(T_\delta) = \int_{\mathbb{R}^d} |u_n(T_\delta)|^2 \tau\left(\frac{|x| - \rho_\delta}{\kappa_\delta}\right) dx \leqslant \int_{|x|>\rho_\delta} |u_n(T_\delta)|^2 dx \leqslant \frac{\delta}{2}.$$

Combining with (9) we obtain

$$\Upsilon(T_0) \leqslant \delta.$$

Now set $r_\delta := \kappa_\delta + \rho_\delta$. Then from the definition of τ it is easy to see that

$$\int_{|x|>r_\delta} |u_n(T_0)|^2 dx \leqslant \Upsilon(T_0) \leqslant \delta,$$

which proves the claim. □

Proof of Lemma 8. Since $u_n(T_0)$ is bounded in $H^1(\mathbb{R}^d)$, there exists $u_0 \in H^1(\mathbb{R}^d)$ such that up to a subsequence $u_n(T_0) \rightharpoonup u_0$ weakly in $H^1(\mathbb{R}^d)$. Hence, $u_n(T_0) \to u_0$ strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$ and actually strongly in $L^2(\mathbb{R}^d)$ by Claim 9. By interpolation we get the desired conclusion. □

Proof of Theorem 1. Let u_0 be given by Lemma 8 and let $u \in \mathcal{C}([T_0, T^\star), H^1(\mathbb{R}^d))$ be the corresponding maximal solution of (NLS). By (A1)-(A2), there exists $0 < \sigma < 1$ such that $1 < p < 1 + \frac{4}{d-2\sigma}$ and

$$|f(z_1) - f(z_2)| \leqslant C(1 + |z_1|^{p-1} + |z_2|^{p-1})|z_1 - z_2| \text{ for all } z_1, z_2 \in \mathbb{C}.$$

This implies that the Cauchy problem for (NLS) is well-posed in $H^\sigma(\mathbb{R}^d)$ (see [7, 9]). Combined with Lemma 8 this implies that $u_n(t) \to u(t)$ strongly in $H^\sigma(\mathbb{R}^d)$ for any $t \in [T_0, T^\star)$. By boundedness of $u_n(t)$ in $H^1(\mathbb{R}^d)$, we also have $u_n(t) \rightharpoonup u(t)$ weakly in $H^1(\mathbb{R}^d)$ for any $t \in [T_0, T^\star)$. By Proposition 7, for any $t \in [T_0, T^\star)$ we have

$$\|u(t) - R(t)\|_{H^1(\mathbb{R}^d)} \leqslant \liminf_{n \to +\infty} \|u_n(t) - R(t)\|_{H^1(\mathbb{R}^d)} \leqslant e^{-\alpha\omega_\star^{\frac{1}{2}} v_\star t}. \tag{10}$$

In particular, since $R(t)$ is bounded in $H^1(\mathbb{R}^d)$ there exists $C > 0$ such that for any $t \in [T_0, T^\star)$ we have

$$\|u(t)\|_{H^1(\mathbb{R}^d)} \leqslant e^{-\alpha\omega_\star^{\frac{1}{2}} v_\star t} + \|R(t)\|_{H^1(\mathbb{R}^d)} \leqslant C. \tag{11}$$

Recall that, by the blow up alternative (see e.g. [7]), either $T^\star = +\infty$ or $T^\star < +\infty$ and $\lim_{t \to T^\star} \|u\|_{H^1(\mathbb{R}^d)} = +\infty$. Therefore (11) implies that $T^\star = +\infty$. From (10) we infer that for all $t \in [T_0, +\infty)$ we have

$$\|u(t) - R(t)\|_{H^1(\mathbb{R}^d)} \leqslant e^{-\alpha\omega_\star^{\frac{1}{2}} v_\star t}.$$

This concludes the proof. □

2.2. Uniform backward estimates

This section is devoted to the proof of Proposition 7. This proof relies on a bootstrap argument. Indeed, from the definition of the final datum $u_n(T_n)$ and continuity of u_n in time, it follows that (4) holds on an interval $[t^\dagger, T_n]$ for $t^\dagger$ close enough to T_n. Then the following Proposition 10 shows that we can actually improve to a better estimate, hence leaving enough room to extend the interval on which the original estimate holds.

Proposition 10. *There exists $v_\sharp := v_\sharp(\Phi_1, ..., \Phi_N) > 0$ such that if $v_\star > \alpha^{-1} v_\sharp$ then the following holds. There exist $n_0 \in \mathbb{N}$, $T_0 > 0$ such that for all $n \geqslant n_0$ every approximate multi-soliton u_n is defined on $[T_0, T_n]$. Let $t^\dagger \in [T_0, T_n]$ and $n \geqslant n_0$. If for all $t \in [t^\dagger, T_n]$ we have*

$$\|u_n(t) - R(t)\|_{H^1(\mathbb{R}^d)} \leqslant e^{-\alpha\omega_\star^{\frac{1}{2}} v_\star t} \tag{12}$$

then for all $t \in [t^\dagger, T_n]$ we have

$$\|u_n(t) - R(t)\|_{H^1(\mathbb{R}^d)} \leqslant \frac{1}{2} e^{-\alpha\omega_\star^{\frac{1}{2}} v_\star t}. \tag{13}$$

Before proving Proposition 10, we indicate precisely how it is used to obtain Proposition 7.

Proof of Proposition 7. Let T_0, n_0 and $v_\sharp$ be given by Proposition 10, assume $v_\star > \alpha^{-1} v_\sharp$, and let $n \geqslant n_0$. Since $u_n(T_n) = R(T_n)$ and u_n is continuous in $H^1(\mathbb{R}^d)$, for t close enough to T_n we have

$$\|u_n(t) - R(t)\|_{H^1(\mathbb{R}^d)} \leqslant e^{-\alpha\omega_\star^{\frac{1}{2}} v_\star t}. \tag{14}$$

Let $t^\dagger$ be the minimal time such that (14) holds:

$$t^\dagger := \min\{\tau \in [T_0, T_n]; (14) \text{ holds for all } t \in [\tau, T_n]\}.$$

We prove by contradiction that $t^\dagger = T_0$. Indeed, assume that $t^\dagger > T_0$. Then

$$\|u_n(t^\dagger) - R(t^\dagger)\|_{H^1(\mathbb{R}^d)} \leqslant e^{-\alpha\omega_\star^{\frac{1}{2}} v_\star t^\dagger}$$

and by Proposition 10 we can improve this estimate in

$$\|u_n(t^\dagger) - R(t^\dagger)\|_{H^1(\mathbb{R}^d)} \leqslant \frac{1}{2} e^{-\alpha\omega_\star^{\frac{1}{2}} v_\star t^\dagger}.$$

Hence, by continuity of $u_n(t)$ in $H^1(\mathbb{R}^d)$, there exists $T_0 \leqslant t^\ddagger < t^\dagger$ such that (14) holds for all $t \in [t^\ddagger, t^\dagger]$. This contradicts the minimality of $t^\dagger$ and finishes the proof. □

The proof of Proposition 10 is done in two steps. First, assuming (12) we prove that we can control the $L^2(\mathbb{R}^d)$-norm of $(u_n - R)$. To obtain the full control on the $H^1(\mathbb{R}^d)$-norm of $(u_n - R)$ as in (13) we use the linearization of an action-like functional. This linearization is coercive (i.e. controls the $H^1(\mathbb{R}^d)$-norm) up to a finite number of non-positive directions that can all be controlled due to the $L^2(\mathbb{R}^d)$-estimate.

Let $T_0 > 0$ large enough and fix $n \in \mathbb{N}$ such that $T_n > T_0$. For notational convenience, the dependency on n is understood for u and we drop the subscript n. Set $v := u - R$. Let $t^\dagger \in [T_0, T_n]$ and assume that for all $t \in [t^\dagger, T_n]$ we have

$$\|v(t)\|_{H^1(\mathbb{R}^d)} \leqslant e^{-\alpha \omega_\star^{\frac{1}{2}} v_\star t}.$$

Step 1: $L^2(\mathbb{R}^d)$-control

Lemma 11. *For all $K > 0$ and $m \in \mathbb{N} \setminus \{0\}$ there exists $v_\sharp = v_\sharp(K, m, \Phi_1, ..., \Phi_N) > 0$ such that if $v_\star > \alpha^{-1} v_\sharp$ then for all $t \in [t^\dagger, T_n]$ we have*

$$\|v(t)\|_{L^2(\mathbb{R}^d)} \leqslant \frac{1}{\sqrt{2mK}} e^{-\alpha \omega_\star^{\frac{1}{2}} v_\star t}.$$

Notice that the reason why we introduce such K and m will appear later in the proof.

Proof. First note that by identifying $\mathbb{C}$ to $\mathbb{R}^2$ and viewing $f : \mathbb{R}^2 \to \mathbb{R}^2$ we can consider

$$df(z).w = g(|z|^2)w + 2\mathscr{R}e(z\bar{w})g'(|z|^2)z.$$

The function v satisfies

$$iv_t + \mathcal{L}v + \mathcal{N}(v) = 0,$$

where

$$\mathcal{L}v := \Delta v + df(R).v$$

and the remaining nonlinear term $\mathcal{N}(v)$ verifies

$$|\,(i\mathcal{N}(v), v)_{L^2(\mathbb{R}^d)}\,| \leqslant \eta(\|v\|_{H^1(\mathbb{R}^d)})\|v\|^2_{H^1(\mathbb{R}^d)},$$

where η is a decreasing function satisfying $\eta(s) \to 0$ when $s \to 0$. Take any $t \in [t^\dagger, T_n]$. We have

$$\frac{1}{2}\frac{d}{dt}\|v\|^2_{L^2(\mathbb{R}^d)} = (v_t, v)_{L^2(\mathbb{R}^d)} = (i\mathcal{L}v, v)_{L^2(\mathbb{R}^d)} + (i\mathcal{N}(v), v)_{L^2(\mathbb{R}^d)}\,.$$

We have

$$\begin{aligned}
&(i\mathcal{L}v, v)_{L^2(\mathbb{R}^d)} \\
&\quad = \mathscr{R}e \int_{\mathbb{R}^d} i\left(\Delta v + df(R).v)\right)\bar{v}dx, \\
&\quad = \mathscr{R}e \int_{\mathbb{R}^d} i\left(\Delta v + g(|R|^2)v + 2g'(|R|^2)\mathscr{R}e(R\bar{v})R\right)\bar{v}dx, \\
&\quad = \mathscr{R}e \int_{\mathbb{R}^d} i(-|\nabla v|^2 + g(|R|^2)|v|^2 + 2g'(|R|^2)\mathscr{R}e(R\bar{v})R\bar{v})dx, \\
&\quad = -\int_{\mathbb{R}^d} 2g'(|R|^2)\mathscr{R}e(R\bar{v})\mathscr{I}m(R\bar{v})dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\,(i\mathcal{L}v, v)_{L^2(\mathbb{R}^d)}\,| &\leqslant \int_{\mathbb{R}^d} 2|g'(|R|^2)||R|^2|v|^2dx. \\
&\leqslant \frac{C_{\mathcal{L}}}{2}\|v\|^2_{L^2(\mathbb{R}^d)},
\end{aligned}$$

where this last constant $C_{\mathcal{L}}$ depends only on g and $\|R\|_{L^\infty(\mathbb{R}^d)}$. By the bootstrap assumption on v, this implies

$$|\,(i\mathcal{L}v, v)_{L^2(\mathbb{R}^d)}\,| \leqslant \frac{C_{\mathcal{L}}}{2} e^{-2\alpha\omega_\star^{\frac{1}{2}} v_\star t}.$$

In addition, it is easy to see that

$$|\,(i\mathcal{N}(v), v)_{L^2(\mathbb{R}^d)}\,| \leqslant \eta(\|v\|_{H^1(\mathbb{R}^d)})\|v\|^2_{H^1(\mathbb{R}^d)} \leqslant \eta(e^{-\alpha\omega_\star^{\frac{1}{2}} v_\star t}) e^{-2\alpha\omega_\star^{\frac{1}{2}} v_\star t}.$$

In short, if T_0 is large enough so that $\eta(e^{-\alpha\omega_\star^{\frac{1}{2}} v_\star t}) \leqslant \frac{C_{\mathcal{L}}}{2}$, we have obtained that

$$\left|\frac{d}{dt}\|v\|^2_{L^2(\mathbb{R}^d)}\right| \leqslant 2C_{\mathcal{L}} e^{-2\alpha\omega_\star^{\frac{1}{2}} v_\star t}.$$

Therefore, by integration between t and T_n we get

$$\|v(t)\|^2_{L^2(\mathbb{R}^d)} - \|v(T_n)\|^2_{L^2(\mathbb{R}^d)} \leqslant \frac{C_{\mathcal{L}}}{\alpha\omega_\star^{\frac{1}{2}} v_\star}(e^{-2\alpha\omega_\star^{\frac{1}{2}} v_\star t} - e^{-2\alpha\omega_\star^{\frac{1}{2}} v_\star T_n}). \tag{15}$$

Now, we take $v_\sharp$ such that

$$\frac{C_{\mathcal{L}}}{\omega_\star^{\frac{1}{2}} v_\sharp} < \frac{1}{2mK}.$$

If $v_\star > \alpha^{-1} v_\sharp$ and since $v(T_n) = 0$ we get from (15) that

$$\|v(t)\|_{L^2(\mathbb{R}^d)} \leqslant \frac{1}{\sqrt{2mK}} e^{-\alpha\omega_\star^{\frac{1}{2}} v_\star t},$$

which is the desired conclusion. □

Step 2: $H^1(\mathbb{R}^d)$-control

The idea of the second step of the proof of Proposition 10 is reminiscent of the technique used to prove stability for a single soliton in the subcritical case (see e.g. [21, 22, 27, 39, 40]). Indeed, it is well-known that the linearization of the action functional S_0 (see the definition of S_0 p. 1), whose critical points are the solutions of (1), is coercive on a subspace of $H^1(\mathbb{R}^d)$ of finite codimension in $L^2(\mathbb{R}^d)$. At large time, the components of the multi-soliton are well-separated and thus it is possible to localize the analysis around each soliton to gain an $H^1(\mathbb{R}^d)$-local control, up to a space of finite dimension in $L^2(\mathbb{R}^d)$. But due to Lemma 11 we are able to control the remaining $L^2(\mathbb{R}^d)$-directions, hence to close the proof. The idea of looking at localized versions of the invariants of (NLS) was introduced in [32] and later developed in [13, 29, 30, 31]. We shall therefore be sketchy in the proofs, highlighting only the main differences with the previous works.

We start with the case of a single soliton.

Lemma 12 (Coercivity for a soliton)**.** *Let $\omega_0 > 0$, $\gamma_0 \in \mathbb{R}$, $x_0, v_0 \in \mathbb{R}^d$ and a solution $\Phi_0 \in H^1(\mathbb{R}^d)$ of* (1)*. Then there exist $K_0 = K_0(\Phi_0) > 0$, $\nu_0 \in \mathbb{N}\setminus\{0\}$ and $\tilde{X}_0^1, ..., \tilde{X}_0^{\nu_0} \in L^2(\mathbb{R}^d)$ such that for $k = 1, ..., \nu_0$ we have $\|\tilde{X}_0^k\|_{L^2(\mathbb{R}^d)} = 1$ and for any $w \in H^1(\mathbb{R}^d)$ we have*

$$\|w\|^2_{H^1(\mathbb{R}^d)} \leqslant K_0 H_0(t,w) + K_0 \sum_{k=1}^{\nu_0} \big(w, X_0^k(t)\big)^2_{L^2(\mathbb{R}^d)} \text{ for all } t \in \mathbb{R},$$

where

$$\begin{aligned} X_0^k(t) :=& e^{i(\frac{1}{2}v_0\cdot x - \frac{1}{4}|v_0|^2 t + \omega_0 t + \gamma_0)} \tilde{X}_0^k(x - v_0 t - x_0), \\ H_0(t,w) :=& \|\nabla w\|^2_{L^2(\mathbb{R}^d)} + \left(\omega_0 + \frac{|v_0|^2}{4}\right)\|w\|^2_{L^2(\mathbb{R}^d)} - v_0 \cdot \mathscr{Im}\int_{\mathbb{R}^d} \bar{w}\nabla w dx \\ & - \int_{\mathbb{R}^d} \big(g(|R_0|^2)|w|^2 + 2g'(|R_0|^2)\mathscr{Re}(R_0\bar{w})^2\big)\,dx, \end{aligned}$$

and $R_0(t,x)$ is the soliton given by (2)*.*

Lemma 12 follows from standard arguments. We included a proof in Appendix C for the reader's convenience.

We introduce now the localization procedure around each component of the multi-soliton.

We begin by the selection of a particular direction of propagation.

Claim 13. *Let* $0<\alpha<\sin\left(\frac{\sqrt{\pi}\Gamma(\frac{d-1}{2})}{N(N-1)\Gamma(\frac{d}{2})}\right)$. *Then there exists an orthonormal basis* $(e_1,...,e_d)$ *of* $\mathbb{R}^d$ *such that for all* $j,k=1,...,N$*, we have*

$$|(v_j-v_k,e_1)_{\mathbb{R}^d}|\geqslant\alpha|v_j-v_k|.$$

Proof. For $j\neq k$, set $v_{jk}:=\frac{v_j-v_k}{|v_j-v_k|}$. The claim will be proved if we show that the measure of the set

$$\Lambda:=\bigcup_{\substack{j,k=1,\dots,N\\ j\neq k}}\{w\in\mathbb{S}^{d-1},|(v_{jk},w)_{\mathbb{R}^d}|\leqslant\alpha\}$$

is smaller than the measure of the surface of the unit sphere $\mathbb{S}^{d-1}$.

Take $j,k=1,...,N$; $j\neq k$. Without loss of generality, assume that $v_{jk}=(1,0,...,0)$. Take $w\in\mathbb{S}^{d-1}$ and let $(\theta_1,...,\theta_{d-1})$ be the spherical coordinates of w. Then we have

$$(v_{jk},w)_{\mathbb{R}^d}=\cos\theta_1.$$

Therefore, after easy calculations we get

$$\mu(\{w\in\mathbb{S}^{d-1},|(v_{jk},w)_{\mathbb{R}^d}|\leqslant\alpha\})\leqslant 2\arcsin(\alpha)\frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$$

where μ is the Lebesgues measure on $\mathbb{S}^{d-1}$ and $\frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}$ is the area of the $(d-2)$-unit sphere. By subadditivity of the measure this leads to

$$\mu(\Lambda)\leqslant N(N-1)\arcsin(\alpha)\frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}.$$

Now, remember that

$$0<\alpha<\sin\left(\frac{\sqrt{\pi}\Gamma(\frac{d-1}{2})}{N(N-1)\Gamma(\frac{d}{2})}\right).$$

This implies

$$\mu(\Lambda)\leqslant N(N-1)\arcsin(\alpha)\frac{\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}<\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2})}=\mu(\mathbb{S}^{d-1}).$$

Therefore $\mu(\mathbb{S}^{d-1}\setminus\Lambda)>0$ and we can pick up $e_1\in\mathbb{S}^{d-1}$ such that for all $j,k=1,...,N$, we have

$$|(v_j-v_k,e_1)_{\mathbb{R}^d}|\geqslant\alpha|v_j-v_k|.$$

Completing e_1 into an orthonormal basis $(e_1,...,e_d)$ of $\mathbb{R}^d$ finishes the proof. □

By invariance of (NLS) with respect to orthonormal transformations we can assume without loss of generality that the basis $(e_1,...,e_d)$ is the canonical basis of $\mathbb{R}^d$. Up to a changes of indices, we can also assume that $v_1^1<...<v_N^1$ where the exponent 1 in v_j^1 denote the first coordinate of $v_j=(v_j^1,...,v_j^d)$.

Let $\psi:\mathbb{R}\to\mathbb{R}$ be a $\mathcal{C}^\infty$ cut-off function such that $\psi(s)=0$ for $s<-1$, $\psi(s)\in[0,1]$ if $s\in[-1,1]$ and $\psi(s)=1$ for $s>1$. We define

$$m_j:=\frac{1}{2}(v_{j-1}^1+v_j^1)\text{ for }j=2,...,N,$$

$$\psi_1(t,x):=1,\ \psi_j(t,x):=\psi(\frac{1}{\sqrt{t}}(x^1-m_jt))\text{ for }j=2,...,N.$$

Then we can define

$$\phi_j = \psi_j - \psi_{j+1} \text{ for } j = 1, ..., N-1, \ \phi_N := \psi_N.$$

We introduce localized versions of the energy, charge and momentum. For $j = 1, ..., N$ we define

$$E_j(t,w) := \frac{1}{2}\int_{\mathbb{R}^d} |\nabla w|^2 \phi_j dx - \int_{\mathbb{R}^d} F(w)\phi_j dx,$$

$$M_j(t,w) := \int_{\mathbb{R}^d} |w|^2 \phi_j dx, \ P_j(t,w) := \mathscr{Im}\int_{\mathbb{R}^d} (\nabla w)\bar{w}\phi_j dx.$$

We denote by S_j a localized action defined for $w \in H^1(\mathbb{R}^d)$ by

$$S_j(t,w) := E_j(t,w) + \frac{1}{2}\left(\omega_j + \frac{|v_j|^2}{4}\right) M_j(t,w) - \frac{1}{2} v_j \cdot P_j(t,w)$$

and by H_j a localized linearized defined for $w \in H^1(\mathbb{R}^d)$ by

$$H_j(t,w) := \int_{\mathbb{R}^d} |\nabla w|^2 \phi_j dx - \int_{\mathbb{R}^d} \left(g(|R_j|^2)|w|^2 + 2g'(|R_j|^2)\mathscr{Re}(R_j\bar{w})^2\right)\phi_j dx$$
$$+ \left(\omega_j + \frac{|v_j|^2}{4}\right)\int_{\mathbb{R}^d} |w|^2\phi_j dx - v_j \cdot \mathscr{Im}\int_{\mathbb{R}^d} \bar{w}\nabla w \phi_j dx.$$

We define an action-like functional for multi-solitons

$$\mathcal{S}(t,w) := \sum_{j=1}^{N} S_j(t,w)$$

and a corresponding linearized

$$\mathcal{H}(t,w) := \sum_{j=1}^{N} H_j(t,w).$$

We have the following coercivity property on $\mathcal{H}$.

Lemma 14 (Coercivity for the multi-soliton)**.** *There exists $K = K(\Phi_1, ..., \Phi_N) > 0$ such that for all t large enough and for all $w \in H^1(\mathbb{R}^d)$ we have*

$$\|w\|^2_{H^1(\mathbb{R}^d)} \leqslant K\mathcal{H}(t,w) + K\sum_{j=1}^{N}\sum_{l=1}^{\nu_j}\left(w, X_j^l(t)\right)^2_{L^2(\mathbb{R}^d)},$$

where (ν_j), (X_j^l) are given for each R_j by Lemma 12.

Proof. It is a consequence of Lemma 12 (see [31, Lemma 4.1]). □

Lemma 15. *The following equality holds*

$$S_j(t,u(t)) = S_j(t,R_j) + H_j(t,v) + O(e^{-3\alpha\omega_\star^{\frac{1}{2}} v_\star t}) + o(\|v\|^2_{H^1(\mathbb{R}^d)}).$$

The proof relies on the following claim.

Claim 16. *For all $x \in \mathbb{R}^d$ and $j,k = 1,...,N$ the following inequalities holds.*

$$(|R_k(t,x)| + |\nabla R_k(t,x)|)\phi_j(t,x) \leqslant Ce^{-2\alpha\omega_\star^{\frac{1}{2}} v_\star t} e^{-\frac{\omega_\star^{\frac{1}{2}}}{2}|x - v_k t - x_k|} \text{ for } j \neq k,$$

$$(|R_j(t,x)| + |\nabla R_j(t,x)|)(1 - \phi_j(t,x)) \leqslant Ce^{-2\alpha\omega_\star^{\frac{1}{2}} v_\star t} e^{-\frac{\omega_\star^{\frac{1}{2}}}{2}|x - v_j t - x_j|}.$$

Proof. The claim follows immediately from the support properties of ϕ_j, the definitions of $\omega_\star$ and $v_\star$ and exponential decay of Φ_j. $\square$

Proof of Lemma 15. The proof is done by writing $u(t) = R(t) + v(t)$ and expanding in the definition of S_j. We start with the terms of order 0 in v. By Claim 16 we have

$$S_j(t,R) = S_j(t,R_j) + O(e^{-4\alpha\omega_\star^{\frac{1}{2}} v_\star t}). \tag{16}$$

We now look at the terms of order 1 in v. Still by Claim 16, taking in addition into account that $\|v\|_{H^1(\mathbb{R}^d)} = O(e^{-\alpha\omega_\star^{\frac{1}{2}} v_\star t})$ and remembering the equation solved by R_j (see (C.1)) we obtain,

$$\left\langle S_j'(t,R),v\right\rangle = \left\langle S_j'(t,R_j),v\right\rangle + O(e^{-3\alpha\omega_\star^{\frac{1}{2}} v_\star t}) = O(e^{-3\alpha\omega_\star^{\frac{1}{2}} v_\star t}), \tag{17}$$

$$\left\langle S_j''(t,R)v,v\right\rangle = H_j(t,v) + O(e^{-3\alpha\omega_\star^{\frac{1}{2}} v_\star t}) + o(\|v\|^2_{H^1(\mathbb{R}^d)}). \tag{18}$$

Gathering (16)-(18) we obtain the following expansion

$$S_j(t,u(t)) = S_j(t,R_j) + H_j(t,v) + O(e^{-3\alpha\omega_\star^{\frac{1}{2}} v_\star t}) + o(\|v\|^2_{H^1(\mathbb{R}^d)}),$$

which concludes the proof. $\square$

We can now write a Taylor-like expansion for $\mathcal{S}$.

Lemma 17. *We have*

$$\mathcal{S}(t,u) - \mathcal{S}(t,R) = \mathcal{H}(t,v) + o(\|v\|^2_{H^1(\mathbb{R}^d)}) + O(e^{-3\alpha\omega_\star^{\frac{1}{2}} v_\star t}).$$

Proof. In view of Lemma 15 all we need to prove is

$$\mathcal{S}(t,R) = \sum_{j=1}^{N} S_j(t,R_j) + O(e^{-3\alpha\omega_\star^{\frac{1}{2}} v_\star t}),$$

which follows immediately from Claim 16. $\square$

Lemma 18. *The following estimate holds.*

$$\left|\frac{\partial \mathcal{S}(t,u(t))}{\partial t}\right| \leqslant \frac{C}{\sqrt{t}} e^{-2\alpha\omega_\star^{\frac{1}{2}} v_\star t}.$$

Proof. We remark that

$$\mathcal{S}(t,w) = E(w) + \sum_{j=1}^{N}\left(\frac{1}{2}\left(\omega_j + \frac{|v_j|^2}{4}\right) M_j(t,w) - \frac{1}{2} v_j \cdot P_j(t,w)\right).$$

Since the energy E is conserved by the flow of (NLS), to estimate the variations of $\mathcal{S}(t,u(t))$ we only have to study the variations of the localized masses $M_j(t,u(t))$ and momentums $P_j(t,u(t))$. Take any $j = 2,...,N$. We have

$$\begin{aligned}&\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbb{R}^d} |u(t)|^2 \psi_j(t,x)dx \\ &\quad = \frac{1}{\sqrt{t}}\int_{\mathbb{R}^d}\left(\mathscr{I}\!m(\bar{u}\partial_1 u) - |u|^2 \frac{x^1 + m_j t}{4t}\right)\psi'(\frac{1}{\sqrt{t}}(x^1 - m_j t))dx.\end{aligned} \tag{19}$$

Define $I_j := [m_j t - \sqrt{t}, m_j t + \sqrt{t}] \times \mathbb{R}^{d-1}$. From (19) and the support properties of ψ we obtain

$$\left| \frac{\partial}{\partial t} \int_{\mathbb{R}^d} |u(t)|^2 \psi_j(t,x) dx \right| \leqslant \frac{C}{\sqrt{t}} \int_{I_j} |\nabla u|^2 + |u|^2 dx.$$

Similarly, for the first component of P_j we have

$$\begin{aligned} \frac{1}{2}\frac{\partial}{\partial t} \int_{\mathbb{R}^d} \bar{u} \partial_1 u \psi_j dx = \\ \frac{1}{\sqrt{t}} \int_{\mathbb{R}^d} \left(|\partial_1 u|^2 - g(|u|^2)|u|^2 + F(u) - \bar{u}\partial_1 u \frac{x^1 + m_j t}{2t} \right) \psi'(\frac{1}{\sqrt{t}}(x^1 - m_j t)) \\ - \frac{1}{4t}|u|^2 \psi'''(\frac{1}{\sqrt{t}}(x^1 - m_j t) dx. \end{aligned} \quad (20)$$

Combining (20) with the support properties of ψ and (A1)-(A2) we obtain

$$\left| \frac{\partial}{\partial t} \int_{\mathbb{R}^d} \bar{u} \partial_1 u \psi_j dx \right| \leqslant \frac{C}{\sqrt{t}} \left(\int_{I_j} |\nabla u|^2 + |u|^2 dx + \left(\int_{I_j} |\nabla u|^2 + |u|^2 dx \right)^{\frac{p+1}{2}} \right).$$

Similar arguments lead for $k \geqslant 2$ to

$$\left| \frac{\partial}{\partial t} \int_{\mathbb{R}^d} \bar{u} \partial_k u \psi_j dx \right| \leqslant \frac{C}{\sqrt{t}} \int_{I_j} |\nabla u|^2 + |u|^2 dx.$$

Now, we remark that

$$\int_{I_j} \left(|\nabla u|^2 + |u|^2 \right) dx \leqslant \int_{I_j} |\nabla R|^2 + |R|^2 dx + \|u - R\|^2_{H^1(\mathbb{R}^d)}.$$

Recall that by hypothesis we have

$$\|u - R\|_{H^1(\mathbb{R}^d)} = \|v\|_{H^1(\mathbb{R}^d)} \leqslant e^{-\alpha \omega_\star^{\frac{1}{2}} v_\star t}.$$

In addition, the decay properties of each Φ_k and the definition of I_j imply

$$\int_{I_j} \left(|\nabla u|^2 + |u|^2 \right) dx \leqslant C e^{-2\alpha \omega_\star^{\frac{1}{2}} v_\star t}.$$

Consequently,

$$\left| \frac{\partial}{\partial t} \int_{\mathbb{R}^d} |u(t)|^2 \psi_j(t,x) dx \right| + \left| \frac{\partial}{\partial t} \int_{\mathbb{R}^d} \bar{u} \nabla u \psi_j dx \right| \leqslant \frac{C}{\sqrt{t}} e^{-2\alpha \omega_\star^{\frac{1}{2}} v_\star t}.$$

Note that the previous inequality is trivial for $j = 1$ since $\psi_1 = 1$ and the mass and momentum are conserved. Plugging the previous into the expressions of M_j and P_j gives

$$\left| \frac{\partial}{\partial t} \left(M_j(t,u) + P_j(t,u) \right) \right| \leqslant \frac{C}{\sqrt{t}} e^{-2\alpha \omega_\star^{\frac{1}{2}} v_\star t}$$

and the desired conclusion readily follows. □

Proof of Proposition 10. Let $K = K(\Phi_1, ..., \Phi_N)$ and $m := \sum_{j=1}^N \nu_j$ be given by Lemma 14. Since $\|X_j^k(t)\|_{L^2(\mathbb{R}^d)} = 1$ for any t, j, k, by Lemma 11, there exists $v_\sharp = v_\sharp(\Phi_1, ..., \Phi_N)$ such that if $v_\star > \alpha^{-1} v_\sharp$ we have for $j = 1, ..., N$, $k = 1, ..., \nu_j$ that

$$\left(v(t), X_j^k(t) \right)^2_{L^2(\mathbb{R}^d)} \leqslant \|v(t)\|^2_{L^2(\mathbb{R}^d)} \leqslant \frac{1}{2mK} e^{-2\alpha \omega_\star^{\frac{1}{2}} v_\star t}. \quad (21)$$

Using Lemma 18 we obtain

$$\mathcal{S}(t,u(t)) - \mathcal{S}(T_n,u(T_n)) \leqslant \int_t^{T_n} \left|\frac{\partial \mathcal{S}(s,u(s))}{\partial s}\right| ds \leqslant \frac{C}{\sqrt{t}} e^{-2\alpha\omega_\star^{\frac{1}{2}} v_\star t}. \tag{22}$$

Note that since $u_n(T_n) = R(T_n)$ we have

$$\mathcal{S}(T_n,u(T_n)) - \mathcal{S}(T_n,R(T_n)) = 0 \tag{23}$$

By Lemma 17, (22)-(23) imply

$$\mathcal{H}(t,v) \leqslant \frac{Ce^{-2\alpha\omega_\star^{\frac{1}{2}} v_\star t}}{\sqrt{t}} + o(\|v\|_{H^1(\mathbb{R}^d)}^2). \tag{24}$$

Combining (21)-(24) and Lemma 14 we get

$$\|v\|_{H^1(\mathbb{R}^d)}^2 \leqslant \left(\frac{C}{\sqrt{t}} + \frac{1}{2}\right) e^{-2\alpha\omega_\star^{\frac{1}{2}} v_\star t} + o(\|v\|_{H^1(\mathbb{R}^d)}^2)$$

and we easily obtain the desired conclusion if T_0 is chosen large enough. □

3. Non-uniqueness and instability

In this section, we assume $g \in \mathcal{C}^\infty$ and (A1)-(A4) are satisfied. We take $N \in \mathbb{N} \setminus \{0,1\}$, and for $j = 1,...,N$, $\omega_j > 0$, $\gamma_j \in \mathbb{R}$, $v_j \in \mathbb{R}^d$, $x_j \in \mathbb{R}^d$ and $\Phi_j \in H^1(\mathbb{R}^d)$ a solution of (1) (with ω_0 replaced by ω_j). Recall that

$$R_j(t,x) = \Phi_j(x - v_j t - x_j) e^{i(\frac{1}{2}v_j\cdot x - \frac{1}{4}|v_j|^2 t + \omega_j t + \gamma_j)},$$
$$\omega_\star = \frac{1}{2}\min\{\omega_j, j = 1,...,N\}, \quad v_\star = \frac{1}{9}\min\{|v_j - v_k|; j,k = 1,...,N, j \neq k\}.$$

3.1. Construction of approximation profiles

Since (NLS) is Galilean invariant, we can assume without loss of generality that $v_1 = 0, \gamma_1 = 0, x_1 = 0$. For notational brevity we drop in this subsection the subscript 1 indicating that we work we the first excited state. Hence we will write (in this subsection only) $R_1(t,x) = R(t,x)$, $\Phi_1 = \Phi$, etc.

Note first $df(z).w = g(|z|^2)w + 2\mathscr{R}_e(z\bar{w})g'(|z|^2)z$ is not $\mathbb{C}$-linear. This is why we shall identify $\mathbb{C}$ with $\mathbb{R}^2$ and use the notation $a + ib = \begin{pmatrix} a \\ b \end{pmatrix}$ $(a, b \in \mathbb{R})$, so as to consider operators with real entries. Given a vector $v \in \mathbb{C}^2$, we denote v^+ and v^- its components (so that if v represents a complex number, v^+ is the real part and v^- the imaginary part). To avoid confusion, we will denote with an index whether we consider the operator with $\mathbb{C}$, $\mathbb{R}^2$, or $\mathbb{C}^2$-valued functions.

Thus, as we consider

$$\mathscr{L}_\mathbb{C} v = -i\Delta v - idf(R).v, \quad L_\mathbb{C} v = -i\Delta v + i\omega v - idf(\Phi).v,$$

and the non-linear operators

$$\mathscr{N}_\mathbb{C}(v) = if(R+v) - if(R) - idf(R).v,$$
$$\mathscr{M}_\mathbb{C}(v) = e^{-i\omega t}\mathscr{N}(e^{i\omega t}v) = if(\Phi+v) - if(\Phi) - idf(\Phi).v,$$

then for instance

$$L_{\mathbb{R}^2}\begin{pmatrix} v^+ \\ v^- \end{pmatrix} = \begin{pmatrix} J & \Delta - \omega + I^- \\ -\Delta + \omega - I^+ & -J \end{pmatrix}\begin{pmatrix} v^+ \\ v^- \end{pmatrix}.$$

with Φ^+ and Φ^- the real and imaginary parts of Φ and

$$J = 2\Phi^+\Phi^- g'(|\Phi|^2), \quad I^\pm = g(|\Phi|^2) + 2{\Phi^\pm}^2 g'(|\Phi|^2).$$

Now $L_{\mathbb{R}^2}$ is as an (unbounded) $\mathbb{R}$-linear operator on $H^2(\mathbb{R}^d,\mathbb{R}^2) \to L^2(\mathbb{R}^d,\mathbb{R}^2)$. So as to have some eigenfunctions, we can complexify, and we are interested in $L_{\mathbb{C}^2} : H^2(\mathbb{R}^d,\mathbb{C}^2) \to L^2(\mathbb{R}^d,\mathbb{C}^2)$, which is a $\mathbb{C}$-linear operator with real entries.

Let $\alpha > 0$ be the decay rate given by Proposition 25 for eigenfunctions of L with eigenvalue λ (see (A4)). Possibly taking a smaller value of α, we can assume $\alpha \in (0, \sqrt{\omega})$. For $\mathbb{K} = \mathbb{R}, \mathbb{R}^2, \mathbb{C}$ or $\mathbb{C}^2$, denote

$$\mathscr{H}(\mathbb{K}) = \{v \in H^\infty(\mathbb{R}^d,\mathbb{K}) |\ e^{\alpha|x|}|D^a v| \in L^\infty(\mathbb{R}^d) \text{ for any multi-index } a\}. \tag{25}$$

We have gathered in the following proposition some properties of $L_{\mathbb{C}^2}$ that shall be needed for our analysis.

Proposition 19 (Properties of $L_{\mathbb{C}^2}$)**.**

(i) The eigenvalue $\lambda = \rho + i\theta \in \mathbb{C}$ of $L_{\mathbb{C}^2}$ can be chosen with maximal real part. We denote $Z(x) = \begin{pmatrix} Z^+(x) \\ Z^-(x) \end{pmatrix} \in H^2(\mathbb{R}^d,\mathbb{C}^2)$ an associated eigenfunction.

(ii) $\Phi \in \mathscr{H}(\mathbb{R}^2)$ and $Z \in \mathscr{H}(\mathbb{C}^2)$.

(iii) Let $\mu \notin \mathrm{Sp}(L_{\mathbb{R}^2})$, and $A \in \mathscr{H}(\mathbb{C}^2)$. Then there exists a solution $X \in \mathscr{H}(\mathbb{C}^2)$ to $(L-\mu I)X = A$, and $(L-\mu I)^{-1}$ is a continuous operator on $\mathscr{H}(\mathbb{C}^2)$.

Exponential decay of eigenvalues of L is a fact of independent interest. Hence we have stated the result under general assumptions in the Appendix A (see Proposition 25). Notice that we treat all possible eigenvalues (in particular without assuming $|\mathscr{I}\!m\,\lambda| < \omega$, as it is the case for example in [23]).

Proof. (i) It is well known that the spectrum of $L_{\mathbb{C}^2}$ is composed of essential spectrum on $\{iy, y \in \mathbb{R}, |y| \geqslant \omega\}$ and eigenvalues symmetric with respect to the real and imaginary axes (see e.g. [19, 23]). The set of eigenvalues with positive real part is non-empty due to (A4). As $L_{\mathbb{C}^2}$ is a compact perturbation of $\begin{pmatrix} 0 & \Delta - \omega \\ -\Delta + \omega & 0 \end{pmatrix}$ there exists an eigenvalue λ with maximal real part.

(ii) Exponential decay of Φ, $\nabla\Phi$ is a well-known fact (see e.g. [7]). Then using the equation satisfied by Φ, one deduces that $\Phi \in \mathscr{H}(\mathbb{R}^2)$. The decay and regularity of the eigenfunction Z rely essentially on the decay and regularity of Φ. Therefore, we leave the proof to Appendix A, Proposition 25 and Proposition 30.

(iii) Regularity of X follows from a simple bootstrap argument. For the exponential decay, we use the properties of fundamental solutions of Helmoltz equations (see Proposition 30). □

To conclude with the notations, we define the decay class $O(\chi(t))$, which we will use for functions decaying exponentially in time.

Definition 20. *Let $\xi \in \mathcal{C}^\infty(\mathbb{R}^+, H^\infty(\mathbb{R}^d))$ and $\chi : \mathbb{R}^+ \to (0,+\infty)$. Then we denote*

$$\xi(t) = O(\chi(t)) \quad \text{as } t \to +\infty,$$

if, for all $s \geqslant 0$, there exists $C(s) > 0$ such that

$$\forall t \geqslant 0, \quad \|\xi(t)\|_{H^s(\mathbb{R}^d)} \leqslant C(s)\chi(t).$$

Let $Y_1 := \mathscr{R}\!e(Z) = \begin{pmatrix} \mathscr{R}\!e(Z^+) \\ \mathscr{R}\!e(Z^-) \end{pmatrix}$ and $Y_2 := \mathscr{I}\!m(Z) = \begin{pmatrix} \mathscr{I}\!m(Z^+) \\ \mathscr{I}\!m(Z^-) \end{pmatrix}$. Then $Y_1, Y_2 \in \mathscr{H}(\mathbb{R}^2)$, and

$$\begin{cases} L_{\mathbb{R}^2} Y_1 &= \rho Y_1 - \theta Y_2, \\ L_{\mathbb{R}^2} Y_2 &= \theta Y_1 + \rho Y_2. \end{cases}$$

Denote

$$Y(t) = e^{-\rho t}(\cos(\theta t)Y_1 + \sin(\theta t)Y_2). \tag{26}$$

Lemma 21. *The function Y verifies for all $t \in \mathbb{R}$ the following equation.*

$$\partial_t Y + L_{\mathbb{R}^2} Y = 0. \tag{27}$$

Proof. Indeed, we compute

$$\begin{aligned}
&\partial_t(e^{-\rho t}(\cos(\theta t)Y_1 + \sin(\theta t)Y_2)),\\
&\quad = e^{-\rho t}\left((-\rho\cos(\theta t) - \theta\sin(\theta t))Y_1 + (-\rho\sin(\theta t) + \theta\cos(\theta t))Y_2\right),\\
&L_{\mathbb{R}^2}(e^{-\rho t}(\cos(\theta t)Y_1 + \sin(\theta t)Y_2)),\\
&\quad = e^{-\rho t}(\cos(\theta t)LY_1 + \sin(\theta t)LY_2)),\\
&\quad = e^{-\rho t}(\cos(\theta t)(\rho Y_1 - \theta Y_2) + \sin(\theta t)(\theta Y_1 + \rho Y_2),\\
&\quad = e^{-\rho t}((\rho\cos(\theta t) + \theta\sin(\theta t))Y_1 + (\rho\sin(\theta t) - \theta\cos(\theta t))Y_2.
\end{aligned}$$

So that $(\partial_t + L_{\mathbb{R}^2})(Y(t)) = 0$. □

Proposition 22. *Let $N_0 \in \mathbb{N}$ and $a \in \mathbb{R}$. Then there exists a profile $W^{N_0} \in \mathcal{C}^\infty([0,+\infty), \mathscr{H}(\mathbb{R}^2))$, such that as $t \to +\infty$,*

$$\partial_t W^{N_0} + L_{\mathbb{R}^2} W^{N_0} = \mathscr{M}_{\mathbb{R}^2}(W^{N_0}) + O(e^{-\rho(N_0+1)t}),$$

and $W^{N_0}(t) = aY(t) + O(e^{-2\rho t})$.

Remark 23. *Notice that $W^{N_0}(t,x)$ is a real valued vector. If we go back and consider W^{N_0} as a function taking values in $\mathbb{C}$, we then have, by definition of $\mathscr{M}$, with $U^{N_0}(t) = R(t) + e^{i\omega t}W^{N_0}(t)$,*

$$i\partial_t U^{N_0} + \Delta U^{N_0} + f(U^{N_0}) = O(e^{-\rho(N_0+1)t}).$$

For the proof of Proposition 22, we write W for W^{N_0} (for simplicity in notation) and we look for W in the following form

$$W(t,x) = \sum_{k=1}^{N_0} e^{-\rho k t}\left(\sum_{j=0}^{k} A_{j,k}(x)\cos(j\theta t) + B_{j,k}(x)\sin(j\theta t)\right), \tag{28}$$

where $A_{j,k} = \begin{pmatrix} A^+_{j,k} \\ A^-_{j,k}\end{pmatrix}$ and $B_{j,k} = \begin{pmatrix} B^+_{j,k} \\ B^-_{j,k}\end{pmatrix}$ are some functions of $\mathscr{H}(\mathbb{R}^2)$ to be determined.

We start by the expansion of $\mathscr{M}(W)$.

Claim 24. *We have*

$$\mathscr{M}_{\mathbb{R}^2}(W) = \sum_{\kappa=2}^{N_0} e^{-\kappa\rho t}\sum_{j=0}^{\kappa}\left(\tilde{A}_{j,\kappa}(x)\cos(j\theta t) + \tilde{B}_{j,\kappa}(x)\sin(j\theta t)\right) + O(e^{-(N_0+1)\rho t})$$

where $\tilde{A}_{j,\kappa}, \tilde{B}_{j,\kappa} \in \mathscr{H}(\mathbb{R}^2)$ depend on $A_{l,n}$ and $B_{l,n}$ only for $l \leqslant n \leqslant \kappa - 1$.

Proof. First we use a Taylor expansion. Due to smoothness of f and $\Phi \in \mathscr{H}(\mathbb{R}^2)$, and as $\mathscr{M}_{\mathbb{R}^2}$ is at least quadratic in v, there exists a polynomial $P_{N_0} \in \mathscr{H}(\mathbb{R}^2)[X,Y]$ with coefficients in $\mathscr{H}(\mathbb{R}^2)$, and valuation at least 2, such that :

$$\mathscr{M}_{\mathbb{R}^2}(v) = P_{N_0}(v^+, v^-) + O(|v|^{N_0+1}) = \sum_{m=2}^{N_0}\sum_{j=0}^{m}\begin{pmatrix} P_{j,m}(x)v_+^j v_-^{m-j} \\ Q_{j,m}(x)v_+^j v_-^{m-j}\end{pmatrix} + O(v^{N_0+1}),$$

where $P_{j,m},\ Q_{j,m} \in \mathscr{H}(\mathbb{R})$.

Consider now the term $W_+^n W_-^{m-n}$ and use (28). It writes

$$\left(\sum_{k=1}^{N_0} e^{-\rho k t}\left(\sum_{l=0}^{k} A_{l,k}^+ \cos(l\theta t) + B_{l,k}^+ \sin(l\theta t)\right)\right)^n \times \left(\sum_{k=1}^{N_0} e^{-\rho k t}\left(\sum_{l=0}^{k} A_{l,k}^- \cos(l\theta t) + B_{l,k}^- \sin(l\theta t)\right)\right)^{m-n}.$$

Now, the multinomial development gives

$$\sum_{\substack{i_1+\cdots+i_{N_0}=n \\ j_1+\cdots+j_{N_0}=m-n}} \frac{n!}{i_1!\cdots i_{N_0}!}\frac{(m-n)!}{j_1!\cdots j_{N_0}!} e^{-\rho t\sum_{k=1}^{N_0} k(i_k+j_k)} \times \prod_{k=1}^{N_0}\left[\left(\sum_{l=0}^{k}\left(A_{l,k}^+(x)\cos(l\theta t) + B_{l,k}^+(x)\sin(l\theta t)\right)\right)^{i_k} \times \left(\sum_{l=0}^{k}\left(A_{l,k}^-(x)\cos(l\theta t) + B_{l,k}^-(x)\sin(l\theta t)\right)\right)^{j_k}\right].$$

Fix some $(i_k)_k$, $(j_k)_k$ and define the decay rate $\kappa = \sum_{k=1}^{N_0} k(i_k + j_k)$. Then

$$\kappa \geqslant \sum_{k=1}^{N_0}(i_k + j_k) = n + (m-n) = m \geqslant 2.$$

The product factor is a trigonometric polynomial in t, it can be linearized into a sum of sin and cos with frequency $\ell\theta$ and $\ell \leqslant \sum_k k(i_k + j_k) = \kappa$.

Of course, as $W \in \mathscr{H}(\mathbb{R}^2)$, the higher order terms (i.e. with $\kappa \geqslant N_0 + 1$) all fit into $O(e^{-(N_0+1)\rho t})$.

It is now clear that $\tilde{A}_{j,\kappa}$ and $\tilde{B}_{j,\kappa}$ are polynomial in $A_{j,k}$, $B_{j,k}$, $P_{n,m}$, and $Q_{n,m}$. It remains to see that the $A_{j,k}$ or $B_{j,k}$ that intervene (i.e $i_k + j_k > 0$) come with $k \leqslant \kappa - 1$. Let a be the maximal index such that $i_a + j_a > 0$. Recall $i_1 + \cdots + i_{N_0} + j_1 + \cdots + j_{N_0} = m \geqslant 2$. If $i_a + j_a \geqslant 2$, we have $2a \leqslant a(i_a + j_a) \leqslant \kappa$ so that (as $\kappa \geqslant m \geqslant 2$) $a \leqslant \kappa - 1$. If $i_a + j_a = 1$, there exist $b \geqslant 1$, $b \neq a$, such that $i_b + j_b \geqslant 1$ and

$$\kappa = \sum_k k(i_k + j_k) \geqslant a(i_a + j_a) + b(i_b + j_b) \geqslant a + 1.$$

Finally the product has the desired properties. □

Proof of Proposition 22. By definition of W, we can compute:

$$(\partial_t W + L_{\mathbb{R}^2} W) = \sum_{k=1}^{N_0} e^{-\rho k t}\left(\sum_{j=0}^{k}(L_{\mathbb{R}^2} A_{j,k} + j\theta B_{j,k} - k\rho A_{j,k})\cos(j\theta t) + (L_{\mathbb{R}^2} B_{j,k} - j\theta A_{j,k} - k\rho B_{j,k})\sin(j\theta t)\right).$$

From the computations of Claim 24, it suffices to solve for all $0 \leqslant j \leqslant k \leqslant N_0$

$$\begin{cases} L_{\mathbb{R}^2} A_{j,k} + j\theta B_{j,k} - k\rho A_{j,k} = \tilde{A}_{j,k}, \\ L_{\mathbb{R}^2} B_{j,k} - j\theta A_{j,k} - k\rho B_{j,k} = \tilde{B}_{j,k}. \end{cases} \tag{29}$$

Obviously, one starts to solve for $k=1$, then from this $k=2$ etc. so that at all stages $\tilde{A}_{j,k}$ and $\tilde{B}_{j,k}$ are well defined (remark that $\tilde{A}_{j,1}=\tilde{B}_{j,1}=0$).

We initialized the induction process by setting $A_{1,1}=aY_1$, $B_{1,1}=aY_2$, and $A_{0,1}=B_{0,1}=0$. Assume that $A_{j,k}$ and $B_{j,k}$ are constructed up to $k\leqslant k_0-1$ and belong to $\mathscr{H}(\mathbb{R}^2)$, we now construct A_{j,k_0}, B_{j,k_0} for all $j\leqslant k_0$. By Claim 24, all $\tilde{A}_{j,k_0}$ and $\tilde{B}_{j,k_0}$ are constructed for $j\leqslant k_0$ and belong to $\mathscr{H}(\mathbb{R}^2)$.

Consider now the operator $L_{j,k_0}=L_{\mathbb{C}^2}-(k_0\rho+ij\theta)\operatorname{Id}$, $L_{j,k_0}:\mathscr{H}(\mathbb{C}^2)\to\mathscr{H}(\mathbb{C}^2)$. As $e=\rho+i\theta$ is an eigenvalue of $L_{\mathbb{C}^2}$ with maximal real part, for all $k_0\geqslant 2$ and all j, $k_0\rho+ij\theta\notin \operatorname{Sp}(L)$ so that L_{j,k_0} is invertible. Let $X=L_{j,k_0}^{-1}(\tilde{A}_{j,k_0}+i\tilde{B}_{j,k_0})$, and define $C:=\mathscr{Re}(X)=\begin{pmatrix}\mathscr{Re}(X^+)\\ \mathscr{Re}(X^-)\end{pmatrix}$, $D:=\mathscr{Im}(X)=\begin{pmatrix}\mathscr{Im}(X^+)\\ \mathscr{Im}(X^-)\end{pmatrix}$, so that $C,D\in\mathscr{H}(\mathbb{R}^2)$ and $X=C+iD$. Then we compute

$$\begin{aligned}\tilde{A}_{j,k_0}+i\tilde{B}_{j,k_0}&=L_{j,k_0}(C+iD)\\&=L_{\mathbb{R}^2}C+iL_{\mathbb{R}^2}D-k_0\rho C-ik_0D-ij\theta C+j\theta D\\&=(L_{\mathbb{R}^2}C-k_0\rho C+j\theta D)+i(L_{\mathbb{R}^2}D-j\theta C-k_0\rho D).\end{aligned}$$

Hence $A_{j,k_0}=C$ and $B_{j,k_0}=D$ are solutions to the system (29). □

We now switch back notation from vector valued functions to complex valued functions and summarize what we have obtained. We use again the subscript 1. Hence we can consider $V_1^{N_0}$, $U_1^{N_0}$ defined by

$$V_1^{N_0}(t,x):=e^{i\omega t}W^{N_0}(t,x),\quad U_1^{N_0}(t,x):=R_1(t,x)+V_1^{N_0}(t,x).$$

Then we define

$$\begin{aligned}Err_1^{N_0}(t,x)&:=i\partial_tU_1^{N_0}+\Delta U_1^{N_0}+f(U_1^{N_0})\\&=i\partial_tV_1^{N_0}+\Delta V_1^{N_0}+f(R_1(t)+V_1^{N_0})-f(R_1(t))\\&=i(\partial_tV_1^{N_0}+\mathscr{L}_{\mathbb{C}}V_1^{N_0}-\mathscr{N}_{\mathbb{C}}(V_1^{N_0}))\\&=ie^{i\omega t}(\partial_tW^{M_0}+L_{\mathbb{C}}W^{N_0}-\mathscr{M}_{\mathbb{C}}(W^{N_0})).\end{aligned}$$

By Proposition 22, $Err_1^{N_0}(t,x)=O(e^{-(N_0+1)\rho t})$. Also, from (28) we deduce $V_1^{N_0}(t)=ae^{i\omega t}Y(t)+O(e^{-2\rho t})$, so that for all $s\geqslant 0$, there exists $C(N_0,s)$ such that

$$\forall t\geqslant 0,\quad \|V_1^{N_0}(t)\|_{H^s(\mathbb{R}^d)}\leqslant C(N_0,s)e^{-\rho t}. \tag{30}$$

3.2. Proofs of Theorems 2 and 3

Proof of Theorem 2. Let N_0 to be determined later, we do a fixed point around $U_1^{N_0}(t)$. Suppose $u=U_1^{N_0}(t)+w(t)$ (with $w(t)\to 0$ as $t\to+\infty$) is a solution to (NLS), then

$$i\partial_tw+\Delta w+f(U_1^{N_0}+w)-f(U_1^{N_0})-Err_1^{N_0}(t)=0$$

From this, Duhamel's Formula gives, for $t\leqslant s$,

$$w(s)=e^{i\Delta(s-t)}w(t)+i\int_t^s e^{i\Delta(s-\tau)}\Big(f((U_1^{N_0}+w)(\tau))-f(U_1^{N_0}(\tau))-Err_1^{N_0}(\tau)\Big)\,d\tau,$$

so that

$$e^{-i\Delta s}w(s)=e^{-i\Delta t}w(t)+i\int_t^s e^{-i\Delta\tau}\Big(f((U_1^{N_0}+w)(\tau))-f(U_1^{N_0}(\tau))-Err_1^{N_0}(\tau)\Big)\,d\tau.$$

Letting $s\to+\infty$, as $w(s)\to 0$, we are looking for a solution to the fixed point equation

$$w(t)=-i\int_t^{+\infty}e^{i\Delta(t-\tau)}(f((U_1^{N_0}+w)(\tau))-f(U_1^{N_0}(\tau))-Err_1^{N_0}(\tau))d\tau.$$

Hence, we define the map

$$v \mapsto \Psi(v) = -i\int_t^{+\infty} e^{i\Delta(t-\tau)}(f((R_1+V_1^{N_0}+v)(\tau)) - f((R_1+V_1^{N_0})(\tau)) - Err_1^{N_0}(\tau))d\tau.$$

Fix $\sigma > \frac{d}{2}$, so that $H^\sigma(\mathbb{R}^d)$ is an algebra, and let B, T_0 to be determined later. For $w \in \mathcal{C}((T_0,+\infty), H^\sigma(\mathbb{R}^d))$ define

$$\|w\|_{X^\sigma_{T_0,N_0}} = \sup_{t\geqslant T_0} e^{(N_0+1)\rho t}\|w(t)\|_{H^\sigma(\mathbb{R}^d)},$$

to be the norm of the Banach space

$$X^\sigma_{T_0,N_0} := \left\{ w \in \mathcal{C}((T_0,+\infty),H^\sigma(\mathbb{R}^d)) \middle| \|w\|_{X^\sigma_{T_0,N_0}} < +\infty \right\}.$$

Consider the ball of radius B of $X^\sigma_{T_0,N_0}$

$$X^\sigma_{T_0,N_0}(B) := \left\{ w \in X^\sigma_{T_0,N_0} \middle| \|w\|_{X^\sigma_{T_0,N_0}} \leqslant B \right\}.$$

By (30), we can assume T_0 is large enough so that

$$\|V_1^{N_0}\|_{H^\sigma(\mathbb{R}^d)} \leqslant 1 \text{ and also } Be^{-(N_0+1)\rho T_0} \leqslant 1.$$

Our problem is to find a fixed point for Ψ, we will find it in $X^\sigma_{T_0,N_0}(B)$ for adequate parameters.

Notice that for $t \geqslant T_0$, $\|V_1^{N_0}(t)\|_{H^\sigma(\mathbb{R}^d)} \leqslant 1$. Hence, we will always work in the $H^\sigma(\mathbb{R}^d)$-ball of radius $r_\sigma = \|\Phi_1\|_{H^\sigma(\mathbb{R}^d)} + 2$. Due to $\mathcal{C}^{\sigma+1}$ smoothness of f, there exists a constant K_σ such that

$$\forall a,b \in B_{H^\sigma(\mathbb{R}^d)}(r_\sigma), \quad \|f(a)-f(b)\|_{H^\sigma(\mathbb{R}^d)} \leqslant K_\sigma\|a-b\|_{H^\sigma(\mathbb{R}^d)}.$$

In particular, for all t,

$$\|f(R_1(t)+V_1^{N_0}(t)+v) - f(R_1(t)+V_1^{N_0}(t))\|_{H^\sigma(\mathbb{R}^d)} \leqslant K_\sigma\|v\|_{H^\sigma(\mathbb{R}^d)}.$$

Hence, as $e^{i\Delta(t-s)}$ is an isometry in $H^\sigma(\mathbb{R}^d)$, for any $v \in X^\sigma_{T_0,N_0}(B)$ we have

$$\begin{aligned}
&\|\Psi(v)(t)\|_{H^\sigma(\mathbb{R}^d)} \\
&\quad = \left\| \int_t^{+\infty} e^{i\Delta(t-\tau)} \left[f(R_1+V_1^{N_0}+v) - f(R_1+V_1^{N_0}) - Err_1^{N_0} \right](\tau)d\tau \right\|_{H^\sigma(\mathbb{R}^d)} \\
&\quad \leqslant \int_t^{+\infty} (\|f(R_1+V_1^{N_0}+v) - f(R_1+V_1^{N_0})\|_{H^\sigma(\mathbb{R}^d)} + \|Err_1^{N_0}(\tau)\|_{H^\sigma(\mathbb{R}^d)})d\tau \\
&\quad \leqslant \int_t^{+\infty} (K_\sigma\|v\|_{H^\sigma(\mathbb{R}^d)} + C(N_0,\sigma)e^{-(N_0+1)\rho\tau})d\tau \\
&\quad \leqslant \frac{K_\sigma B + C(N_0,\sigma)}{(N_0+1)\rho} e^{-(N_0+1)\rho t}.
\end{aligned}$$

First choose N_0 large enough so that $\frac{K_\sigma}{(N_0+1)\rho} \leqslant \frac{1}{2}$. Then choose $B = 2\frac{C(N_0,\sigma)}{(N_0+1)\rho}$. Finally choose T_0 large enough so that $C(N_0,\sigma)e^{-\rho T_0} \leqslant 1$. Hence we get

$$\|\Psi(v)(t)\|_{H^\sigma(\mathbb{R}^d)} \leqslant Be^{-(N_0+1)\rho t}.$$

This shows that Ψ maps $X^\sigma_{T_0,N_0}(B)$ to itself. Let us now show that Ψ is a contraction in $X^\sigma_{T_0,N_0}(B)$. Let $v, w \in X^\sigma_{T_0,N_0}(B)$ then we have

$$\Psi(v)(t) - \Psi(w)(t) = -i\int_t^{+\infty} e^{i\Delta(t-s)}(f(R_1+V_1^{N_0}+v) - f(R_1+V_1^{N_0}+w))ds.$$

As previously, we have

$$\begin{aligned}
&e^{(N_0+1)\rho t}\|\Psi(v)(t)-\Psi(w)(t)\|_{H^\sigma(\mathbb{R}^d)}\\
&\quad = e^{(N_0+1)\rho t}\left\|\int_t^{+\infty} e^{i\Delta(t-s)}(f(R_1+V_1^{N_0}+v)-f(R_1+V_1^{N_0}+w))ds\right\|_{H^\sigma(\mathbb{R}^d)}\\
&\quad \leqslant e^{(N_0+1)\rho t}\int_t^{+\infty}\|f(R_1+V_1^{N_0}+v)-f(R_1+V_1^{N_0}+w)\|_{H^\sigma(\mathbb{R}^d)}ds\\
&\quad \leqslant e^{(N_0+1)\rho t}\int_t^{+\infty} K_\sigma\|v(s)-w(s)\|_{H^\sigma(\mathbb{R}^d)}ds\\
&\quad \leqslant K_\sigma e^{(N_0+1)\rho t}\int_t^{+\infty} e^{-(N_0+1)\rho s}\|v-w\|_{X^\sigma_{T_0,N_0}}ds\\
&\quad \leqslant K_\sigma e^{(N_0+1)\rho t}\|v-w\|_{X^\sigma_{T_0,N_0}}\frac{e^{-(N_0+1)\rho t}}{(N_0+1)\rho}\\
&\quad \leqslant \frac{K_\sigma}{(N_0+1)\rho}\|v-w\|_{X^\sigma_{T_0,N_0}}.
\end{aligned}$$

Taking the supremum over $t\geqslant T_0$, we deduce that

$$\|\Psi(v)-\Psi(w)\|_{X^\sigma_{T_0,N_0}}\leqslant\frac{K_\sigma}{(N_0+1)\rho}\|v-w\|_{X^\sigma_{T_0,N_0}}\leqslant\frac{1}{2}\|v-w\|_{X^\sigma_{T_0,N_0}}.$$

Hence, Ψ is a contraction on $X^\sigma_{T_0,N_0}(B)$, and has a unique fixed point $\bar v$. Notice that we have obtained a unique fixed point for any $\sigma\geqslant\frac{d}{2}$: from this we deduce that $\bar v$ does not depend on σ, and hence, $\bar v\in\mathcal{C}^\infty([T_0,+\infty),H^\infty(\mathbb{R}^d))$. Then $\bar u=R_1+V_1^{N_0}+\bar v$ is the desired solution. □

Proof of Theorem 3. The proof is essentially a generalization of that of Theorem 2. Let $v_\natural$ to be fixed later and assume that $v_\star>v_\natural$. Let N_0 to determined later and $a\in\mathbb{R}$, from this we dispose of a profile $V_1^{N_0}(t)$, $U_1^{N_0}(t)$, an error term $Err_1^{N_0}(t)$ associated to $R_1(t)$, and an eigenvalue $\lambda=\rho+i\theta$ of L. We look for a solution of the form $u(t)=U_1^{N_0}(t)+\sum_{j\geqslant 2}R_j(t)+w(t)$. Then w satisfies

$$i\partial_t w+\Delta w+f(U_1^{N_0}+\sum_{j\geqslant 2}R_j+w)-f(U_1^{N_0})-\sum_{j\geqslant 2}f(R_j)-Err_1^{N_0}=0.$$

Hence considering the map

$$v\mapsto\Psi(v)=-i\int_t^{+\infty}e^{i\Delta(t-s)}(f((U_1^{N_0}+\sum_{j\geqslant 2}R_j+v)(s))-f(U_1^{N_0}(s))-\sum_{j\geqslant 2}f(R_j(s))-Err_1^{N_0}(s))ds,$$

we are looking for a fixed point for Ψ, in the set $X^\sigma_{T_0,N_0}(B)$ (defined in the proof of Theorem 2) for adequate parameters T_0,N_0,B,σ. Let $\sigma>\frac{d}{2}$. As previously, let T_0 large enough so that $\|V_1^{N_0}(t)\|_{H^s(\mathbb{R}^d)}\leqslant 1$ for $t\geqslant T_0$, and $Be^{-(N_0+1)\rho T_0}\leqslant 1$, so that we remain in a ball of radius 1 in $H^\sigma(\mathbb{R}^d)$.

Using exponential localization of the solitons R_j and of the profile $U_1^{N_0}$, we deduce as in the proof of Theorem 2 that for some $K_\sigma=K(f,\|U_1^{N_0}\|_{H^\sigma(\mathbb{R}^d)}+\sum_{j\geqslant 2}\|R_j\|_{H^\sigma(\mathbb{R}^d)}+1)$, we have

$$\|f((U_1^{N_0}+\sum_{j\geqslant 2}R_j+v))-f(U_1^{N_0})-\sum_{j\geqslant 2}f(R_j)\|_{H^\sigma(\mathbb{R}^d)}\leqslant K_\sigma\|v\|_{H^\sigma(\mathbb{R}^d)}+O(e^{-2\alpha\sqrt{\omega_\star}v_\star t}),$$

possibly by taking a smaller value of $\omega_\star$ such that $\omega_\star\leqslant\alpha_1$, where α_1 is the (exponential) decay rate of $U_1^{N_0}$.

Notice that α_1 is independent of N_0, due to the construction of $U_1^{N_0}$. Hence we have as in Theorem 2:

$$\begin{aligned}
&\|\Psi(v)(t)\|_{H^\sigma(\mathbb{R}^d)}\\
&\quad= \left\| \int_t^{+\infty} e^{i\Delta(t-s)}(f(U_1^{N_0} + \sum_{j\geqslant 2} R_j + v) - f(U_1^{N_0}) - \sum_{j\geqslant 2} f(R_j) - Err_1^{N_0})ds, \right\|_{H^\sigma(\mathbb{R}^d)}\\
&\quad\leqslant \int_t^{+\infty} \|f(U_1^{N_0} + \sum_{j\geqslant 2} R_j + v) - f(U_1^{N_0}) - \sum_{j\geqslant 2} f(R_j)\|_{H^\sigma(\mathbb{R}^d)} + \|Err_1^{N_0}\|_{H^\sigma(\mathbb{R}^d)} ds\\
&\quad\leqslant \int_t^{+\infty} (K_\sigma \|v\|_{H^\sigma(\mathbb{R}^d)} + C(N_0,\sigma)e^{-(N_0+1)\rho s} + C(\sigma)e^{-2\alpha\sqrt{\omega_\star}v_\star s})ds\\
&\quad\leqslant \frac{K_\sigma B + C(N_0,\sigma)}{(N_0+1)\rho} e^{-(N_0+1)\rho t} + \frac{C(\sigma)}{2\alpha\sqrt{\omega_\star}v_\star} e^{-2\alpha\sqrt{\omega_\star}v_\star t}.
\end{aligned}$$

First choose N_0 large enough so that $\frac{K_\sigma}{(N_0+1)\rho} \leqslant \frac{1}{3}$ and set $B := \frac{3C(N_0,\sigma)}{(N_0+1)\rho}$. Recall that $v_\star > v_\natural$. We chose $v_\natural$ large enough so that from the choice of $\omega_\star, v_\star$, we have

$$\frac{C(\sigma)}{2\alpha\sqrt{\omega_\star}v_\star} \leqslant \frac{B}{3}, \text{ and } 2\alpha\sqrt{\omega_\star}v_\star \geqslant (N_0+1)\rho.$$

Finally choose T_0 large enough so that

$$Be^{-(N_0+1)\rho T_0} \leqslant 1, \quad \text{and} \quad C(N_0,\sigma)e^{-\rho T_0} \leqslant 1.$$

From this, $\|\Psi(v)(t)\|_{H^\sigma(\mathbb{R}^d)} \leqslant Be^{-(N_0+1)\rho t}$ for $t \geqslant T_0$, i.e. Ψ maps $X^\sigma_{T_0,N_0}(B)$ to itself. Similar computations show that Ψ is a contracting map, so that it has a unique fixed point $\bar{w}$. Again as in Theorem 2, $\bar{w}$ does not depend on σ and $\bar{w} \in \mathcal{C}^\infty([T_0,+\infty), H^\infty(\mathbb{R}^d))$. Then $\bar{u} = U_1^{N_0} + \sum_{j\geqslant 2} R_j(t) + \bar{w}(t)$ fulfills the requirements. □

Appendix A. Exponential decay of eigenfunctions to matrix Schrödinger operators

We consider an operator $L : H^2(\mathbb{R}^d,\mathbb{C}^2) \subset L^2(\mathbb{R}^d,\mathbb{C}^2) \to L^2(\mathbb{R}^d,\mathbb{C}^2)$ of the form

$$L = \begin{pmatrix} W_1 & -\Delta + \omega + V_1 \\ \Delta - \omega + V_2 & W_2 \end{pmatrix}$$

where $\omega > 0$ and V_1, V_2, W_1, W_2 are complex-valued potentials satisfying the following assumptions.

(VW1) There exists $q \in (\max\{2, \frac{d}{2}\}, +\infty]$ such that $V_k, W_k \in L^q(\mathbb{R}^d)$ for $k = 1, 2$.

(VW2) $\lim_{|x|\to+\infty} V_k(x) = \lim_{|x|\to+\infty} W_k(x) = 0$ for $k = 1, 2$.

Assumptions (VW1)-(VW2) are probably not optimal, but they are sufficient in the context in which we want to apply the following Proposition 25.

Our goal is to prove that if L has an eigenvalue which does not belong to the set $\{iy, y \in \mathbb{R}, |y| \geqslant \omega\}$ (which is the essential spectrum of L, see e.g. [23]) then the corresponding eigenvectors are exponentially decaying at infinity. Note that it was previously known only for eigenfunctions corresponding to eigenvalues lying in the strip $\{z \in \mathbb{C}, |\mathscr{Im}(z)| < \omega\}$ and with a restricted class of potentials (see [23]).

Proposition 25. *Assume that (VW1)-(VW2) hold. Take $u, v \in H^2(\mathbb{R}^d,\mathbb{C})$, $\lambda \in \mathbb{C} \setminus \{iy, y \in \mathbb{R}, |y| \geqslant \omega\}$, and suppose that for $U := \begin{pmatrix} u \\ v \end{pmatrix}$ we have $LU = \lambda U$. Then there exist $C > 0$ and $\alpha > 0$ such that for all $x \in \mathbb{R}^d$ we have*

$$|u(x)| + |v(x)| \leqslant Ce^{-\alpha|x|}.$$

Our proof consists in obtaining estimates on fundamental solutions to Helmholtz equations and considering the eigenvalue problem $LU = \lambda U$ as an inhomogeneous problem.

Appendix A.1. Fundamental solutions

For a given $\mu \in \mathbb{C}$, a fundamental solution of the Helmholtz equation in $\mathbb{R}^d$ is a solution of

$$(-\Delta - \mu) g_\mu^d = \delta_0.$$

Setting $\nu := \frac{d-2}{2}$ fundamental solutions of the Helmholtz equation are given by

$$g_\mu^d(x) := \frac{i\pi\mu^{\frac{\nu}{2}}}{2|x|^\nu (2\pi)^{\frac{d}{2}}} H_\nu^1(\sqrt{\mu}|x|),$$

where H_ν^1 is the first Hankel function (see e.g. [1]). For $\mu = \rho e^{i\theta}$ with $\rho \geqslant 0$ and $\theta \in [0, 2\pi)$ we defined $\sqrt{\mu}$ by $\sqrt{\mu} := \rho^{\frac{1}{2}} e^{i\frac{\theta}{2}}$. Defining $\sqrt{\cdot}$ in this way ensures in particular that g_μ^d is square integrable for $\mu \notin \mathbb{R}^+$. The fundamental solutions g_μ^d verify the recurrence relation

$$g_\mu^{d+2}(x) = -\frac{\frac{\partial}{\partial r} g_\mu^d(x)}{2\pi|x|}.$$

We deduce the following formula for the fundamental solution. For $d = j + 2l$ where $j = 1, 2$ and $l \in \mathbb{N} \setminus \{0\}$, we have

$$g_\mu^{j+2l} = \sum_{k=1}^{l} a_l^k (-1)^k (g_\mu^j)^{(k)} |x|^{-2l+k}, \tag{A.1}$$

where the coefficients (a_l^k) are positive and the exponent (k) denotes the k^{th} derivative.

Lemma 26 (Estimates on fundamental solutions)**.** *Let $\mu \in \mathbb{C} \setminus \mathbb{R}^+$. Then there exists $\tau > 0$ and $C > 0$ such that*

$$|g_\mu^d(x)| \leqslant C g_{-\tau}^d(x) \text{ for all } x \in \mathbb{R}^d \setminus \{0\}.$$

In particular, g_μ^d is exponentially decaying at infinity with decay rate $\sqrt{\tau}$, i.e. $|g_\mu^d(x)| \leqslant C e^{-\sqrt{\tau}|x|}$ for $|x|$ large enough.

We separated the proof of Lemma 26 into two proofs depending on the oddness of d.

Proof for odd d. We have $\sqrt{\mu} = \rho^{\frac{1}{2}} e^{i\frac{\theta}{2}}$. Choose $\tau > 0$ such that $\sqrt{\tau} = \rho^{\frac{1}{2}} \sin\frac{\theta}{2}$. It is well-known that $g_\mu^1(x) = \frac{i}{2\sqrt{\mu}} e^{i\sqrt{\mu}|x|}$. It follows from easy computations that

$$\left|g_\mu^1(x)\right| \leqslant \frac{1}{2\sqrt{\rho}} e^{-\rho^{\frac{1}{2}} \sin\frac{\theta}{2}|x|}.$$

Since

$$g_{-\tau}^1(x) = \frac{1}{2\sqrt{\rho}\sin\frac{\theta}{2}} e^{-\rho^{\frac{1}{2}} \sin\frac{\theta}{2}|x|}$$

this readily implies that for all $x \in \mathbb{R}^d$ we have

$$|g_\mu^1(x)| \leqslant C g_{-\tau}^1(x),$$

which proves the lemma for $d = 1$.

Similar calculations lead to

$$|(g_\mu^1)^{(k)}| \leqslant C(-1)^k (g_{-\tau}^1)^{(k)} \text{ for all } k \in \mathbb{N}. \tag{A.2}$$

Assume now that $d \geqslant 3$ and take $l \in \mathbb{N} \setminus \{0\}$ such that $d = 1 + 2l$. Combining (A.1) and (A.2) gives

$$|g_\mu^{1+2l}(x)| \leqslant C g_{-\tau}^{1+2l}(x) \text{ for all } x \in \mathbb{R}^d \setminus \{0\},$$

which is the desired conclusion. □

Proof for even d. Let $\nu \in \mathbb{N}$ and $z \in \mathbb{C}$. We have the following asymptotic expansions on the Hankel functions (see [1]).

$$\begin{aligned} iH_0^1(z) &\approx -\frac{2}{\pi}\ln(z) && \text{for } |z| \text{ close to } 0, \\ iH_\nu^1(z) &\approx \frac{\nu! z^{-\nu}}{2^{-\nu}\pi} && \text{for } |z| \text{ close to } 0, \nu \neq 0, \\ H_\nu^1(z) &\approx \sqrt{\frac{2}{\pi z}} e^{i(z-\frac{\nu\pi}{2}-\frac{\pi}{4})} && \text{for } |z| \text{ close to } +\infty. \end{aligned}$$

Therefore, we can infer the following estimates on the fundamental solutions. Recall that $d = 2+2\nu$ and $\mu = \rho e^{i\theta}$.

$$|g_\mu^2(x)| \leqslant C|\ln(\rho^{\frac{1}{2}}|x|)| \qquad \text{for } |x| \text{ close to } 0, \tag{A.3}$$

$$|g_\mu^d(x)| \leqslant C|x|^{-\nu} \qquad \text{for } |x| \text{ close to } 0, \nu \neq 0, \tag{A.4}$$

$$|g_\mu^d(x)| \leqslant C|x|^{-(\nu+1)} e^{-\rho^{\frac{1}{2}}\sin(\frac{\theta}{2})|x|} \qquad \text{for } |x| \text{ close to } +\infty. \tag{A.5}$$

For $\tau > 0$, the function $g_{-\tau}^d$ verifies $g_{-\tau}^d > 0$ and

$$g_{-\tau}^2(x) \approx C|\ln(\tau^{\frac{1}{2}}|x|)| \qquad \text{for } |x| \text{ close to } 0, \tag{A.6}$$

$$g_{-\tau}^d(x) \approx C|x|^{-\nu} \qquad \text{for } |x| \text{ close to } 0, \nu \neq 0 \tag{A.7}$$

$$g_{-\tau}^d(x) \approx C|x|^{-(\nu+1)} e^{-\tau^{\frac{1}{2}}|x|} \qquad \text{for } |x| \text{ close to } +\infty. \tag{A.8}$$

Choose $\tau > 0$ such that $\tau^{\frac{1}{2}} = \sqrt{\rho}\sin\frac{\theta}{2}$. Then we infer from (A.3)-(A.8) and the continuity of fundamental solutions that there exists $C > 0$ such that

$$|g_\mu^d(x)| \leqslant C g_{-\tau}^d(x) \text{ for all } x \in \mathbb{R}^d \setminus \{0\},$$

which finishes the proof. □

Appendix A.2. Exponential decay

We start with a regularity result on eigenfunctions.

Lemma 27. *Assume that (VW1) is satisfied. Take* $\lambda \in \mathbb{C} \setminus \{iy, y \in \mathbb{R}, |y| \geqslant \omega\}$, $u, v \in H^2(\mathbb{R}^d, \mathbb{C})$ *and assume that for* $U := \begin{pmatrix} u \\ v \end{pmatrix}$ *we have* $LU = \lambda U$. *Then* $u, v \in W^{2,r}(\mathbb{R}^d)$ *for any* $r \in [2, q]$. *In particular,* $u, v \in \mathcal{C}^0(\mathbb{R}^d)$ *and* $\lim_{|x|\to+\infty} u(x) = \lim_{|x|\to+\infty} v(x) = 0$.

Proof. The result follows from a classical bootstrap argument. Let the sequence (r_n) be defined by

$$\begin{cases} r_0 &= 2, \\ \frac{1}{r_{j+1}} &= \frac{1}{q} + \frac{d-2r_j}{dr_j}, \end{cases}$$

where q is given by (VW1). An elementary analysis of (r_j) shows that there exists j_0 such that for all $0 \leqslant j < j_0$ we have $r_{j+1} > r_j$, $\frac{d-2r_j}{dr_j} > 0$ and $\frac{d-2r_{j_0}}{dr_{j_0}} < 0$.

By induction, it is easy to see that for all $j = 0, ..., j_0$ we have $u, v \in W^{2,r_j}(\mathbb{R}^d)$. For $j = 0$ it is by definition of u, v. Take any $0 \leqslant j < j_0$ and assume that $u, v \in W^{2,r_j}(\mathbb{R}^d)$. Since $\frac{d-2r_j}{dr_j} > 0$, by Sobolev embeddings we infer that $u, v \in L^{\frac{dr_j}{d-2r_j}}(\mathbb{R}^d)$. Then, (VW1) and Hölder inequality imply

$$W_1 u, V_1 v, V_2 u, W_1 v \in L^{r_{j+1}}(\mathbb{R}^d).$$

Combined with $U = (u, v)^T$ satisfying $LU = \lambda U$, this leads to $u, v \in W^{2,r_{j+1}}(\mathbb{R}^d)$.

In particular, we have $u, v \in W^{2,r_{j_0}}(\mathbb{R}^d)$. Since $\frac{d-2r_{j_0}}{dr_{j_0}} < 0$, from Sobolev embeddings we infer $u, v \in L^\infty(\mathbb{R}^d)$. Then by (VW1) we get

$$W_1 u, V_1 v, V_2 u, W_1 v \in L^q(\mathbb{R}^d).$$

As before, combined with $LU = \lambda U$, this leads to $u, v \in W^{2,q}(\mathbb{R}^d)$. The conclusion follows by interpolation. □

For the rest of the proof, it is easier to work with the operator

$$L' := iPLP^{-1} = \begin{pmatrix} -\Delta + \omega + V_1' & W_1' \\ W_2' & \Delta - \omega + V_2' \end{pmatrix},$$

where $P = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$. The potentials V_1', V_2', W_1', W_2' verify also (VW1)-VW2). The spectrum of L' is $\mathrm{Sp}(L') = \mathrm{Sp}(iPLP^{-1}) = i\,\mathrm{Sp}(L)$. Hence if $\lambda \in \mathbb{C}$ is an eigenvalue of L with eigenvector U then $\lambda' := i\lambda$ is an eigenvalue of L' with eigenvector $U' = \begin{pmatrix} u' \\ v' \end{pmatrix} := PU$.

Write $L' - \lambda' I = H + K$ where

$$H := \begin{pmatrix} -\Delta + \omega - \lambda' & 0 \\ 0 & \Delta - \omega - \lambda' \end{pmatrix} \text{ and } K := \begin{pmatrix} V_1' & W_1' \\ W_2' & V_2' \end{pmatrix}.$$

Take

$$F := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} := KU' = \begin{pmatrix} V_1' u + W_1' v \\ W_2' u + V_2' v \end{pmatrix}.$$

It is well known that we can represent u' and v' in the following way

$$u' = g^d_{-\omega+\lambda'} * f_1 \text{ and } v' = -g^d_{-\omega-\lambda'} * f_2.$$

Let $\mu_1 := -\omega + \lambda'$ and $\mu_2 := -\omega - \lambda'$. From the assumptions on λ' we infer that μ_1, μ_2 satisfy the hypothesis of Lemma 26. Let τ_1, τ_2 be given by Lemma 26 and set $\tau := \min\{\tau_1, \tau_2\}$. Take

$$\tilde{F} := \begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \end{pmatrix} := \begin{pmatrix} |f_1| \\ |f_2| \end{pmatrix},$$

$$\tilde{u} := g^d_{-\tau} * \tilde{f}_1 \text{ and } \tilde{v} := g^d_{-\tau} * \tilde{f}_2.$$

Claim 28. *There exists $C > 0$ such that*

$$|u'| \leqslant C\tilde{u} \text{ and } |v'| \leqslant C\tilde{v}.$$

Proof. This readily follows from Lemma 26. □

Lemma 29. *Set $w := \tilde{u} + \tilde{v}$. There exists $C > 0$ and $\alpha > 0$ such that*

$$w(x) \leqslant Ce^{-\alpha|x|} \text{ for all } x \in \mathbb{R}^d.$$

The proof of Lemma 29 follows closely the proof of Theorem 1.1 in [14].

Proof. Set $f := \tilde{f}_1 + \tilde{f}_2$. We first note that $w \in \mathcal{C}^0(\mathbb{R}^d)$. Indeed, by definition w satisfies

$$-\Delta w + \tau w = f. \tag{A.9}$$

Since, by (VW1) and Lemma 27, $f \in L^q(\mathbb{R}^d)$ this implies $w \in W^{2,q}(\mathbb{R}^d)$ and in particular $w \in \mathcal{C}^0(\mathbb{R}^d)$.

Now, we claim that there exists $R > 0$ such that for all $x \in \mathbb{R}^d$ verifying $|x| > R$ we have

$$\frac{\tau w(x) - f(x)}{w(x)} \geqslant \frac{\tau}{2}. \tag{A.10}$$

Indeed, setting $T(x) := (|V_1'| + |V_2'| + |W_1'| + |W_2'|)$, by Claim 28 we have

$$f \leqslant T(x)(|u'| + |v'|) \leqslant CT(x)(\tilde{u} + \tilde{v}) = CT(x)w.$$

Therefore

$$\frac{\tau w(x) - f(x)}{w(x)} \geqslant \tau - CT(x).$$

By (VW2), we can take R large enough so that $CT(x) \leqslant \frac{\tau}{2}$ for $|x| > R$, which proves (A.10).

Note that $w \geqslant 0$ by definition. Since $w \in \mathcal{C}^0(\mathbb{R}^d) \cap W^{2,q}(\mathbb{R}^d)$, there exists C_R such that for all $x \in \mathbb{R}^d$ with $|x| < R$ we have

$$0 \leqslant w(x) \leqslant C_R.$$

Define $\psi(x) := C_R e^{-\sqrt{\frac{\tau}{2}}(|x|-R)}$. It is easy to see that

$$\begin{aligned} -\Delta\psi + \frac{\tau}{2}\psi \geqslant 0 \text{ on } \mathbb{R}^d \setminus \{0\}, \\ w(x) - \psi(x) \leqslant 0 \text{ on } \{x \in \mathbb{R}^d, |x| < R\}. \end{aligned} \tag{A.11}$$

Therefore we only have to prove that $w(x) \leqslant \psi(x)$ for $|x| > R$. We proceed by contradiction. Assume that there exists $x_0 \in \mathbb{R}^d$ with $|x_0| > R$ such that $w(x_0) > \psi(x_0)$. Define the set

$$\Omega := \{x \in \mathbb{R}^d, w(x) > \psi(x)\}.$$

Then Ω is a non-empty open set, for all $x \in \Omega$ we have $|x| > R$ and for all $x \in \partial\Omega$ we have $w(x) - \psi(x) = 0$. On Ω, by (A.9), (A.10) and (A.11) we have

$$\begin{aligned} \Delta(w - \psi) &= \Delta w - \Delta\psi = \tau w - f - \Delta\psi \\ &= \frac{\tau w - f}{w} w - \Delta\psi \geqslant \frac{\tau}{2}(w - \psi) > 0. \end{aligned}$$

By the maximum principle, this implies that $w - \psi \leqslant 0$ on Ω, a contradiction. Thus, for all $x \in \mathbb{R}^d$ we have

$$w(x) \leqslant \psi(x) = C_R e^{-\sqrt{\frac{\tau}{2}}(|x|-R)} = C_R e^{\sqrt{\frac{\tau}{2}}R} e^{-\sqrt{\frac{\tau}{2}}|x|} = Ce^{-\sqrt{\frac{\tau}{2}}|x|}.$$

This ends the proof. ☐

Proof of Proposition 25. The statement is an immediate consequence of Lemma 27, Claim 28 and Lemma 29. ☐

Appendix A.3. Higher regularity and decay

Upon assuming more regularity and decay, we can obtain more regularity and decay on the solutions to $(L - \lambda I) = A$.

The new assumption is the following.

(VW3) $V_1, V_2, W_1, W_2 \in \mathscr{H}(\mathbb{C})$.

Recall that $\mathscr{H}$ was defined in (25).

Proposition 30. *Assume that (VW1)-(VW3) hold.*

(i) Let λ, u and v be as in Proposition 25. Then $u, v \in \mathscr{H}(\mathbb{C})$.

(ii) Let $\lambda \notin \mathrm{Sp}(L)$ and take $A \in \mathscr{H}(\mathbb{C}^2)$. Then there exists $X \in \mathscr{H}(\mathbb{C}^2)$ such that $(L - \lambda \, \mathrm{Id})X = A$.

Proof. (i) The assertion follows from similar arguments to those used in the proof of Proposition 25, provided we remark that (using the same notations) $D^a u' = g^d_{-\omega+\lambda'} * D^a f_1$, $D^a v' = -g^d_{-\omega-\lambda'} * D^a f_2$ and $D^a f_1, D^a f_2$ satisfy the same properties as f_1 and f_2.

(ii) Since $\lambda \notin \mathrm{Sp}(L)$ the operator $L - \lambda \, \mathrm{Id}$ is invertible, hence the existence of $X \in H^2(\mathbb{R}^d, \mathbb{C}^2)$ such that $(L - \lambda \, \mathrm{Id})X = A$. Regularity of X follows from a standard bootstrap argument as explained in the proof of Proposition 25 (ii). We now recall that $L = -iP^{-1}L'P$. Hence, if we define $X' = PX$, $\lambda' = i\lambda$, and $A' = iPA$ then

$$(L' - \lambda' \, \mathrm{Id})X' = A'.$$

Recall that $L' - \lambda' I = H + K$. Set $Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} := KX'$ and $A' = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$. Then we can represent $X' = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in the following way

$$x_1 = g^d_{-\omega+\lambda'} * (y_1 + a_1) \text{ and } x_2 = -g^d_{-\omega-\lambda'} * (y_2 + a_2).$$

The terms $g^d_{-\omega+\lambda'} * a_1$ and $g^d_{-\omega-\lambda'} * a_2$ are clearly exponentially decaying, with decay rate α. Since $V_1, V_2, W_1, W_2 \in \mathscr{H}(\mathbb{C}^2)$, it follows that each component of Y is also exponentially decaying with rate α. Hence $g^d_{-\omega+\lambda'} * y_1$ and $g^d_{-\omega-\lambda'} * y_2$ are exponentially decaying with decay rate α. The decay rate of the derivatives of X' is follows immediately if we remark that for any multiindex a we have $D^a x_k = g^d_{-\omega+\lambda'} * D^a(y_k + a_k)$ for $k = 1, 2$. □

Appendix B. Instability of solitons and multi-solitons

Since (NLS) is Galilean invariant, we can assume in this section without loss of generality that $v_1 = x_1 = \gamma_1 = 0$. Hence $R_1(t,x) = e^{i\omega_1 t}\Phi_1(x)$.

Recall that, as defined in Section 3.1, $Y(t)$ is of the form $e^{-\rho t}(\cos(\theta t)Y_1(x) + \sin(\theta t)Y_2(x))$, where Y_1, Y_2 are smooth, exponentially decaying functions, along with their derivatives. Notice that if $u(t,x)$ is a solution to (NLS) and $T, \vartheta \in \mathbb{R}$, then so is $\bar{u}(T-t,x)e^{i\vartheta}$. The hypotheses of Theorem 2 are verified by Φ_1 and therefore also by $\bar{\Phi}_1$. Hence the conclusion of Theorem 2 holds for $\tilde{R}_1(t,x) := \bar{R}_1(-t,x) = e^{i\omega_1 t}\bar{\Phi}_1$. Let $\mathfrak{u} \in \mathscr{C}([T_0, \infty), H^1(\mathbb{R}^d)$ be the solution constructed in Theorem 2 associated with the soliton $\tilde{R}_1(t,x)$ and correction $e^{-\rho t}(\cos(\theta t)Y_1(x) + \sin(\theta t)Y_2(x)) + O(e^{-2\rho t})$ (i.e. $\mathfrak{u} = u_1$ in the notations of Theorem 2). In particular, for all $\sigma \geqslant 0$,

$$\forall t \geqslant T_0, \quad \|\mathfrak{u}(t) - \tilde{R}_1(t) - Y(t)\|_{H^\sigma(\mathbb{R}^d)} \leqslant Ce^{-2\rho t}.$$

Note that we construct $\mathfrak{u}$ on $\tilde{R}_1$ and not R_1 so as to have instability forward in time.

Appendix B.1. Orbital instability of one soliton

First let us prove a modulation lemma.

Lemma 31. *There exist $\varepsilon > 0$, $t_0 \geqslant T_0$ and $M \geqslant 0$ such that*

$$\inf_{y \in \mathbb{R}^d, \vartheta \in \mathbb{R}} \|\mathfrak{u}(t_0) - \bar{\Phi}_1(x-y)e^{i\vartheta}\|_{L^2(B(0,M))} = \varepsilon > 0.$$

Proof. Let $t_0 > T_0$ to be determined later. Up to increasing t_0, we can assume that $\omega_1 t_0 \equiv 0 (2\pi)$.

Consider $\Theta(y, \vartheta) = \|\mathfrak{u}(t_0) - \bar{\Phi}_1(x-y)e^{i\vartheta}\|_{L^2(\mathbb{R}^d)}$. The function Θ is continuous on $\mathbb{R}^{d+1}$. Notice that for $\vartheta = 0$ and $y = 0$, one gets $\Theta(0,0) \leqslant Ce^{-\rho t_0}$.

Now, we have that $\liminf_{|y| \to \infty} \inf_{\vartheta \in \mathbb{R}} \Theta(y, \vartheta) \geqslant 2\|\bar{\Phi}_1\|_{L^2(\mathbb{R}^d)} - Ce^{-\rho t_0}$ due to space localization of $\bar{\Phi}_1$, so that, as $\vartheta \in \mathbb{R}/2\pi\mathbb{Z}$ compact, if t_0 is large enough, $\inf_{y \in \mathbb{R}^d, \vartheta \in \mathbb{R}} \Theta(y, \vartheta)$ is attained at some point (y_0, ϑ_0).

Assume $\Theta(y_0, \vartheta_0) = 0$, i.e. $\mathfrak{u}(t_0) = \bar{\Phi}_1(x - y_0)e^{i\vartheta_0}$.

Claim: There exists a continuous function η such that $\eta(0) = 0$ and $|y_0| + |\vartheta_0| \leqslant \eta(e^{-\rho t_0})$.

Indeed, first consider y_0. Denote $g(y) = \||\bar{\Phi}_1| - |\bar{\Phi}_1(\cdot - y)|\|^2_{L^2(\mathbb{R}^d)}$. We have

$$0 = \Theta(y_0, \vartheta_0) \geqslant \||\mathfrak{u}(t_0)| - |\bar{\Phi}_1(\cdot - y_0)|\|_{L^2(\mathbb{R}^d)} \geqslant \||\bar{\Phi}_1| - |\bar{\Phi}_1(\cdot - y_0)|\|_{L^2(\mathbb{R}^d)} - C\|Y(t_0)\|_{L^2(\mathbb{R}^d)}.$$

As $\|Y(t_0)\|_{L^2(\mathbb{R}^d)} \leqslant Ce^{-\rho t_0}$, we get $g(y_0) \leqslant C^2 e^{-2\rho t_0}$. Now, due to space localization of $\bar{\Phi}_1$, $g(y) \to 2\|\bar{\Phi}_1\|^2_{L^2(\mathbb{R}^d)} > 0$ as $|y| \to +\infty$. Let (y_n) be such that $g(y_n) \to 0$, and $y_n \not\to 0$. Then up to a subsequence, $y_n \to y^\infty$ and $g(y^\infty) = 0$, so that $|\bar{\Phi}_1|$ is periodic and as $\bar{\Phi}_1 \in L^2(\mathbb{R}^d)$, $\bar{\Phi}_1 \equiv 0$, a contradiction. This shows that $y \to 0$ as $g(y) \to 0$, and it gives the bound on y_0. For ϑ_0,

$$0 = \|\mathfrak{u}(t_0) - \bar{\Phi}_1(\cdot - y_0)\|_{L^2(\mathbb{R}^d)} \geqslant -\|\mathfrak{u}(t_0) - \bar{\Phi}_1\|_{L^2(\mathbb{R}^d)} + \|\bar{\Phi}_1 - \bar{\Phi}_1 e^{i\vartheta_0}\|_{L^2(\mathbb{R}^d)} - \|\bar{\Phi}_1 - \bar{\Phi}_1(\cdot - y_0)\|_{L^2(\mathbb{R}^d)},$$

As $\|\bar{\Phi}_1 - \bar{\Phi}_1 e^{i\vartheta_0}\|_{L^2(\mathbb{R}^d)} = |1 - e^{i\vartheta_0}|\|\bar{\Phi}_1\|_{L^2(\mathbb{R}^d)}$, we deduce that $|\vartheta_0| \leqslant Ce^{-\rho t_0} + Cg(y_0)$. This concludes the proof of the claim.

Denote $T_{\bar{\Phi}_1}\mathscr{F}$ the tangent space of $\mathscr{F} = \{\bar{\Phi}_1(\cdot - y)e^{i\vartheta} | (y, \vartheta) \in \mathbb{R}^d\}$ at point $\bar{\Phi}_1$. Note that, due to the *Claim*, $\mathscr{F}$ is a manifold. It is easy to see that $T_{\bar{\Phi}_1}\mathscr{F} \subset \ker L_{\mathbb{C}}$ (by differentiating the relation $\Delta\bar{\Phi}_1(x-y) + g(|\bar{\Phi}_1(x-y)|^2)\bar{\Phi}_1(x-y) = \omega_1\bar{\Phi}_1(x-y)$). But for all t, $(\cos(\theta t)Y_1(x) + \sin(\theta t)Y_2(x)) \notin \ker L_{\mathbb{C}}$ (as Y_1, Y_2 are build on an eigenvector for an eigenvalue of positive real part of $L_{\mathbb{C}}$). As $\mathfrak{u}(t_0) = \bar{\Phi}_1 + e^{\rho t_0}(\cos(\theta t_0)Y_1(x) + \sin(\theta t_0)Y_2(x)) + O(e^{-2\rho t_0})$, up to choosing $t_0 + 2k\pi/\theta$, ($k \in \mathbb{N}$ large) instead of t_0, this proves that $\mathfrak{u}(t_0) \notin \mathscr{F}$. We proved that for t_0 large enough,

$$\inf_{y \in \mathbb{R}^d, \vartheta \in \mathbb{R}} \|\mathfrak{u}(t_0) - \bar{\Phi}_1(x-y)e^{i\vartheta}\|_{H^1(\mathbb{R}^d)} > 0.$$

Assume that this does not hold when we restrict to $L^2(B(0,M))$, for any large M. This would mean that for all $m \geqslant 0$, there exist $y_m \in \mathbb{R}^d, \vartheta_m \in \mathbb{R}$ such that

$$\|\mathfrak{u}(t_0) - \bar{\Phi}_1(x - y_m)e^{i\vartheta_m}\|_{L^2(B(0,m))} \leqslant \frac{1}{m}.$$

Then by localization arguments, (y_m) remains bounded, so that up to a subsequence, $y_m \to y_\infty$, $\vartheta_m \to \vartheta_\infty$. Therefore $\|\mathfrak{u}(t_0, x) - \bar{\Phi}_1(x - y_\infty)e^{i\vartheta_\infty}\|_{L^2(\mathbb{R}^d)} = 0$, so that $\mathfrak{u}(t_0, x) = \bar{\Phi}_1(x - y_\infty)e^{i\vartheta_\infty}$, a contradiction. □

Proof of Corollary 2. Let t_0 and ε be given by Lemma 31. Take an increasing sequence (S_n) so that $S_n \to +\infty$ as $n \to +\infty$, and define $T_n := S_n - t_0$ and

$$u_n(t,x) := \bar{\mathfrak{u}}(S_n - t, x)e^{-i\omega_1 S_n}.$$

Then $u_n \in \mathscr{C}([0,T_n], H^1(\mathbb{R}^d))$ is a solution of (NLS), and

$$\begin{aligned} u_n(0,x) &= \bar{\mathfrak{u}}(S_n, x)e^{-i\omega_1 S_n} = \Phi_1(x) + O_{H^\sigma}(e^{-\rho S_n}), \\ u_n(T_n, x) &= \bar{\mathfrak{u}}(t_0, x)e^{-i\omega_1 S_n}. \end{aligned}$$

Therefore, $\|u_n(0) - R_1(0)\|_{H^\sigma(\mathbb{R}^d)} \leqslant Ce^{-\rho S_n} \to 0$ as $n \to +\infty$. Due to Lemma 31, we deduce that for all $n \in \mathbb{N}$ we have

$$\inf_{y \in \mathbb{R}^d, \vartheta \in \mathbb{R}} \|u_n(T_n) - e^{i\vartheta}\Phi_1(\cdot - y)\|_{L^2(\mathbb{R}^d)} \geqslant \inf_{y \in \mathbb{R}^d, \vartheta \in \mathbb{R}} \|\mathfrak{u}(t_0) - \bar{\Phi}_1(x-y)e^{i\vartheta}\|_{L^2(B(0,M))} \geqslant \varepsilon,$$

which is the desired conclusion. □

Appendix B.2. Instability of multi-solitons

Proof of Corollary 4. Let $T > 0$, M be given by Lemma 31 and ε, (u_n), (T_n) be given by Corollary 2.

The idea is the following. We use the fact that $u_n(T_n)$ is ε-away from the orbit of the soliton R_1. Given a parameter I, we consider at time I an initial data $w(I)$ which is $u_n(0)$ adequately shifted, denoted by $\tilde{u}_n(I)$, plus the sum of the $R_j(I)$, $j \geqslant 2$. (All the functions will depend on n and I, although we do not always show this dependence for convenience in the notation). We aim at controlling w up to time $I + T_n$. The role of I is to ensure that the interaction of u_n and the R_j are small: as $\{u_n(t) | t \in [0,T_n]\}$ is compact and the $R_j(t)$ ($j \geqslant 2$) are localized away from $\tilde{u}_n(t)$, their interaction goes to 0 as $I \to +\infty$. Using a Gronwall type argument, we are able to show that $w(I + T_n)$ is $\tilde{u}_n(I + T_n) + \sum_{j=2}^N R_j(I + T_n) + o_{I \to +\infty}(1)$. As $u_n(T_n)$ is

ε-away from the soliton family, we deduce that $w(I+T_n)$ is $\varepsilon - o_{I\to+\infty}(1) \geqslant \varepsilon/2$ away from the family of a sum of solitons.

Given $I \geqslant T$, define $\tilde{u} \in \mathscr{C}([I, I+T_n], H^1(\mathbb{R}^d))$ by

$$\tilde{u}_n(t,x) = u_n(t-I,x).$$

Possibly increasing I so that $\omega_1 I = 0(2\pi)$, we have $\|\tilde{u}_n(I) - R_1(I)\|_{H^\sigma(\mathbb{R}^d)} = \|u_n(0) - R_1(0)\|_{H^\sigma(\mathbb{R}^d)} \to 0$ as $n \to +\infty$ and $\tilde{u}_n(I+T_n)$ is ε-away from the Φ_1-soliton family. Consider the solution $w_n \in \mathscr{C}([I,T^*), H^1(\mathbb{R}^d))$ to (NLS) with initial data at time I

$$w_n(I,x) = \tilde{u}_n(I,x) + \sum_{j=2}^N R_j(I,x).$$

If $T^* < +\infty$, the blow-up alternative for (NLS) automatically implies instability on the multi-soliton, hence we assume $T^* = +\infty$. Let $\sigma > d/2$ be an integer. Notice that, as $u_n \in \mathscr{C}([0,T_n], H^\sigma(\mathbb{R}^d))$ and $[0,T_n]$ is compact, the set $\{u_n(t) | t \in [0,T_n]\}$ is compact in $H^\sigma(\mathbb{R}^d)$. In particular, $\sup_{t\in[0,T_n]} \|u_n(t)\|_{H^\sigma(|x|\geqslant R)} \to 0$ as $R \to +\infty$. Hence, as the R_j are decoupling as time grows, there exists a function $\eta(I)$ such that $\eta(I) \to 0$ as $I \to +\infty$ and

$$\forall t \in [I, I+T_n], \quad \sum_{j\geqslant 2} \|\tilde{u}_n(t) R_j(t)\|_{H^\sigma} \leqslant \eta(I).$$

Denote $x_j(t) = v_j t + x_j$. Up to modifying the function η, we can also assume that the $R_j(t)$, $j \geqslant 2$, are far away from $x_1(t) \equiv 0$, and that the multisoliton $R(t)$ is near the sum of solitons $\sum_{j=1}^N R_j(t)$, that is

$$\forall t \geqslant I, \quad \sum_{j=2}^N \|R_j(t)\|_{H^\sigma(B(0,2M))} + \|R(t) - \sum_{j=1}^N R_j(t)\|_{H^\sigma(\mathbb{R}^d)} \leqslant \eta(I).$$

Finally we denote $J = I + T_n$ and

$$z(t) = w_n(t) - (\tilde{u}_n(t) + \sum_{j=2}^N R_j(t)).$$

Now, as f is $\mathscr{C}^\infty$, for all $R > 0$, there exists $C(R)$ such that

$$\forall a,b \in B(0,R), \quad |f(a+b) - f(a) - f(b)| \leqslant C(R)|a||b|. \tag{B.1}$$

Indeed, this expression is symmetric in a, b, so that we can assume without loss of generality that $|b| \leqslant |a|$. As $f(0) = f'(0) = 0$, we have that $|f(b)| \leqslant C|b|^2 \leqslant C|a||b|$, and a Taylor expansion shows that

$$|f(a+b) - f(a)| = b\int_0^1 |f'(a+tb)|dt \leqslant b \sup_{x\in B(0,|a|+|b|)} |f'(x)| \leqslant C|b|(|b|+|a|) \leqslant C|a||b|.$$

Now, as $H^\sigma(\mathbb{R}^d)$ is an algebra, we deduce from (B.1) that there exists a constant $C > 0$ (depending only on the Φ_j) such that for $t \in [I,J]$,

$$\|f(z(t) + \tilde{u}_n(t) + \sum_{j=2}^N R_j(t)) - f(\tilde{u}_n) - \sum_{j=2}^N f(R_j(t))\|_{H^\sigma(\mathbb{R}^d)} \leqslant C\|z(t)\|_{H^\sigma(\mathbb{R}^d)} + C\sum_{j=2}^N \|\tilde{u}_n(t) R_j(t)\|_{H^\sigma(\mathbb{R}^d)}.$$

The function z satisfies the equation

$$iz_t + \Delta z + f\left(z + \tilde{u}_n + \sum_{j=2}^N R_j\right) - f(\tilde{u}_n) - \sum_{j=2}^N f(R_j) = 0,$$

Since $z(I)=0$, Duhamel formula for z gives

$$z(t)=\int_I^t e^{i\Delta(t-s)}\left(f\left(z(s)+\tilde{u}_n(s)+\sum_{j=2}^N R_j(s)\right)-f(\tilde{u}_n(s))-\sum_{j=2}^N f(R_j(s))\right)ds.$$

Hence, for all $t\in[I,J]$

$$\|z(t)\|_{H^s(\mathbb{R}^d)}\leqslant C\int_I^t(\|z(s)\|_{H^\sigma(\mathbb{R}^d)}+\eta)ds\leqslant C\int_I^t\|z(s)\|_{H^\sigma(\mathbb{R}^d)}ds+\eta(I)(t-I).$$

By Grönwall's Lemma, we deduce that for $t\in[I,J]$, we have

$$\|z(t)\|_{H^\sigma(\mathbb{R}^d)}\leqslant C\eta(I)(t-I)e^{C(t-I)}\leqslant C_n\eta(I),$$

where $C_n=CT_ne^{CT_n}$. Thus for all $n\in\mathbb{N}$ we have

$$\left\|w_n(J)-u_n(T_n)-\sum_{j=2}^N R_j(J)\right\|_{H^\sigma(\mathbb{R}^d)}\leqslant C_n\eta(I).$$

Now choose I_n such that $C_n\eta(I_n)\leqslant\varepsilon/3$ and set $J_n=I_n+T_n$. Then $\|z(J_n)\|_{H^\sigma(\mathbb{R}^d)}\leqslant\varepsilon/3$. Then, given $y_j\in\mathbb{R}^d$, $\vartheta_j\in\mathbb{R}$, we have (denote $c_j=c_j(t)=-\frac{1}{4}|v_j|^2t+\omega_jt+\gamma_0$)

$$\begin{aligned}
&\|w_n(J_n)-\sum_{j=1}^N\Phi_j(\cdot-y_j)e^{i(\frac12 v_j\cdot x+\vartheta_j)}\|_{L^2}\\
&\geqslant\left\|u_n(T_n)+\sum_{j=2}^N R_j(J_n)-\sum_{j=1}^N\Phi_j(\cdot-y_j)e^{i(\frac12 v_j\cdot x+\vartheta_j)}\right\|_{L^2}-\left\|w_n(J_n)-u_n(T_n)-\sum_{j=2}^N R_j(J_n)\right\|_{L^2}\\
&\geqslant\left\|u_n(T_n)-\Phi_1(x-y_1)e^{i\vartheta_1}+\sum_{j=2}^N(\Phi_j(x-x_j(J_n))e^{i(\frac12 v_j\cdot x+c_j)}-\Phi_j(x-y_j)e^{i(\frac12 v_j\cdot x+\vartheta_j)})\right\|_{L^2}-\varepsilon/3.
\end{aligned}$$

Now consider y_j, ϑ_j that realize a near infimum, say $\|w_n(J_n)-\sum_{j=1}^N\Phi_j(\cdot-y_j)e^{i(\frac12 v_j\cdot x+\vartheta_j)}\|_{L^2}\leqslant 2\varepsilon$. Then considering the L^2 norm on balls $B(x_j(J_n),R)$ around each exited state R_j, $j\geqslant 2$ (for some large and fixed radius R), we see that, up to a permutation if two Φ_j or more are equal, we must have $y_j-x_j(J_n)=O(1)$ for $j\geqslant 2$. In particular, this implies that

$$\left\|\sum_{j=2}^N(\Phi_j(x-x_j(J_n))e^{i(\frac12 v_j\cdot x+c_j)}-\Phi_j(x-y_j)e^{i(\frac12 v_j\cdot x+\vartheta_j)}\right\|_{H^\sigma(B(0,M))}=o_{I_n\to+\infty}(1)\leqslant\varepsilon/3,$$

up to increasing again I_n. Thus,

$$\begin{aligned}
\inf_{\substack{y_j\in\mathbb{R}^d,\vartheta_j\in\mathbb{R},\\ j=1,\dots,N}}\|w_n(J_n)-\sum_{j=1}^N\Phi_j(\cdot-y_j)e^{i(\frac12 v_j\cdot x+\vartheta_j)}\|_{L^2}&\geqslant\|w_n(J_n)-\sum_{j=1}^N\Phi_j(\cdot-y_j)e^{i(\frac12 v_j\cdot x+\vartheta_j)}\|_{L^2(B(0,M))}\\
&\geqslant\|u_n(T_n)-\Phi_1(x-y_1)e^{i\vartheta_1}\|_{L^2(B(0,M))}-2\varepsilon/3\\
&\geqslant\varepsilon-2\varepsilon/3\geqslant\varepsilon/3,
\end{aligned}$$

where we used Corollary 2 on the last line. As

$$\|w_n(I_n)-R(I_n)\|_{H^\sigma(\mathbb{R}^d)}\leqslant\|w_n(I_n)-\sum_{j=1}^N R_j(I_n)\|_{H^\sigma(\mathbb{R}^d)}+\|\sum_{j=1}^N R_j(I_n)-R(I_n)\|_{H^\sigma(\mathbb{R}^d)}\to 0,$$

w_n, I_n and J_n satisfies the conditions of Corollary 4. □

Remark 32. *Notice that we did not use any high speed condition on the v_j. The most delicate point here is that we have no uniform spatial decay on u_n (as well as on the multi-soliton constructed in Theorem 3), apart that coming from $H^\sigma(\mathbb{R}^d)$ compactness. We conjecture it should be exponentially decaying (in space) around the soliton (resp. every soliton R_j); a proof of this should be related to uniqueness of the multi-soliton in the L^2 sub-critical case, which is currently an open problem.*

Appendix C. Coercivity for a soliton

This section is devoted to the proof of Lemma 12.

Proof of Lemma 12. We first remark that R_0 is solution of

$$-\Delta R_0 + \left(\omega_0 + \frac{|v_0|^2}{4}\right) R_0 - f(R_0) + iv_0 \nabla R_0 = 0. \tag{C.1}$$

Therefore it is a critical point of the functional $\tilde{S}_0$ defined for $w \in H^1(\mathbb{R}^d)$ by

$$\tilde{S}_0(w) := \frac{1}{2}\|\nabla w\|^2_{L^2(\mathbb{R}^d)} + \frac{1}{2}\left(\omega_0 + \frac{|v_0|^2}{4}\right)\|w\|^2_{L^2(\mathbb{R}^d)} - \int_{\mathbb{R}^d} F(w)dx - \frac{1}{2}v_0 \cdot \mathscr{I}m \int_{\mathbb{R}^d} \bar{w}\nabla w dx.$$

The quadratic form H_0 is precisely

$$H_0(t,w) = \left\langle \tilde{S}_0''(R_0)w, w\right\rangle.$$

Consider z such that $w = e^{-i(\frac{1}{2}v_0\cdot x - \frac{1}{4}|v_0|^2 t + \omega_0 t + \gamma_0)} z(x + v_0 t + x_0)$. Then it is easy to see that

$$H_0(t,w) = \tilde{H}_0(z) := \|\nabla z\|^2_{L^2(\mathbb{R}^d)} + \omega_0 \|z\|^2_{L^2(\mathbb{R}^d)} - \int_{\mathbb{R}^d} \left(g(|\Phi_0|^2)|z|^2 + 2g'(|\Phi_0|^2)\mathscr{R}e(\Phi_0 \bar{z})^2\right) dx.$$

It is well-known that up to a finite number of non-positive directions $\tilde{H}_0(z)$ controls the $H^1(\mathbb{R}^d)$-norm of z. Indeed, the self-adjoint operator corresponding to the quadratic form $\tilde{H}_0$ (viewed on $H^1(\mathbb{R}^d, \mathbb{R}^2)$) is a compact perturbation of $\begin{pmatrix} -\Delta + \omega_0 & 0 \\ 0 & -\Delta + \omega_0 \end{pmatrix}$, hence its spectrum lies on the real line and its essential spectrum is $[\omega_0, +\infty)$. Since in addition the quadratic form $\tilde{H}_0$ is bounded from below on the unit $L^2(\mathbb{R}^d)$-sphere, the corresponding operator admits only a finite number of eigenvalues in $(-\infty, \omega_0')$ for any $\omega_0' < \omega_0$. In particular, there exist $\tilde{K}_0 > 0$, $\nu_0 \in \mathbb{N}$ and $\tilde{X}_0^1, ..., \tilde{X}_0^{\nu_0} \in L^2(\mathbb{R}^d)$ such that $\|\tilde{X}_0^k\|_{L^2(\mathbb{R}^d)} = 1$ for any k and

$$\|z\|^2_{H^1(\mathbb{R}^d)} \leqslant \tilde{K}_0 \tilde{H}_0(z) + \tilde{K}_0 \sum_{k=1}^{\nu_0} \left(z, \tilde{X}_0^k\right)^2_{L^2(\mathbb{R}^d)}.$$

Since

$$\|\nabla w\|^2_{L^2(\mathbb{R}^d)} = \frac{3}{2}\|\nabla z\|^2_{L^2(\mathbb{R}^d)} + \frac{3|v_0|^2}{4}\|z\|^2_{L^2(\mathbb{R}^d)}$$

there exists $K_0 > 0$ such that

$$\|w\|^2_{H^1(\mathbb{R}^d)} \leqslant K_0 H_0(t,w) + K_0 \sum_{k=1}^{\nu_0} \left(w, X_0^k(t)\right)^2_{L^2(\mathbb{R}^d)},$$

where $X_0^k(t) := e^{i(\frac{1}{2}v_0\cdot x - \frac{1}{4}|v_0|^2 t + \omega_0 t + \gamma_0)} \tilde{X}_0^k(x - v_0 t - x_0)$. □

[1] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series.* U.S. Government Printing Office, Washington, D.C., 1964.

[2] H. Berestycki and T. Cazenave. Instabilité des états stationnaires dans les équations de Schrödinger et de Klein-Gordon non linéaires. *C. R. Acad. Sci. Paris Sér. I Math.*, 293(9):489–492, 1981.

[3] H. Berestycki, T. Gallouët, and O. Kavian. Équations de champs scalaires euclidiens non linéaires dans le plan. *C. R. Acad. Sci. Paris*, 297:307–310, 1983.
[4] H. Berestycki and P.-L. Lions. Nonlinear scalar field equations I. *Arch. Ration. Mech. Anal.*, 82:313–346, 1983.
[5] H. Berestycki and P.-L. Lions. Nonlinear scalar field equations II. *Arch. Ration. Mech. Anal.*, 82(4):347–375, 1983.
[6] F. A. Berezin and M. A. Shubin. *The Schrödinger equation*, volume 66 of *Mathematics and its Applications (Soviet Series)*. Kluwer Academic Publishers Group, Dordrecht, 1991.
[7] T. Cazenave. *Semilinear Schrödinger equations.* New York University – Courant Institute, New York, 2003.
[8] T. Cazenave and P.-L. Lions. Orbital stability of standing waves for some nonlinear Schrödinger equations. *Comm. Math. Phys.*, 85(4):549–561, 1982.
[9] T. Cazenave and F. B. Weissler. The Cauchy problem for the critical nonlinear Schrödinger equation in H^s. *Nonlinear Anal.*, 14(10):807–836, 1990.
[10] S.-M. Chang, S. Gustafson, K. Nakanishi, and T.-P. Tsai. Spectra of linearized operators for NLS solitary waves. *SIAM J. Math. Anal.*, 39(4):1070–1111, 2007/08.
[11] V. Combet. Construction and characterization of solutions converging to solitons for supercritical gKdV equations. *Differential Integral Equations*, 23(5-6):513–568, 2010.
[12] V. Combet. Multi-soliton solutions for the supercritical gKdV equations. *Comm. Partial Differential Equations*, 36(3):380–419, 2011.
[13] R. Cote, Y. Martel, and F. Merle. Construction of multi-soliton solutions for the L2-supercritical gKdV and NLS equations. *Revista Matematica Iberoamericana*, to appear.
[14] Y. Deng and Y. Li. Exponential decay of the solutions for nonlinear biharmonic equations. *Commun. Contemp. Math.*, 9(5):753–768, 2007.
[15] T. Duyckaerts and F. Merle. Dynamics of threshold solutions for energy-critical wave equation. *Int. Math. Res. Pap. IMRP*, (4):Art. ID rpn002, 67 pp. (2008), 2007.
[16] T. Duyckaerts and F. Merle. Dynamics of threshold solutions for energy-critical wave equation. *Int. Math. Res. Pap. IMRP*, pages Art ID rpn002, 67, 2008.
[17] T. Duyckaerts and F. Merle. Dynamic of threshold solutions for energy-critical NLS. *Geom. Funct. Anal.*, 18(6):1787–1840, 2009.
[18] T. Duyckaerts and S. Roudenko. Threshold solutions for the focusing 3D cubic Schrödinger equation. *Rev. Mat. Iberoam.*, 26(1):1–56, 2010.
[19] M. Grillakis. Linearized instability for nonlinear Schrödinger and Klein-Gordon equations. *Comm. Pure Appl. Math.*, 41(6):747–774, 1988.
[20] M. Grillakis. Existence of nodal solutions of semilinear equations in $\mathbf{R}^N$. *J. Differential Equations*, 85(2):367–400, 1990.
[21] M. Grillakis, J. Shatah, and W. A. Strauss. Stability theory of solitary waves in the presence of symmetry I. *J. Funct. Anal.*, 74(1):160–197, 1987.
[22] M. Grillakis, J. Shatah, and W. A. Strauss. Stability theory of solitary waves in the presence of symmetry II. *J. Funct. Anal.*, 94(2):308–348, 1990.
[23] D. Hundertmark and Y.-R. Lee. Exponential decay of eigenfunctions and generalized eigenfunctions of a non-self-adjoint matrix Schrödinger operator related to NLS. *Bull. Lond. Math. Soc.*, 39(5):709–720, 2007.
[24] C. K. R. T. Jones. An instability mechanism for radially symmetric standing waves of a nonlinear Schrödinger equation. *J. Differential Equations*, 71(1):34–62, 1988.
[25] C. K. R. T. Jones and T. Küpper. On the infinitely many solutions of a semilinear elliptic equation. *SIAM J. Math. Anal.*, 17(4):803–835, 1986.
[26] G. L. Lamb, Jr. *Elements of soliton theory.* John Wiley & Sons Inc., New York, 1980.
[27] S. Le Coz. Standing waves in nonlinear Schrödinger equations. In *Analytical and numerical aspects of partial differential equations*, pages 151–192. Walter de Gruyter, Berlin, 2009.
[28] P.-L. Lions. Solutions complexes d'équations elliptiques semilinéaires dans $\mathbf{R}^N$. *C. R. Acad. Sci. Paris Sér. I Math.*, 302(19):673–676, 1986.
[29] Y. Martel. Asymptotic N-soliton-like solutions of the subcritical and critical generalized Korteweg-de Vries equations. *Amer. J. Math.*, 127(5):1103–1140, 2005.
[30] Y. Martel and F. Merle. Multi solitary waves for nonlinear Schrödinger equations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 23(6):849–864, 2006.
[31] Y. Martel, F. Merle, and T.-P. Tsai. Stability in H^1 of the sum of K solitary waves for some nonlinear Schrödinger equations. *Duke Math. J.*, 133(3):405–466, 2006.
[32] F. Merle. Construction of solutions with exactly k blow-up points for the Schrödinger equation with critical nonlinearity. *Comm. Math. Phys.*, 129(2):223–240, 1990.
[33] T. Mizumachi. Instability of bound states for 2D nonlinear Schrödinger equations. *Discrete Contin. Dyn. Syst.*, 13(2):413–428, 2005.
[34] T. Mizumachi. Vortex solitons for 2D focusing nonlinear Schrödinger equation. *Differential Integral Equations*, 18(4):431–450, 2005.
[35] T. Mizumachi. Instability of vortex solitons for 2D focusing NLS. *Adv. Differential Equations*, 12(3):241–264, 2007.
[36] P. C. Schuur. *Asymptotic analysis of soliton problems*, volume 1232 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986.
[37] T. Tao. Why are solitons stable? *Bull. Amer. Math. Soc.*, 46(1):1–33, 2009.
[38] M. I. Weinstein. Nonlinear Schrödinger equations and sharp interpolation estimates. *Comm. Math. Phys.*, 87(4):567–576, 1982/83.

[39] M. I. Weinstein. Modulational stability of ground states of nonlinear Schrödinger equations. *SIAM J. Math. Anal.*, 16:472–491, 1985.

[40] M. I. Weinstein. Lyapunov stability of ground states of nonlinear dispersive evolution equations. *Comm. Pure Appl. Math.*, 39(1):51–67, 1986.