Solitons et dispersion

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Introduction

L'OBJET de ce mémoire est de présenter quelques aspects de la dynamique des solutions d'équations aux dérivées partielles dispersives. Le fil conducteur liant les différents travaux exposés ici est l'étude des solitons. Ce sont des ondes progressives ou stationnaires, des objets non linéaires qui préservent leur forme au cours du temps. Elles possèdent des propriétés de rigidité remarquables, et jouent un rôle prééminent dans la description de solutions génériques en temps long.

Ce mémoire débute en préambule par un rappel de quelques propriétés des solitons. On profite de l'introduction de notations et de concepts utilisés dans toute la suite pour esquisser la preuve de la stabilité orbitale des solitons, un résultat clé obtenu dans les années 80 indépendamment par Cazenave, Lions [28] et Weinstein [131]. On présente ici le point de vue de Weinstein.

Le premier chapitre de ce mémoire est consacré à la construction de multi-solitons. Il s'agit de solutions se comportant asymptotiquement comme une somme de solitons découplées. Elles ont la particularité d'être non dispersives dans un certain sens: toute leur masse reste concentrée localement dans des boules; on s'attend à ce qu'elles soient également rigides à l'instar des solitons.
Les multi-solitons furent construits tout d'abord pour des équations intégrables, et notamment pour l'équation de Korteweg-de Vries (KdV): il y a alors une formule explicite, et l'on constate que dans ce cas, les multi-solitons se découplent en somme de solitons en $t \to +\infty$ et en $t \to -\infty$ (les collisions sont élastiques).
Le premier cas de construction pour une équation non intégrale est dû à Merle [98] dans le cas de l'équation de Schrödinger non linéaire (NLS) L^2-critique; puis Martel [84] s'est intéressé aux cas des équations de Korteweg-de Vries généralisées, suivi de plusieurs autres.
Notre principale contribution a été de construire des multi-solitons dans le cas L^2 sur-critique (le plus instable). Cette construction est suffisamment robuste pour s'adapter à de nombreuses équations: notamment (NLS), (gKdV), Klein-Gordon (cas de type ondes)... Nous avons également étudié les ondes progressives basées sur des états excités, également instables.

Le deuxième chapitre rassemble un certain nombre de résultats concernant la décomposition en solitons pour des équations de type ondes. La conjecture de décomposition en solitons suggère que toute solution générique se décompose en temps long en une somme d'objets non linéaires rigides découplé (typiquement, des solitons), et d'un objet purement dispersif (typiquement linéaire).
Encore une fois, c'est dans le cas d'équations intégrables que les premières décompositions en solitons ont été obtenu, notamment pour les équations de Korteweg-de Vries et de Korteweg-de Vries modifiée.
On présente dans ce chapitre la suite de travaux tout à fait remarquable due à Duyckaerts, Kenig, Merle [40–45] concernant l'équation des ondes $\dot{H}^1 \times L^2$ critique, et en particulier le cas 3d avec données radiales, où la conjecture est résolue.
Nous avons obtenu des versions d'une telle décomposition pour les wave maps équivariantes à valeurs dans la sphère, ainsi que pour l'équation des ondes $\dot{H}^1 \times L^2$ critique radiale en dimension 4. Ce chapitre débute par un rappel d'une construction centrale dans la preuve, la décomposition en profile, qui fut introduite indépendamment par Bahouri, Gérard [18]

et Merle, Vega [104]; et par l'étude de la répartition de l'énergie d'une solution radiale de l'équation des ondes linéaire autour du cône de lumière: ce résultat, en sus de son rôle pour la résolution en solitons, a son intérêt propre.

Le troisième chapitre est consacré à l'étude des solutions explosives pour les équations des ondes semi-linéaires sous-conformes; on se consacre essentiellement à la dimension 1. Merle, Zaag [106–108, 110] ont montré que les points d'explosion peuvent se classifier en deux familles: l'espace se partitionne en points réguliers et singuliers, qui se caractérisent à la fois du point de vue de la géométrie de la courbe au point considéré, et suivant le profil à l'explosion de la solution. Il est tout à fait étonnant que les deux caractérisations définissent la même partition. Une conséquence tout à fait frappante de l'analyse menée est que l'ensemble des points singuliers est discret.
Notre contribution à ce sujet fut double. Nous avons précisé le profil explosif de la solution en un point singulier, en exhibant une fonctionnelle de Lyapunov qui assure la convergence de tous les paramètres de liberté. Et nous avons construit une solution possédant un point singulier où le profil explosif est prescrit.

Dans le quatrième chapitre, nous étudions le flot de l'équation de Zakharov-Kuznetsov (ZK) au voisinage d'un soliton. Nous montrons en particulier des théorèmes de rigidité de type Liouville: si une solution qui reste proche pour tout temps du soliton ne disperse pas, alors c'est le soliton. Ce résultat est essentiel dans la preuve de la stabilité asymptotique du soliton L^2 sous-critique, que nous montrons ensuite. Enfin nous étudions des propriétés similaires pour les multi-solitons.
Nous retraçons ainsi dans le cas de (ZK) l'étude faite par Martel, Merle [86–88], associés à Tsai [96] dans le cas de (gKdV). De tels résultats sont des problèmes ouverts importants pour (NLS). L'étude de (ZK) constitue une extension de (gKdV) au cas de dimensions d'espace supérieures $d \geqslant 2$, pour une équation qui présente, dans $d-1$ directions, beaucoup de similarités avec (NLS).

Le cinquième et dernier chapitre élargit notre champ de recherche: il est dédié à quelques questions en micro-magnétisme. L'objet central ici est la paroi de Néel, un objet rigide qui apparaît à la transition du spin $+$ en spin $-$ comme minimiseur d'une fonctionnelle d'énergie, dans la limite où de petits paramètres tendent vers 0. Ces parois de Néel, de codimension 1, sont observées physiquement, et sont en compétition avec d'autres objets singuliers, les vortex, de codimension 2, et d'énergie supérieure. Mathématiquement, l'une des difficultés principale est la présence d'un terme non local dans l'énergie d'une magnétisation, qui rend compte de l'effet du champ magnétique induit par les charges intérieures.
Dans un cas où les petits paramètres sont favorables à l'émergence de parois de Néel, nous montrons effectivement un résultat de compacité pour les magnétisations et la convergence vers une paroi de Néel, ainsi que leur optimalité. Ceci est inspiré par Ignat, Otto [65, 66].
Dans un second temps, nous étudions l'évolution des magnétisations suivant le modèle de Landau-Lifshitz-Gilbert, qui est à la fois non-hamiltonien et non-dissipatif. Après avoir construit des solutions faibles, nous montrons que, dans un régime physiquement pertinent, les parois de Néel restent statiques.

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The purpose of this thesis is to present a few aspects of the dynamics of solutions of dispersive partial differential equations. The connecting thread that runs through the various works described below is the study of solitons. These are travelling or stationary waves, non linear objects that preserve their shape along time. They enjoy remarkable rigidity properties, and play a preeminent role in the long time description of generic solutions.

This thesis starts with a preamble recalling some properties of the solitons. We draw benefit from the introduction of notations and concepts used throughout all the following to sketch the proof of the orbital stability of solitons, a key result obtained in the 80s independently by Cazenave, Lions [28] and Weinstein [131]. We present here the point of view of Weinstein.

The first chapter of this memoir is devoted to the construction to multi-solitons. These are solutions behaving asymptotically as a sum of decoupled solitons. They have special feature of being non dispersive in some sense: all their mass remain concentrated in balls; one also expect them to be rigid, as solitons are.
Multi-solitons were first constructed for integrable equations, and most notably of the Korteweg-de Vries equation (KdV): in that case, there is an explicit formula, and one observe by inspection that multi-solitons decouple into a sum of solitons as $t \to +\infty$ and as $t \to -\infty$ (collisions are elastics).
The first case of construction for a non integrable equation is due to Merle [98] in the case of the L^2 critical non linear Schrödinger equation (NLS); after that, Martel [84] studied the case of generalized Korteweg-de Vries equation (gKdV), followed by others.
Our main contribution was to construct multi-solitons in the L^2 supercritical case (the most unstable). This construction is sufficiently robust to adapt to various equations: in particular (NLS), (gKdV), Klein-Gordon (wave type equation) . . . We also studied travelling wave based on excited states, which are also unstable.

The second chapter gather various results regarding soliton decomposition for wave type equations. The soliton resolution conjecture suggest that any generc solution decomposes for large times into a sum of decoupled rigid nonlinear objects (typically solitons) and a purely dispersive term (typically linear).
Once again, the first solitons decompositions were obtained in the setting of integrable equations, most notably for the Korteweg-de Vries and modified Korteweg-de Vries equations.
We present in this chapter the remarkable sequence of papers by Duyckaerts, Kenig, Merle [40–45] regarding the $\dot{H}^1 \times L^2$ critical wave equation, in particular the 3d case with radial data, where the conjecture is solved.
We obtained a version of such a decomposition for equivariant wave maps into the sphere, and for the $\dot{H}^1 \times L^2$ critical wave equation in 4 dimension with radial data. This chapter starts by recalling a central construction in the proof, the profile decomposition which was introduced independently by Bahouri, Gérard [18] and Merle, Vega [104]; and by the study of the energy partition of a radial solution of the linear wave equation around the light cone: this last result, atop of its role for the soliton resolution, has its own interest.

The third chapter is devoted to the study of blow up solutions of the sub-conformal semi-linear wave equation; we essentially concentrate on dimension 1. Merle, Zaag [106–108, 110] proved that the blow up points can be divided in two families: regular and singular points, which can be caracterized both by the geometry of the curve and by blow up profile of the solution at the considered point. It is trully surprising that these two charaterizations define the same partition. One striking consequence of the analysis which was carried out, is that the set of singular points is discrete.
Our contribution on this topic is twofold. We precised the blow up behavior of the solution at a singular point by exhibiting a Lyapunov functionnal which makes all degrees of freedom converge. And we built a solution admitting a singular point with a prescribed blow up profile.

In the fourth chapter, we study the flow of the Zakharov-Kuznetsov equation (ZK) in the neighbourhood of a soliton. We show in particular Liouville type rigidity theorems: if a solution remains for all times near the soliton and does not disperse, then it is the soliton. This result is essential in the proof of the asymptotic stability of the L^2 subcritical soliton, which we prove next. We then study related properties for multi-solitons.
We thus retrace in the case of (ZK) the analysis conducted by Martel, Merle [86–88], together with Tsai [96] for (gKdV). Such results are important open problems for (NLS). The study carried out on (ZK) is an extension of (gKdV) to higher space dimensions $d \geqslant 2$, for an equation which presents, in $d-1$ directions, many similarities with (NLS).

The fifth and final chapter widen our research themes: it is dedicated to some questions on micromagnetism. The central object here is Néel walls, a rigid object appearing at the transition of spin + to spin -, in the limit when small parameters tend to 0. These Néel walls, of codimension 1, are physically observed, and in competition with other singular objects, the vortices, which are of codimension 2, and of higher energy. One of the main mathematical difficulties is the presence of a nonlocal term in the energy of a magnetization, which gives account for the magnetic field induced by interior charges.

In the case when the regime of the small parameters is favorable to the emergence of Néel walls, we show indeed a compactness result for the magnetizations and the convergence to a Néel wall, along with its optimality. This is inspired by Ignat, Otto [65, 66]. In a second step, we study the evolution of magnetizations under the Landau-Lifschitz-Gilbert model, which is both non-hamiltonian and non-dissipative. After having constructed global weak solutions, we shows that, in a physically relevant regime, Néel walls remain static.

Preamble: Orbital stability of solitons

Solitons

SOLITONS are remarquable special solutions to various nonlinear *focusing* dispersive partial differential equations. For these equations, two elements play a role in the long time dynamics of solutions: the dispersive properties of the linear differential operator tend to make the solution scatter, whereas the focusing nature of the nonlinearity, to the contrary, tend to concentrate the solution. Solitons appear as a delicate equilibrium between these two forces: they are nonlinear objects that keep their form along time, neither scattering of concentrating.

Soliton also enjoy many outstanding properties when viewed as particular solutions of a PDEs: for example stability and rigidity properties, and a prominent role in the long time profile of general solutions, both blow up or global. The purpose of this preamble is to collect some well known facts about solitons, culminating in a description of orbital stability. The other features mentioned above will be studied in the following chapters.

Solitons are defined mathematically as travelling wave solutions. We will give two representative examples. First consider the generalized Korteweg-de Vries equation (gKdV)

$$\begin{cases} \partial_t u + \partial_x(\partial_{xx} u + f(u)) = 0, \\ u(t=0,x) = u_0(x), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{gKdV}$$

The solitons form a 2 parameter family of travelling wave solution to (gKdV), denoted $Q[c_0, x_0]$ for $(c_0, x_0) \in (0, +\infty) \times \mathbb{R}$, where

$$Q[c_0, x_0](t,x) := Q_{c_0}(x - x_0 - c_0 t), \tag{0.1}$$

and $Q_{c_0} \in H^1$ is a solution to the elliptic equation

$$-\Delta Q_{c_0} + c_0 Q_{c_0} = f(Q_{c_0}). \tag{0.2}$$

In the pure power case $f(u) = |u|^{p-1}u$, Q_c can be deduced from Q_1 via scaling and in one space dimension, this last function is explicit:

$$Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x), \quad \text{where} \quad Q(x) = Q_1(x) = \left(\frac{p+1}{2\cosh^2(\frac{p-1}{2})} \right)^{\frac{1}{p-1}}.$$

The existence of Q_c, and its uniqueness up to translations, is guaranteed by the following result. Let F be the standard integral of f:

$$F(s) := \int_0^s f(\sigma) d\sigma. \tag{0.3}$$

The assumptions we make regarding the nonlinearity f and speed $c > 0$ are the following

(A1) f is $\mathscr{C}^1$, odd, and satisfies $f(0) = f'(0) = 0$.

(A2) $s \mapsto F(s) - cs^2/2$ admits a smallest positive zero $s_c > 0$ and $f(s_c) - cs_c > 0$.

Observe that (A2) makes the nonlinearity focusing.

Proposition 0.1 (Berestycki, Lions [19, Theorem 5 and Remark 6.3]). *Let a function f and $c > 0$ satisfy the assumption* (A1).
Then there exists a non trivial solution $Q_c \in H^1(\mathbb{R})$ to (0.2) *if and only if f satisfies* (A2). *In that case, Q_c is $\mathscr{C}^2$, unique up to translation, and can be chosen even and decreasing on $[0, +\infty)$, with $Q_c(0) = s_c$; this implies $Q_c > 0$. Furthermore, $Q_c e^{\sqrt{c}|x|}$ and $|\partial_x Q_c| e^{\sqrt{c}|x|}$ are bounded on $\mathbb{R}$.*

Our second example is the d dimensional nonlinear Schrödinger equation:

$$\begin{cases} i\partial_t u + \Delta u - f(u) = 0, \\ u(t=0,x) = u_0(x), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad u(t,x) \in \mathbb{C}. \tag{NLS}$$

We will always assume that the nonlinearity $f : \mathbb{C} \to \mathbb{C}$ is gauge invariant, that is

$$\forall \theta \in \mathbb{R},\ \rho \geqslant 0, \quad f(\rho e^{i\theta}) = f(\rho) e^{i\theta}.$$

Then we can formally compute that solitons of the form

$$Q[c_0, \gamma_0, v_0, x_0](t,x) = e^{i(\frac{1}{2} v_0 \cdot x - \frac{1}{4}\|v_0\|^2 t + c_0 t + \gamma_0)} Q_{c_0}(x), \tag{0.4}$$

with four parameters: phase $\gamma_0 \in \mathbb{R}$, speed $v_0 \in \mathbb{R}^d$, translation $x_0 \in \mathbb{R}^d$ and frequency $c_0 > 0$ are travelling wave solutions to (NLS) if and only if Q_{c_0} is a solution to the d dimensional elliptic equation

$$-\Delta Q_{c_0} + c_0 Q_{c_0} = f(Q_{c_0}). \tag{0.5}$$

For such a solution Q_{c_0} to exist in dimension $d \geqslant 2$, one need to assume (A1), (A2) and a growth assumption that ensures the nonlinearity be $\dot{H}^1$ subcritical. However this growth condition is not quite enough to study the flow of (NLS) around solitons. We need eigenfunctions for the linearized operator around Q_c, which where studied in the (NLS) cases by Weinstein [131], Grillakis [59] and Schlag [118]. We also need a strong knowledge of spectrum of linearized energy around Q_c. In the case of a general nonlinearity f, these results are open: this is why we stick to the pure power nonlinearity (this could be relaxed under a suitable spectral assumption, see (0.10) below). Therefore, our assumption reads

(A3) $f(u) = |u|^{p-1}u$, where $1 < p$ if $d = 2$, and $1 < p < \dfrac{d+2}{d-2}$ if $d \geqslant 3$.

We emphasize that in the $\dot{H}^1$ supercritical case (i.e. $d \geqslant 3$ and $p > \dfrac{d+2}{d-2}$), the elliptic equation (0.5) admits no non trivial solution, due to Pohozahev identities, let alone that the Cauchy problem for (NLS) is ill posed. In the $\dot{H}^1$ critical case $p = \dfrac{d+2}{d-2}$, the Cauchy problem for (NLS) is well posed (see Cazenave, Weissler [29]), but solutions to (0.5) have only algebraic decay (and the ground state is usually denoted W); see Chapter 2 for further details.

Proposition 0.2. *Let $d \geqslant 2$, $f(u) = |u|^{p-1}u$ satisfy* (A3) *and $c_0 > 0$.*
There exist a positive solution $Q_{c_0} \in H^1$ to (0.5)*, unique up to translation, called the* ground state*: it is also radial, and radially (exponentially) decreasing.*

Before we turn to stability issue, let us recall a few important facts about (NLS) and (gKdV). The Cauchy problem for both equations is well posed in H^1; we refer to Kenig, Ponce, Vega [70] (see also [69]) for (gKdV) and Ginibre, Velo [57] for (NLS). Both equations also admit two conservation laws: an H^1 solution u preserves the L^2 mass and the energy

$$\|u(t)\|_{L^2}^2 = \|u(0)\|_{L^2}^2 \quad (L^2\text{mass}) \tag{0.6}$$

$$E(u(t)) = \int \left(\frac{1}{2} |\nabla u(t,x)|^2 - F(u(t,x)) \right) dx = E(u(0)), \quad \text{(energy)} \tag{0.7}$$

At least formally, (gKdV) preserves the L^1-mass $\int u(t,x)dx$, and (NLS) preserves the momentum $P(u) = \operatorname{Im} \int \bar{u}(t,x) \nabla u(t,x) dx$.

Orbital stability of solitons

The main matter of this preamble is to sketch the following orbital stability result. It was obtained independently by Weinstein [130, 131] (via modulation) and Cazenave, Lions [28] (via concentration-compactness); we also refer to Grillakis Shatah and Strauss [60] for an abstract exposition of Weinstein's method. This is the starting point of many of the dynamical aspects developed in the subsequent chapters.

When considering the stability property of a soliton, one can not expect the usual stability statement for dynamical systems to hold true: indeed if c_0 and c_1 are very close, then $Q[c_0, 0](0)$ and $Q[c_1, 0](0)$ are very close solutions of (gKdV) at initial time, but they travel at different speed and eventually decouple:

$$\|Q[c_0,0](t) - Q[c_1,0](t)\|_{L^2} \to \|Q[c_0,0](t)\|_{L^2} + \|Q[c_1,0](t)\|_{L^2} \quad \text{as} \quad t \to +\infty.$$

However, the shape of each soliton is kept along time, so that up to a translation, they remain close. This motivates the definition:

Definition 0.3. Let B be a Banach space and G be a group of symmetries acting on G. We say that a solution $u \in \mathscr{C}([0,+\infty), B)$ to a partial differential equation (PDE), globally defined for positive times, is *G-orbitally stable* (or simply orbitally stable if there is no ambiguity), if

$$\forall \varepsilon > 0, \exists \delta > 0, \quad \|v(0) - u(0)\|_B \leqslant \delta \Longrightarrow \left(\forall t \geqslant 0, \quad \inf_{g \in G} \|v(t) - g.u(t)\|_B \leqslant \varepsilon \right),$$

where $v(t)$ is the solution to (PDE) with initial data $v(0)$.
If this is not true, we say that u is orbitally unstable.

Usually G does *not* contain scaling. Observe that the definition implies that any initial data close enough to $u(0)$ yields a solution to (PDE) globally defined for positive times.

We can now state the orbital stability of the solitons, here in the case of (gKdV) (a similar statement holds for (NLS)). Here G is simply $\mathbb{R}$ acting on H^1 via space translations.

Theorem 0.4. *Let $c_0 > 0$ and assume that f satisfies (A1) and (A2) for all c in a neighbourhood of c_0.*

1. *If $\frac{d}{dc}\|Q_c\|_{L^2}|_{c=c_0} < 0$, then $Q[c_0, 0]$ is orbitally stable.*
2. *If $\frac{d}{dc}\|Q_c\|_{L^2}|_{c=c_0} > 0$, then $Q[c_0, 0]$ is orbitally unstable.*

The critical case $\frac{d}{dc}\|Q_c\|_{L^2}|_{c=c_0} = 0$ can be much more delicate, and usually the soliton $Q[c_0, 0]$ is unstable in that case. For (gKdV), we refer to Martel, Merle [90] for the pure power nonlinearity $f(u) = u^5$. For (NLS), the simpler way to see this is a Virial identity (leading to finite time blow up solution, but *without* any qualitative description of the blow up).
Actually, the dynamics of the flow around a soliton were studied in great detail, leading to a sharp description of the blow up. Although we won't go further on that subject, let us refer to the works by Martel, Merle [89, 92] and Martel, Merle, Raphael [93–95] for (gKdV), and by Merle, Raphael [99–103] for (NLS).

Theorem 0.4 contains two statements. We will focus on the stability result. It is essentially a static result, consequence of the variational properties of Q_c, and the conservation of the two conservations laws: L^2 mass and energy.

Proposition 0.5. *Q_c minimizes the energy among function with a given L^2-norm, more precisely, it is a minimizer of the following elliptic problem:*

$$\inf\{E(u) \mid \|u\|_{L^2} = \|Q_c\|_{L^2}\}.$$

Notice that $\nabla E(Q) = -\Delta Q + f(Q)$, so that via Lagrange multipliers, we recover the equation for Q_c (0.5). Introduce the Weinstein functional

$$E_c(u) = E(u) + \frac{c}{2}\|u\|_{L^2}^2. \tag{0.8}$$

Then $Q[c_0, x_0]$ is a critical point of E_{c_0}: $\nabla E_{c_0}(Q) = 0$. And we have the expansion, for $v \in H^1$ small

$$E_{c_0}(Q[c_0, x_0](0) + v) = E_{c_0}(Q_{c_0}) + \langle L_{c_0,x_0} v, v\rangle + O(\|v\|_{H^1}^3), \tag{0.9}$$

where $L_{c_0,x_0} = d^2 E_{c_0}(Q[c_0, x_0](0))$ is the Hessian defined by

$$L_{c_0,x_0} v = -\Delta v + f'(Q_{c_0}(\cdot - x_0))v + cv.$$

As a Q_c is minimizer, we also have to coercivity property:

Proposition 0.6 ([131])**.** *Assume (A1), (A2), (A3). Then the following statements are equivalent*

1. $\frac{d}{dc}\|Q_c\|_{L^2}^2|_{c=c_0} < 0$.
2. *there exists $\lambda > 0$ such that:*

$$\forall v \in H^1, \quad \langle L_{c_0} v, v\rangle \geqslant \lambda\|v\|_{H^1}^2 - \frac{1}{\lambda}\left(\langle v|Q_{c_0}\rangle^2 + \sum_{i=1}^{d}\langle v|\partial_{x_i} Q_{c_0}\rangle^2\right). \tag{0.10}$$

So we have $d+1$ "bad" directions. We take care of the first d by modulation theory:

Proposition 0.7. *There exist $C, \delta > 0$ such that the following holds true. Let $x_0 \in \mathbb{R}^d$ and $v \in H^1$ such that $\|v\|_{H^1} \leqslant \delta$. Then there exists a unique $\tilde{x}_0 \in \mathbb{R}^d$, $\|\tilde{x}_0 - x_0\| \leqslant C\|v\|_{H^1}$ such that defining $\tilde{v}$ by*

$$Q_c(\cdot - \tilde{x}_0) + \tilde{v} = Q_c(\cdot - x_0) + v,$$

there holds

$$\langle \tilde{v}|(\partial_x Q_c)(\cdot - \tilde{x}_0)\rangle = 0. \tag{0.11}$$

Furthermore, the map $(x_0, v) \mapsto (\tilde{x}_0, \tilde{v})$ is smooth.

This proposition follows from a suitable use of the implicit function theorem. The last bad direction is dealt with the conservation of the L^2 mass:

$$\langle v|Q_c\rangle = \frac{1}{2}\left(\|Q_c + v\|_{L^2}^2 - \|Q_c\|_{L^2}^2 - \|v\|_{L^2}^2\right).$$

The proof goes then as follows. Let $u(t)$ be a solution to (gKdV) such that

$$\|u(0) - Q[c_0, x_0]\|_{H^1} \leqslant \delta.$$

Fix some large A, we work on a time interval I on which $\inf_x \|u(t) - Q[c_0, x]\|_{H^1} \leqslant A\delta$ is small for all $t \in I$. A is only meant to allow the use of the modulation: we write $u(t) = Q[c_0, x(t)] + v(t)$, where $v(t)$ satisfies the orthogonality condition (0.11); also we have $\|v(t)\|_{H^1} = O(\delta)$ is small. Due to the conservation of mass and energy, we have

$$\begin{aligned}
\|v(t)\|_{H^1}^2 &\leqslant \frac{1}{\lambda}\langle L_{c_0,x(t)} v, v\rangle \leqslant \lambda\|v\|_{H^1}^2 + \frac{1}{\lambda^2}\langle v|Q[c_0, x(t)]\rangle^2 \\
&\leqslant \frac{1}{\lambda}\left(E(u(t)) - E(Q_{c_0}) + O(\|v\|_{H^1}^3)\right) + \frac{1}{2\lambda^2}\left((\|u(t)\|_{L^2}^2 - \|Q_{c_0}\|_{L^2}^2)^2 + O(\|v\|_{L^2}^3)\right) \\
&\leqslant C\|v(0)\|_{H^1}^2 + O(\|v\|_{H^1}^3) \leqslant C\delta^2 + O(\|v\|_{H^1}^3).
\end{aligned}$$

As $\|v\|_{H^1} = O(\delta)$ is small, we infer $\|v(t)\|_{H^1} \leqslant C\delta$, with C independent of A. A bootstrap argument allows to conclude that $v(t)$ remains small for all time, and then to conclude to orbital stability.

CHAPTER 1

Multi-solitons

In this chapter we consider the following generalization of solitons, namely *trains of solitons* or *multi-solitons*: these are solutions, defined for t large enough, and which decouple as $t \to +\infty$ into a sum of weakly interacting solitons. The most natural way to ensure weak interaction is to ask that the final solitons have distinct speeds.

A multi-soliton is then a solution $U \in \mathscr{C}([T_-(U), +\infty), H^1)$ to (gKdV) such that

$$\left\| U(t) - \sum_{j=1}^{N} Q[c_j, x_j](t) \right\|_{H^1} \to 0 \quad \text{as} \quad t \to +\infty, \tag{1.1}$$

where $N \geqslant 1$, $c_1 < \cdots < c_N$ and $x_1, \ldots, x_n \in \mathbb{R}$ are given parameters.
For other equations, multi-solitons are defined similarly, adapting the set of parameters to the symmetries of the equation.

The relevance of studying multi-solitons is that they appear naturally when studying the long time dynamics of dispersive equation, especially of view to the soliton resolution conjecture. Furthermore, multi-solitons enjoy remarquable dynamical properties. For example, a multi-soliton U does not disperse, in the sense that in physical space, all the mass remains in a fixed number of ball (moving with time, but not increasing in diameter). Also, in Fourier space, there is no transfer of mass to high frequencies.

The assumptions (A1),(A2),(A3) on the nonlinearity refer to those introduced in the preamble.

1 Construction of multi-solitons

We now present the construction of multi-solitons to various nonlinear (focusing) dispersive equations admitting soliton solutions. Let us start with the case of (gKdV).

Multi-solitons for the generalized Korteweg-de Vries equation

Combining the proof of Martel [84, Theorem 1] in the L^2 subcritical and critical cases, and C., Martel, Merle [9, Theorem 1] in the L^2 supercritical case, we claim the following existence result of multi-soliton for (gKdV).

Theorem 1.1 (Multi-solitons for (gKdV))**.** *Let $f \in \mathscr{C}^3$ be convex on $[0, +\infty)$, and let $0 < c_1 < \ldots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$ be such that* (A1) *and* (A2) *hold for all c in a neighbourhood of $\{c_1, \ldots, c_n\}$.*
There exist $T_0 \in \mathbb{R}$, $C, \sigma_0 > 0$, and a solution $U \in \mathscr{C}([T_0, \infty), H^1)$ to (gKdV) *such that*

$$\forall t \in [T_0, \infty), \qquad \left\| U(t) - \sum_{j=1}^{N} Q[c_j, x_j](t) \right\|_{H^1} \leqslant C e^{-\sigma_0^{3/2} t}.$$

Let us first say a few words on the assumptions: the conditions on f ensure both local well-posedness in H^1 from [70] for (gKdV), and the existence of eigenvalues for the linearized operator in the instable case from [116]. The condition that (A2) should hold for all c in a neighbourhood of any c_j allows to use modulation theory, which is a very efficient tool to relate the nonlinear problem to the linearized flow.

Let us emphasize that the solitons $Q[c_j, x_j]$ can actually be orbitally unstable. In the pure power case $f(u) = |u|^{p-1}u$, for $p < 5$ L^2 subcritical case, solitons are stables. Seen from one solitons, the others solitons appear as exponential perturbations: in the L^2 subcritical case, it seems reasonable to construct multi-solitons. In the L^2 supercritical case, each soliton is (linearly) unstable, and the perturbation, even exponentially small, could result in large effects: the previous theorem ensures that there is a way to make these perturbations vanish as $t \to +\infty$, and is more surprising in this latter case. We will give more details on this when classifying (gKdV) multi-solitons, in the next section.

The proof of this result is actually very robust, and can be applied to various equation admitting solitons soution. From the techniques developped by Mizumachi [112], El Dika [50] and El Dika and Martel [51] concerning the (BBM) equation

$$(u - u_{xx})_t + (u + u^p)_x = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}, \tag{BBM}$$

and from the construction of suitable eigenfunctions of the linearized equation by Pego and Weinstein [116, page 74] one can also extend the results obtained in Theorem 1.1 to the (BBM) equation for any $p > 1$.

Multi-solitons for the nonlinear Schrödinger equation

We can also extend the construction to the nonlinear Schrödinger equation (NLS).
Notice that the frequency c_0 and the speed v_0 of solitons are decoupled contrarily to (gKdV). Hence the only dynamical assumption is that solitons have distinct speeds, so that their interaction be exponentially decaying in time; but the frequency of distincts solitons could be the same.

Once these observations are made, we are in a position to state the result concerning the construction of multi-solitons for (NLS).

Theorem 1.2 (Multi-solitons for (NLS))**.** *Let $f(u) = |u|^{p-1}u$ satisfy* (A3).
Let $N \geqslant 1$ be an integer, and parameters $c_1, \dots, c_N > 0$, $\gamma_1, \dots, \gamma_N \in \mathbb{R}$, $x_1, \dots, x_N \in \mathbb{R}^d$, and $v_1, \dots, v_N \in \mathbb{R}^d$ be such that

$$\forall j \neq k, \quad v_j \neq v_k.$$

Then there exist $T_0 \in \mathbb{R}$, $C, \sigma_0 > 0$, and a solution $U \in \mathscr{C}([T_0, \infty), H^1)$ to (NLS) *such that*

$$\forall t \in [T_0, \infty), \qquad \left\| U(t) - \sum_{j=1}^{N} Q[c_j, \gamma_j, v_j, x_j](t) \right\|_{H^1} \leqslant Ce^{-\sigma_0^{3/2} t}.$$

The L^2 critical case $p = 1 + \dfrac{4}{d}$ was in fact the first setting where multisolitons where constructed by Merle [98]. More precisely, this celebrated result constructed solutions where blow-up simultaneously occurs at N distinct points. This is in fact equivalent to the construction of multi-solitons as described above, via the pseudo-conformal transform. The L^2 sub-critical case $p < 1 + \dfrac{4}{d}$ was treated by Martel and Merle [91], following the result by Martel [84] on (gKdV). The remaining L^2 supercritical case $p > 1 + \dfrac{4}{d}$ (and $\dot{H}^1$ subcritical) was done in C., Martel, Merle [9, Theorem 2]; its proof is completely similar to the one of Theorem 1.1 for (gKdV).

A stricking property is that in dimension $d \geqslant 2$, there is not one but infinitely many solutions (even up to the symmetries of the problem) to the elliptic equation (0.5) which we recall:

$$-\Delta Q_c + cQ_c = f(Q_c).$$

We refer to Berestycki, Lions [19], Kwong [77], Gidas, Ni, Nirenberg [54], Serrin, Tang [119] for the following proposition.

Proposition 1.3. *A solution to* (0.5) *in* $H^1(\mathbb{R}^d)$ *is called a* bound state*. There exist infinitely many bound states (even after translation and rotations).*
Furthermore, any bound state Φ_{c_0} *is* $\mathscr{C}^2$ *and exponentially decaying: for all* $c < c_0$, $e^{\sqrt{c}|x|}|\Phi_{c_0}|$ *and* $|\partial_x \Phi_{c_0}|e^{\sqrt{c}|x|}$ *are bounded on* $\mathbb{R}^d$.

Bound states which are not the ground state are called excited states. The denomination soliton is usually reserved for travelling wave solution of the form (0.4) based on the ground state. When based on an excited state Φ (which can be complex valued), we will speak here of excited soliton.

A subsequent question is wether one can achieve the same multi-soliton construction of with excited state Φ instead of solitons.
It turns out that in case of an excited state Φ, the spectral properties of the linearized (NLS) flow around Φ are not known well enough for a sharp construction. However, if the relative speeds are sufficiently high, that is if the excited states decouple fast enough, we can close the estimates and obtain a result.

Theorem 1.4 (Excited state multi-solitons, C., Le Coz [8])**.** *Assume* $d \geqslant 2$ *and (A3).*
Let $N \geqslant 2$ *be an integer, and let parameters* $c_1, \dots, c_N > 0$, $\gamma_1, \dots, \gamma_N \in \mathbb{R}$, $x_1, \dots, x_N \in \mathbb{R}^d$, *and* $v_1, \dots, v_N \in \mathbb{R}^d$, *and excited states* $\Phi_{c_1}, \dots, \Phi_{c_N} \in \dot{H}^1(\mathbb{R}^d)$ *with respective frequencies* $c_1, \dots, c_n$ *be given. Denote*

$$\Phi_j(t,x) = e^{i(\frac{1}{2}v_j\cdot x - \frac{1}{4}\|v_j\|^2 t + c_j t + \gamma_j)}\Phi_{c_j}(x),$$

(this is an excited state of (NLS)*), and*

$$c_* = \min\{c_j \mid j = 1, \dots, N\} \quad \text{and} \quad v_* = \min\{|v_j - v_k| \mid j \neq k\}.$$

Then there exist $\alpha = \alpha(d,N)$ *and* $v_\sharp = v_\sharp(\Phi_1, \dots, \Phi_N)$ *such that if* $v_* \geqslant v_\sharp/\alpha$, *the following holds. There exist* $T_0 \in \mathbb{R}$ *and a solution* $U \in \mathscr{C}([T_0, +\infty), H^1(\mathbb{R}^d, \mathbb{C}))$ *of* (NLS) *such that we have*

$$\forall t \geqslant T_0, \quad \Big\| U(t) - \sum_{j=1}^N \Phi_j(t) \Big\|_{H^1} \leq e^{-\alpha c_*^{\frac{1}{2}} v_* t}.$$

Multi-solitons for the nonlinear Klein-Gordon equation

We also consider multi-solitons for wave type equation. In this context, the natural nonlinear objects are Lorentz boosts of stationary solutions. For the semi linear wave equation, stationary solutions have algebraic decay, and our method of proof can not apply as such (cf. Chapter 2). Therefore, we rather focus on the nonlinear Klein-Gordon equation:

$$\begin{cases} \partial_{tt}u - \Delta u - u - |u|^{p-1}u = 0, \\ (u, \partial_t u)(t=0,x) = (u_0(x), u_1(x)), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d. \tag{KG}$$

Prescribing f to the above class of focusing nonlinearities (i.e. (A1) and (A2) if $d = 1$, and (A3) if $d \geqslant 2$) ensures that the corresponding Cauchy problem for (KG) is locally well-posed in $H^s(\mathbb{R}^d) \times H^{s-1}(\mathbb{R}^d)$, for any $s \geqslant 1$: we refer to Ginibre and Velo [58] and Nakamura and Ozawa [114] (when $d = 2$) for more details.
Then (KG) admits a stationnary solution $Q(t,x) = Q(x)$ which satisfies (0.5) with $c_0 = 1$. Since (KG) is invariant under *Lorentz boosts*, we can define a *boosted ground state* (a *soliton* from now on) with relative velocity $\beta \in \mathbb{R}^d$. More precisely, let $\beta = (\beta^1, \dots, \beta^d) \in \mathbb{R}^d$, with $|\beta| < 1$ (we denote $|\cdot|$ the euclidian norm on $\mathbb{R}^d$), the corresponding Lorentz boost is given by the $(d+1) \times (d+1)$ matrix

$$\Lambda_\beta := \begin{pmatrix} \gamma & -\beta^1\gamma & \cdots & \beta^d\gamma \\ -\beta^1\gamma & & & \\ \vdots & & \mathrm{Id}_d + \dfrac{(\gamma-1)}{|\beta|^2}\beta\beta^T & \\ -\beta^d\gamma & & & \end{pmatrix} \quad \text{where} \quad \gamma := \frac{1}{\sqrt{1-|\beta|^2}}, \tag{1.2}$$

($\beta\beta^T$ is the $d \times d$ rank 1 matrix with coefficient of index (i,j) $\beta^i\beta^j$). Then the boosted ground state with velocity $\beta_0 \in \mathbb{R}^d$ with $|\beta_0| < 1$, and space translation $x_0 \in \mathbb{R}^d$ is

$$Q[\beta_0, x_0](t,x) := Q\left(\Lambda_{\beta_0}\begin{pmatrix} t \\ x - x_0 \end{pmatrix}\right). \tag{1.3}$$

(It is a travelling wave with speed β_0).

Theorem 1.5 (Multi-solitons for (KG), C., Muñoz [10]). *Assume f satisfies* (A). *Let $\beta_1, \beta_2, \dots, \beta_N \in \mathbb{R}^d$ be a set of distinct velocities*

$$\forall j \neq k, \quad \beta_j \neq \beta_k, \quad \text{and} \quad |\beta_k| < 1,$$

and $x_1, x_2, \dots, x_N \in \mathbb{R}^d$ be shift parameters.
Then there exist a time $T_0 \in \mathbb{R}$, constants $C > 0$, and $\gamma_0 > 0$, only depending on the sets $(\beta_j)_j$, $(x_j)_j$, and a solution $(U, \partial_t U) \in \mathscr{C}([T_0, +\infty), H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d))$ of (KG), globally defined for forward times and satisfying

$$\forall t \geqslant T_0, \quad \left\| (U, \partial_t U)(t) - \sum_{j=1}^N (Q[\beta_j, x_j], \partial_t Q[\beta_j, x_j])(t) \right\|_{H^1 \times L^2} \leqslant Ce^{-\gamma_0 t}.$$

Although the nonlinear object Q under consideration is the same as for (NLS) for example, the structure of the flow is different (recall that all solitons are unstable for (KG), irrespective of the nonlinearity). Hence we need to work in a more general framework, the one given by a matrix description of (KG). The main point of this result is to establish an adequate theory for the linearized operator of the (KG) flow and of the coercivity properties of its conserved quantities (energy and momentum).

As we saw, our method of proof applies essentially to any situation where the solitons decay exponentially fast, i.e the $\dot{H}^1$ subcritical case. A nice open problem is to construct multi-solitons for $\dot{H}^1$ critical equations as well, for example (NLS) or (NLW). The situation is then notably different: $\dot{H}^1$ critical solitons decay only algebraically, and so interactions are much more important.

Ideas of proof

For simplicity in the notations, we will focus on the construction of multi-solitons for (gKdV) in the case of a pure power non-linearity $f(u) = |u|^{p-1}u$, and use the notations of Theorem 1.1.
The crux of the argument is to construct a sequence $U_n(t)$ of solution to (gKdV) defined on some interval $[T_0, S_n]$ where T_0 does not depend on n and $S_n \to +\infty$ (S_n can in fact be any such sequence), and such that the following uniform estimate holds: there exists a constant C such that for all n,

$$\forall t \in [T_0, S_n], \quad \left\| U_n(t) - \sum_{j=1}^N Q[c_j, x_j](t) \right\|_{H^1} \leqslant Ce^{\sigma_0 t}. \tag{1.4}$$

Assume for now that such a sequence $(U_n)_n$ is constructed. Notice that $(U_n(T_0))_n$ is bounded in H^1, so that, up to extraction, we can consider a weak limit U_* of $(U_n(T_0))$.
Then define $U(t)$ the solution to (gKdV) with initial data at time T_0: $U(T_0) = U_*$. Recall that the flow of (gKdV) is continuous in the weak-H^1 topology (as a consequence of well posedness in H^{s_0} for some $s_0 < 1$; observe that the problem is $\dot{H}^1$ subcritical). Therefore, taking the weak limit in (1.4), we infer that $T_+(U) = +\infty$ and

$$\forall t \geqslant T_0, \quad \left\| U(t) - \sum_{j=1}^N Q[c_j, x_j](t) \right\|_{H^1} \leqslant Ce^{-\sigma_0^{3/2} t}.$$

U is hence the desired multi-soliton.

We are now left with constructing the solution U_n satisfying (1.4). Let us start with the L^2 subcritical $1 < p < 5$ case, where solitons are stable. In that case we define U_n *backward* in time, as the solution to (gKdV) with final data

$$U_n(S_n) := \sum_{j=1}^{N} Q[c_j, x_j](S_n).$$

We work on the (maximal) interval of time $[T_n, S_n]$ on which (1.4) holds true. For all t on this interval, $U_n(t)$ is close to a sum of solitons, and we can modulate the parameter c_j and x_j to obtain some orthogonality conditions on the remainder: more precisely, there exist unique $\mathscr{C}^1$ functions $y_j(t)$ such that

$$U_n(t,x) = \sum_{j=1}^{N} Q[c_j, y_j(t)](t,x) + \eta(t,x), \quad \text{such that} \quad \int \eta(t,x)\partial_x(Q[c_j, y_j(t)])(t,x)dx = 0, \tag{1.5}$$

for all $j = 1, \dots, N$. Modulation theory also provides bounds on η and the y_j, in particular we have

$$\forall t \in [T_n, S_n], \quad \|\eta(t)\|_{H^1} \leqslant Ce^{-\sigma_0^{3/2}t}. \tag{1.6}$$

Observe that $\eta(S_n) = 0$. The point of the argument is that local quantities related to the mass and energy around each soliton enjoy a monotonicity property. Denote

$$M_j(t) = \int U_n(t,x)^2\varphi_j(t,x)dx \quad \text{where} \quad \varphi_j(t,x) = \psi\left(x - \frac{c_{j+1}+c_{j-1}}{2}t\right) - \psi\left(x - \frac{c_j + c_{j-1}}{2}t\right),$$

and $\psi(y) = \arg\tanh(\exp(-\sqrt{\sigma_0}y))$ is a suitable cut-off function such that $\psi(y) \to 0$ as $y \to +\infty$ and $\psi(y) \to 1$ as $y \to -\infty$ with exponential rate. Then for σ_0 small enough, one has the backward (almost) monotonicity property

$$M_j(t) - M_j(S_n) \geqslant -Ce^{-3\sigma_0^{3/2}t}.$$

The error term $e^{-3\sigma_0^{3/2}t}$ comes essentially from the interaction between solitons; observe that it is super quadratic with respect to (1.6) One can similarly consider the energy related quantity

$$E_j(t) := \int \left(\frac{1}{2}(\partial_x U_n(t,x))^2 - \frac{1}{p+1}|U_n(t,x)|^{p+1}\right)\varphi_j(t,x)dx.$$

(One obtains the desired monotonicity on a sightly modified localized energy). By summation, a similar property holds for the following quantity, related to Weinstein's functional localized around each soliton:

$$W(t) := \sum_{j=1}^{N} \frac{1}{c_j^2}\left(E_j(t) + \frac{c_j}{2}M_j(t)\right).$$

Now we can expand $W(t)$ in terms of η: for this it is convenient to introduce the linear operators

$$Lv = -\Delta v - f'(Q)v + c, \quad L_j(t)v = -\Delta v - f'(Q[c_j, y_j(t)](t))v + c_jv. \tag{1.7}$$

L_j is simply L suitably rescaled and translated. The coefficient c_j of Weinstein's functional makes the linear term vanish, so that for all $t \in [T_n, S_n]$,

$$W(t) = \sum_{j=1}^{N} \frac{1}{c_j^2}\left(E(Q[c_j,0])) + c_j\|Q[c_j,0\|_{L^2}^2\right) + \frac{1}{2}H(t) + O(\|\eta(t)\|_{H^1}^3) + O(e^{-3\sigma_0^{3/2}t}), \tag{1.8}$$

where $H(t) = \sum_{j=1}^{N} \frac{1}{c_j^2}H_j(t)$ and

$$H_j(t) = \int L_j(t)\eta(t,x)\eta(t,x)\varphi_j(t,x)dx.$$

Now recall the coercivity of property of L (Q is a stable soliton): there exists $\lambda_0 > 0$ such that

$$\forall v \in H^1, \quad \langle Lv, v\rangle \geqslant \lambda_0 \|v\|_{H^1}^2 - \frac{1}{\lambda_0}\left(\langle v, \partial_x Q\rangle^2 + \langle v, Q\rangle^2\right). \tag{1.9}$$

Scaling, translation and a usual localization argument yield an analogous result on $H_j(t)$, and by summation – recall that $\langle \eta(t), \partial_x Q[c_j, y_j(t)](t)\rangle = 0$, we get for some $\lambda > 0$ not depending on n or $t \in [T_n, S_n]$:

$$H(t) \geqslant \lambda \|\eta(t)\|_{H^1}^2 - \frac{1}{\lambda}\sum_{j=1}^{N} \langle \eta(t), Q[c_j, y_j(t)](t)\rangle^2.$$

Combining this with the monotonicity property on $W(t)$, we deduce that

$$\|\eta(t)\|_{H^1}^2 \leqslant C\sum_{j=1}^{N} \langle \eta(t), Q[c_j, y_j(t)](t)\rangle^2 + Ce^{-3\sigma_0^{3/2}}t. \tag{1.10}$$

Now, an important feature is that the variation of the scalar product in the previous estimate is in fact *quadratic* in η, more precisely,

$$\left|\frac{d}{dt}\langle \eta(t), Q[c_j, y_j(t)](t)\rangle\right| \leqslant C\|\eta\|_{H^1}^2.$$

(The bound can be improved to an H^1 norm localized around $y_j(t) + c_j t$, the center of mass of$Q[c_j, y_j(t)](t)$). We can integrate this on $t \in [T_n, S_n]$, using the bound (1.6), and pluging it in (1.10), we deduce:

$$\forall t \in [T_n, S_n], \quad \|\eta(t)\|_{H^1}^2 \leqslant Ce^{-3\sigma_0^{3/2}}t.$$

With the bounds coming from modulation theory, a similar inequality is derived for $|y_j(t) - x_j - c_j t|$. In this way, we improved the estimate (1.6): therefore we can put in a place a bootstrap scheme, and via a continuity argument, we conclude that $T_n = T_0$, and that (1.4) holds. This concludes the proof in the L^2 subcritical case $p < 5$.

Observe that (1.4) is an extremely strong bootstrap hypothesis: in particular, exponential decay is preserved by integration. This makes the scheme of proof very robust, and allows to extend to (NLS) for example. For (gKdV) specifically, one develops the same bootstrap argument under the much weaker assumption that the left hand side in (1.4) is bounded by a small ε (uniformly in t). In particular, this shows that any (gKdV) multi-soliton converges exponentially fast to its profile.
Also observe that we could have chosen any initial data $U_n(S_n)$ in a small neighbourhood of $\sum_{j=1}^N Q[c_j, x_j](S_n)$ of size $o(e^{-\sigma_0^{3/2}S_n})$: this also shows some robustness of the method, and a stability property of multi-solitons in the L^2-subcritical setting.

In the L^2 critical case $p = 5$, the method of proof is the same, except that now the coercivity property (1.9) fails, and should be replaced with

$$\forall v \in H^1, \quad \langle Lv, v\rangle \geqslant \lambda_0 \|v\|_{H^1}^2 - \frac{1}{\lambda_0}\left(\langle v, \partial_x Q\rangle^2 + \left\langle v, Q^3\right\rangle^2\right). \tag{1.11}$$

The idea is then to modulate with respect to translations $y_j(t) \sim x_j + c_j t$ and scaling $\gamma_j(t) \sim c_j$, so as to cancel both bad directions which appear in (1.11). The point now is that the variation $\dot\gamma_j$ of the modulated scaling parameter $\gamma_j(t)$ is *quadratic* in η, because we are in the L^2 critical case. Therefore, one can close the blow up estimates as previously.

Let us consider the L^2 critical case $p > 5$. Again, (1.9) fails; even worse, the linearized flow around a soliton admits eigenvalues with non vanishing real part. More precisely, let U be a solution close to the soliton $Q[1,0](t)$, and let $\varepsilon = U(t) - Q[1,0](t)$ be the error term (unmodolated): then the equation on ε reads:

$$\partial_t \varepsilon + \partial_x(L\varepsilon) + O(\varepsilon^2) = 0.$$

In the L^2-supercritical case $\partial_x L$ admits two smooth, exponentially decaying eigenfunctions $Y^\pm$:

$$\partial_x L Y^\pm = \pm\varepsilon_0 Y^\pm, \quad \varepsilon_0 > 0.$$

In fact, we will need to consider the adjoint operator $L\partial_x$, and for which one similarly observes that

$$Z^\pm := LY^\pm \text{are eigenfunctions:} \quad L(\partial_x Z^\pm) = \pm\varepsilon_0 Z^\pm.$$

The first task is to obtain a new coercivity property to make up for (1.9). From Sturm-Liouville theory, one always has that

$$\forall v \in H^1, \quad \langle Lv, v\rangle \geqslant \lambda_0 \|v\|_{H^1}^2 - \frac{1}{\lambda_0}\left(\langle v, \partial_x Q\rangle^2 + \left\langle v, Q^{\frac{p+1}{2}}\right\rangle^2\right),$$

but the last scalar product is not well behaved under the flow: this is why we have to relate to $Z^\pm$.

Proposition 1.6. *Let $p > 5$. There exist $\lambda_0 > 0$ such that*

$$\forall v \in H^1, \quad \langle Lv, v\rangle \geqslant \lambda_0 \|v\|_{H^1}^2 - \frac{1}{\lambda_0}\left(\langle v, \partial_x Q\rangle^2 + \langle v, Z^+\rangle^2 + \langle v, Z^-\rangle^2\right). \tag{1.12}$$

Our goal is still to construct U_n backwards from S_n, and verifying (1.4). As in the L^2 subcritical case, we will work with the error η as in (1.5). However, here, we need to avoid the bad direction Z^+: so instead of imposing *a priori* $\eta(S_n) = 0$, we allow $\eta(S_n)$ to be chosen in a small neighbourhood of $\mathrm{Span}(Z_j^+(S_n), j)$ where $Z_j^+(t)$ is the eigenfunction of $L_j(t)\partial_x$ with positive eigenvalue.
More precisely, we define

$$a_j^\pm(t) := \left\langle \eta(t), Z_j^\pm(t)\right\rangle. \tag{1.13}$$

By inverse mapping, if S_n is large enough, then for any $\mathfrak{a} = (\mathfrak{a}_1, \dots, \mathfrak{a}_N) \in B(0,1)$ in $\mathbb{R}^N$, there exists a unique final data $U_n(S_n)$ such that

$$\eta(S_n) = 0, \quad \text{and} \quad \forall j = 1, \dots, N, \quad y_j(S_n) = x_j + c_j S_n, \quad \text{and} \tag{1.14}$$

$$\forall j = 1, \dots, N, \quad a_j^-(S_n) = 0 \quad \text{and} \quad a_j^+(S_n) = \mathfrak{a}_j e^{-3\sigma_0^{3/2} S_n/2}. \tag{1.15}$$

Our goal is to find, for each n, a suitable $\mathfrak{a}$.
The bootstrap assumption now has two scales, one for η and the y_j and one for the a_j, which is stronger: for $\mathfrak{a} \in B(0,1)$, we consider the minimal time $T_-(\mathfrak{a}) \geqslant T_0$ such that for all $t \in [T_-(\mathfrak{a}), S_n]$,

$$\|\eta(t)\|_{H^1}^2 \leqslant e^{-2\sigma_0^{3/2} t}, \quad \sum_{j=1}^N |y_j(t)|^2 \leqslant e^{-2\sigma_0^{3/2} t} \leqslant 1, \tag{1.16}$$

$$\sum_{j=1}^N |a_j^+(t)|^2 \leqslant e^{-3\sigma_0^{3/2} t}, \quad \sum_{j=1}^N |a_j^+(t)|^2 \leqslant e^{-3\sigma_0^{3/2} t} \tag{1.17}$$

Observe that the constants are important, and we a priori allow $T_(\mathfrak{a}) = S_n$. Nevertheless, our goal it to find $\mathfrak{a}$ such that $T_-(\mathfrak{a}) = T_0$.
We argue by contradiction and assume that for all $\mathfrak{a} \in B(0,1)$, $T_-(\mathfrak{a}) > T_0$. Then this means that at least one of the inequalities in (1.16) is actually an equality at $T_-(\mathfrak{a})$. Let $\mathfrak{a} \in B(0,1)$. With the control provided by the bootstrap hypothesis and the coercivity (1.12), the same argument as in the L^2 subcritical case shows that

$$\|\eta(t)\|_{H^1}^2 + \sum_{j=1}^N |y_j(t)|^2 \leqslant Ce^{-3\sigma_0^{3/2} t}.$$

As we choose to compute the scalar product essentially with $Z^{\pm}$, eigenfunction of the adjoint $L\partial_x$ of the linearized flow around Q, the linear term in the equation on a_j is explicit (and is a_j itself): more precisely, it writes:

$$\left|\dot{a}_j^{\pm}(t) \pm e_0 c_j^{3/2} a_j^{\pm}(t)\right| \leqslant C\|\eta(t)\|_{H^1}^2 + Ce^{-3\sigma_0^{3/2}t}. \tag{1.18}$$

Integrating this shows first

$$\sum_{j=1}^{N} |a_j^+(t)|^2 \leqslant Ce^{-4\sigma_0^{3/2}t}.$$

Therefore, we must have the equality

$$\sum_{j=1}^{N} |a_j^+(T_-(\mathfrak{a}))|^2 = e^{-3\sigma_0^{3/2}T_-(\mathfrak{a})}.$$

Now consider the map

$$\begin{array}{rccl} \Phi: & B(0,1) & \to & \mathbb{S}^{N-1} \\ & \mathfrak{a} & \mapsto & e^{3\sigma_0^{3/2}T_-(\mathfrak{a})}(a_1^+,\dots,a_N^+)(T_-(\mathfrak{a})). \end{array}$$

Using again (1.18), one sees that the flow is *transverse* at exit time $T_-(\mathfrak{a})$: from there one sees that Φ is continuous and that $\Phi|_{\mathbb{S}^{N-1}} = \mathrm{Id}$ (the exit is instantaneous on the sphere). But this contradicts Brouwer retractation theorem. Therefore, there exists $\mathfrak{a} \in B(0,1)$ such that $T_-(\mathfrak{a}) = T_0$, and this provides a final data $U_n(S_n)$ such that (1.4) holds. The proof in the L^2 supercritical case is complete.

2 Families of multi-solitons

Classification of (gKdV) multi-solitons

Once the question of the existence of multi-solitons is settled, the next problem is uniqueness. For simplicity we will now focus on pure power nonlinearity $f(u) = |u|^{p-1}u$ in all the following discussion.
This issue was answered by Martel [84] in the L^2 critical and subcritical cases:

Theorem 1.7 (Martel [84]). *Let $f(u) = |u|^{p-1}u$ where $1 < p \leqslant 5$, and an integer $N \geqslant 1$ and parameters $0 < c_1 < \cdots < c_N$, $x_1,\dots,x_N \in \mathbb{R}$ be given.*
Then the multi-soliton constructed in Theorem 1.1 is unique, i.e. there exists exactly one maximal solution $U \in \mathscr{C}((T_-(U),+\infty))$ to (gKdV) *such that*

$$\left\|U(t) - \sum_{j=1}^{N} Q[c_j,\gamma_j,v_j,x_j](t)\right\|_{H^1} \to 0 \quad as \quad t \to +\infty. \tag{1.19}$$

Due to Theorem 1.1, the convergence is in fact exponentially fast. This uniqueness result is surprising as first sight, but is in fact intimately related to the variationnal properties of Q. Denoting U the multi-soliton constructed in Theorem 1.1 (where convergence is exponential), and V an other multi-soliton in the sense that (1.19) holds, one considers the difference $z(t) = V(t) - U(t)$. As $z(t)$ is the difference of two solutions of (gKdV), it turns out that $z(t)$ satisfies an improved monotonicity property on an localized energy related quantity. After controlling the remaining two bad directions ($\langle z, Q[c_j,x_j]\rangle$ which has quadratic variation, and the second – in the subcritical case $\langle z, \partial_x Q[c_j,x_j]\rangle$ – via modulation), one obtains

$$\|z(t)\|_{H^1} \leqslant Ce^{-\sigma_1 t}\|z(t)\|_{H^1} + C\|z(t)\|_{H^1}^2,$$

for some $\sigma_1 > 0$, and from there, $z(t) = 0$.

In the L^2 supercritical case $p > 5$, multi-solitons are not unique anymore: Combet [33] obtained a complete classification of multi-solitons in the L^2 supercritical case. Using the unstable mode Y^+ on each soliton, it is possible to construct an N-parameter family of multi-solitons. Then by adapting the method to prove uniqueness in the L^2 subcritical, one can show that any multi-soliton must belong to the aformentioned family. These results are summarized below.

Theorem 1.8 (Combet [33]). *Let $p > 5$, and use the notations and results of Theorem 1.1: $U(t)$ denotes a multi-soliton of* (gKdV). *Then:*

1. *For all $\vec{a} = (a_1, \dots, a_N) \in \mathbb{R}^N$, there exists a solution $U_{\vec{a}} \in \mathscr{C}((T_-(U_{\vec{a}}), +\infty), H^1)$ to* (gKdV), *$\sigma > 0$ and C such that*

$$\forall K = 1, \dots, N, \ \forall t \geqslant T_-(U_{\vec{a}}) + 1, \quad \left\| U_{\vec{a}}(t) - U(t) - \sum_{j=K}^{N} a_j e^{-e_j t} Y^+[c_j, x_j](t) \right\|_{H^1} \leqslant C e^{-(e_K + \sigma)t}.$$

   In particular, if $\vec{a} \neq \vec{a}'$, $U_{\vec{a}} \neq U_{\vec{a}'}$.

2. *Conversely, if $u \in \mathscr{C}((T_-(u), +\infty), H^1)$ is a multi-soliton of* (gKdV) *i.e.*

$$\|u(t) - U(t)\|_{H^1} \to 0 \quad as \quad t \to +\infty,$$

   then there exists $\vec{a} \in \mathbb{R}^N$ such that $u = U_{\vec{a}}$.

One could certainly obtain an analoguous result for general nonlinearity f satisfying the conditions of Theorem 1.1: the family of multi-soliton is then parametrized by $\mathbb{R}^K$, where K is the number of indices of unstable solitons, i.e. the indices j such that

$$\frac{d}{dc} \|Q_c\|_{L^2}|_{c=c_j} > 0.$$

Non-uniqueness and instability of L^2 supercritical (NLS) multi-solitons

For the nonlinear Schrödinger equation, Combet [32] was able to construct a similar N-parameter family in the L^2 supercritical case $p > 1 + \frac{4}{d}$ (in 1 dimension).

Theorem 1.9 (Combet [32]). *Let $p \in \left(1 + \frac{d}{4}, \frac{d+2}{d-2}\right)$, and use the notations and results of Theorem 1.2: $U(t)$ denotes a multi-soliton of* (NLS).
Then for all $\vec{a} = (a_1, \dots, a_N) \in \mathbb{R}^N$, there exist a solution $U_{\vec{a}} \in \mathscr{C}((T_-(U_{\vec{a}}), +\infty), H^1)$ to (NLS), *$\sigma > 0$ and C such that*

$$\forall K = 1, \dots, N, \ \forall t \geqslant T_-(U_{\vec{a}}) + 1, \quad \left\| U_{\vec{a}}(t) - U(t) - \sum_{j=K}^{N} a_j e^{-e_j t} Y^+[c_j, x_j](t) \right\|_{H^1} \leqslant C e^{-(e_K + \sigma)t}.$$

In particular, if $\vec{a} \neq \vec{a}'$, $U_{\vec{a}} \neq U_{\vec{a}'}$.

The restriction to 1 space dimension is for technical reason only, most notably one needs some regularity on the nonlinearity, which also has to be $\dot{H}^1$ sub-critical in some sense: in high dimension, no pure power nonlinearity is allowed, only general nonlinearity f are suitable. Then one has to study the eigenfunctions of the linearized operator (notably, exponential decay), which was conducted in [8]. Under the assumptions in Theorem 1.10 below, one could extend Theorem 1.9 to higher dimension.

The question of the classification of (NLS) multi-solitons as in the case of (gKdV) is a widely open problem. Even in the L^2 subcritical case, whether or not multi-solitons are unique is a challenging open question. One crucial argument missing in the (NLS) context is the monotonicity properties which underly all the study of the solitons for (gKdV).
Actually one can see the uniqueness of multi-soliton as a toy problem for several important (and related) open questions: asymptotic stability (say locally in H^1 *without weights*) of (NLS) solitons, Liouville property of non dispersive solutions which remain close to a soliton (cf. Chapter 4 for further detail), and stability of multi-solitons.

One can argue in the same way for multi-solitons based on general bound states, under the assumption that the flow of (at least) one the excited states has an eigenvalue off the imaginary axis, i.e if Φ is a solution to (0.5), then

(A4) $L = -i\Delta + i\omega_1 - idf(\Phi)$ has an eigenvalue $\lambda \in \mathbb{C}$ with $\rho := \mathrm{Re}(\lambda) > 0$.

This assumption is very natural if one expects a travelling wave based on Φ to be unstable. Actually, (A4) holds for any real radial bound state in the L^2-supercritical case (see [59]). For excited states, (A4) is believed to hold for a wide class of non-linearities.

Theorem 1.10 (C., Le Coz [8]). *Let $d \geqslant 2$, and $f \in \mathscr{C}^\infty$ satisfies (A1) and the first part of (A2), and $|f'(s)| \leqslant C(1+|s|^p)$ for some $p \in \left(1, \dfrac{d+2}{d-2}\right)$. We use the notations and results of Theorem 1.4: $U(t)$ denotes a multi-soliton of* (NLS).
Assume that Φ_j satisfies (A4) for some $j \in [\![1,n]\!]$ (denote L_j and ρ_j the linearized flow and real part of the eigenvalue respectively). Then there exists a function $Y(t)$ such that $e^{\rho_j t}Y(t)$ is periodic and non trivial (in fact $\partial_t Y + L_j Y = 0$), such that the following holds.
For all $a \in \mathbb{R}$, there exists a solution $U_a \in \mathscr{C}((T_-(U_a,+\infty)),+\infty), H^1(\mathbb{R}^d))$ to (NLS) *such that*

$$\forall t \geqslant T_0, \quad \|U_a(t) - U(t) - aY[c_j,\gamma_j,v_j,x_j](t)\|_{H^1(\mathbb{R}^d)} \leqslant Ce^{-2\rho_j t}.$$

In particular, for $a \neq b$, $U_a \neq U_b$.

The main difficulty of this result is to construct a very good approximate solution to the multi-soliton. Actually we build such a profile at arbitrary exponential order: this method is inspired by [46–48]. One important issue, as mentioned above, is to have a good knowledge of the eigenfunction of the linearized flow. Also, the fact that the eigenvalue may be complex makes the construction of the profile more involved.

Up to making $v_\sharp(\Phi_1,\dots,\Phi_N) > 0$ smaller, one has the same result with a family of K parameters, where K is the number of bound state Φ_k such that (A4) holds true. However the previous result is enough to prove nonlinear instability of the multi-soliton.

Corollary 1.11 (C., Le Coz [8]). *Under the hypotheses of Theorem 1.10, the following instability property holds. $U(t)$ denotes the multi-soliton.*
There exist $\varepsilon > 0$, $I_n, J_n \to +\infty$, and a sequence of solutions $v_n \in \mathscr{C}([I_n,J_n],H^1(\mathbb{R}^d))$ to (NLS) *such that*

1. *for all $\sigma \geqslant 0$, $\|v_n(I_n) - U(I_n)\|_{H^\sigma(\mathbb{R}^d)} \to 0$ as $n \to +\infty$, and*
2. $\displaystyle\inf_{\substack{y_j\in\mathbb{R}^d,\vartheta_j\in\mathbb{R},\\ j=1,\dots,N}} \left\| w_n(J_n) - \sum_{i=1}^N \Phi_j(x-y_j)e^{i(\frac{1}{2}v_j\cdot x+\vartheta_j)}\right\|_{L^2(\mathbb{R}^d)} \geqslant \varepsilon.$

As mentioned above, this is a strong form of *forward* (orbital) instability (as $J_n \geqslant I_n$). Observe that one way to interpret the non uniqueness results (Theorems 1.9 and 1.10) is to say that there is backward instability (given directly by U_a for $a \neq 0$). Corollary 1.11 is different and requires some further analysis.

CHAPTER 2

Soliton resolution for wave type equations

IN THIS CHAPTER, we study the asymptotic behavior of solutions u *without* size restriction or well prepared assumptions, near the final time of existence $t \to T_+(u)$. We will speak of global solution when $T_+(u) = +\infty$ and blow up solution when $T_+(u) < +\infty$, letting aside the behavior at $T_(u)$.

One of the main conjectures in the field of dispersive equations, is that at least generically, any initial data eventually decouples into a sum of solitons (in the sense of compact solution modulo symmetries), and a (linear) scattering term. This is the so called soliton resolution conjecture.

Some forms of this conjecture where proved in the case of *integrable* equations, most notably (KdV) (cf. Schuur)

$$\begin{cases} \partial_t u + \partial_x(\partial_{xx} u + u^2) = 0, \\ u(t=0,x) = u_0(x), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{KdV}$$

and (mKdV)

$$\begin{cases} \partial_t u + \partial_x(\partial_{xx} u + u^3) = 0, \\ u(t=0,x) = u_0(x), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{mKdV}$$

(cf. Eckhaus, Schuur [49], and Miura [111] for a survey). For example for (KdV), for any generic solution u, there exists N solitons with scaling parameters $c_1, \dots, c_N$ and translations $x_1, \dots x_N$ such that

$$\|u(t) - \sum_{k=1}^{n} Q_{c_i}(x - c_1 t - x_1)\|_{L^\infty(x \geqslant -t^{1/3})} \to 0 \quad \text{quand} \quad t \to +\infty. \tag{2.1}$$

(Recall that (KdV) and (mKdV) are L^2 subcritical and so any H^1 solution is globally defined on $\mathbb{R}$). For (mKdV) a similar statement holds, which includes besides solitons another type of nonlinear object, the breathers:

$$B(t,x) := 2\sqrt{2}\partial_x \left(\arctan\left(\frac{\beta}{\alpha} \frac{\sin(\alpha(x+\gamma t))}{\cosh(\beta(x+\delta t))} \right) \right), \tag{2.2}$$
$$\text{where} \quad \alpha, \beta \in \mathbb{R}^*, \quad \gamma := 3\alpha^2 - \beta^2, \quad \text{and} \quad \delta := \alpha^2 - 3\beta^2.$$

(See Eckhaus, Schuur [49] for further details). Observe that breathers are spatially localized solutions moving to the left, which is definitely not a sum of solitons and linear scattering term.

These results are a consequence of the inverse scattering method, and so rely in an essential way on the integrability of the equations under consideration. It thus seems very unlikely to be able to extend it to more general situations, even pertubative. Also, the stated result itself misses some aspects of the annouced objective: the L^∞ norm is not strong enough to give an account for the linear term, indeed if v is a smooth solution of the linear Korteweg-de Vries equation $\partial_t v + \partial_{xxx} v = 0$, then

$$\|v(t)\|_{L^\infty} \leqslant t^{-1/3} \|v(0)\|_{L^1}.$$

Actually the dispersion relation $\omega - 3k^2 = 0$ shows that the linear waves tend to move to the left (with speed k^2), which explains in part the space localization in the right hand side in the convergence (2.1). This reinforces the idea that linear dispersion is hard to handle precisely.

On another side, the soliton resolution conjecture should be taken in a loose way, as such a rigid decomposition should not hold for all solutions (one can think of excited states for (NLS) in dimension $d \geqslant 2$). Thus the term "generic solution" should be made precise and can not be avoided.

In this chapter, we present soliton resolution results for wave type equations. The reasons for this choice will be made clearer in the proof, but let us underline here that one of the main motivation is *finite speed of propagation*, which replaces the monotonicity properties of (gKdV) type equations. The extension of these types of results to Schrödinger type equation is wide open, and certainly extremely difficult.

1 Profile decomposition

An essential tool to study the long time dynamics is the (linear) profile decomposition, introduced by Bahouri, Gérard [18] in the case of the 3 dimensional energy critical wave equation, and independently by Merle, Vega [104] in the case of the 2 dimensional L^2 critical Schrödinger equation. As we will use it in the wave context, we present it here for the $\dot{H}^1 \times L^2$ critical wave equation:

$$\begin{cases} \partial_{tt} u - \Delta u - |u|^{4/(d-2)} u = 0, \\ (u, \partial_t u)(t = 0, x) = (u_0(x), u_1(x)), \end{cases} \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \qquad \text{(cNLW)}$$

and it linear counterpart:

$$\begin{cases} \partial_{tt} u - \Delta u = 0, \\ (u, \partial_t u)(t = 0, x) = (u_0(x), u_1(x)), \end{cases} \qquad (t, x) \in \mathbb{R} \times \mathbb{R}^d. \qquad \text{(LW)}$$

Notation 2.1. Let $\vec{v} = (v_0, v_1) : I \times \mathbb{R}^d \to \mathbb{R}^2$ be a function of space and time, where $I \subset \mathbb{R}$, and $(t_0, x_0, \lambda_0) \in \mathbb{R} \times \mathbb{R}^d \times (0, +\infty)$ are modulation parameters. Then we denote $\vec{v}[t_0, x_0, \lambda_0]$ function of space and time defined by

$$\vec{v}[t_0, x_0, \lambda_0](t, x) := \left(\frac{1}{\lambda_0^{d/2-1}} v_0 \left(\frac{t - t_0}{\lambda_0}, \frac{x - x_0}{\lambda_0} \right), \frac{1}{\lambda_0^{d/2}} v_1 \left(\frac{t - t_0}{\lambda_0}, \frac{x - x_0}{\lambda_0} \right) \right).$$

We can extend this notation to functions depending on space only by considering them as constant in time.

We will also repetitively use the Strichartz space S, where for an interval of time I, we denote

$$S(I) = L^{\frac{2(d+2)}{d-2}}(I, L^{\frac{2(d+2)}{d-2}}(\mathbb{R}^d)).$$

Theorem 2.2 (Bahouri, Gérard [18]). *Let $\vec{u}_n = (u_{n,0}, u_{n,1})$ be a sequence bounded in $\dot{H}^1 \times L^2$. Up to extracting a subsequence, there exist a sequence of linear profiles $(\vec{U}^j_L)_j$ of $\dot{H}^1 \times L^2$ (solutions to* (LW)*), and of time, shifts and scaling parameters*

$$(t_{j,n}, x_{j,n}, \lambda_{j,n}) \in \mathbb{R} \times \mathbb{R}^d \times (0, +\infty),$$

and linear remainder term $(\vec{w}_n^J)_{J,n}$ *(i.e. solutions to* (LW)*), such that the following decomposition holds for all* $J, n \geqslant 1$

$$\vec{u}_n = \sum_{j=1}^{J} \vec{U}_L^j[t_{j,n}, x_{j,n}, \lambda_{j,n}] + \vec{w}_n^J \tag{2.3}$$

where the remainder terms are asymptotically vanishing in the Strichartz space S

$$\limsup_{n\to+\infty} \|w_n^J\|_{S(\mathbb{R})} \to 0 \quad as \quad J \to +\infty, \tag{2.4}$$

and the modulation parameters are almost orthogonal, in the sense that for any $j \neq k$,

$$\frac{\lambda_{j,n}}{\lambda_{k,n}} + \frac{\lambda_{k,n}}{\lambda_{j,n}} + \frac{|x_{j,n} - x_{k,n}|}{\lambda_{j,n}} + \frac{|t_{j,n} - t_{k,n}|}{\lambda_{j,n}} \to +\infty \quad as \quad n \to +\infty. \tag{2.5}$$

The profiles are obtained by weak limits. Denote $\vec{u}_{\mathrm{L},n}$ the linear solution to (LW) with initial data $\vec{u}_{\mathrm{L},n} = \vec{u}_n$. The $U_{\mathrm{L}}^j(0)$ are chosen among all $\dot{H}^1 \times L^2$ weak limits of $\vec{u}_{\mathrm{L},n}$ up to modulation, in decreasing order of $\dot{H}^1 \times L^2$ norm: for all $j \geqslant 1$,

$$\vec{u}_{\mathrm{L},n}\left[-t_{j,n}, -x_{j,n}, \frac{1}{\lambda_{j,n}}\right](0) \rightharpoonup \vec{U}_{\mathrm{L}}^j(0) \quad \text{weakly in } \dot{H}^1 \times L^2.$$

So that for all $J \geqslant 1$ and $k \leqslant J$

$$\vec{w}_n^J\left[-t_{j,n}, -x_{j,n}, \frac{1}{\lambda_{j,n}}\right](0) \rightharpoonup 0 \quad \text{weakly in } \dot{H}^1 \times L^2,$$

and the almost orthogonality condition is automatic. The heart of Theorem 2.2 is to estimate the remainder term w_n^J in the Strichartz space (2.4): this is a consequence of a precised Sobolev embedding, which describes the default of compactess in the Sobolev embedding $\dot{H}^1 \to L^{\frac{2d}{d-2}}$ in dimension d (we refer also to Brézis, Coron [24], and Gérard [53]).

Proposition 2.3 (Pythagorean expansion)**.** *For all* $J \geqslant 1$,

$$\|\vec{u}_n\|^2_{\dot{H}^1\times L^2} = \sum_{j=1}^{J} \|U_L^j(0)\|^2_{\dot{H}^1\times L^2} + \|\vec{w}_n^J(0)\|^2_{\dot{H}^1\times L^2} + o_n(1).$$

Definition 2.4 (Non linear profile)**.** Let $\vec{U}_{\mathrm{L}}$ be a linear solution to (LW), and $\ell \in [-\infty, +\infty]$. The non linear profile associated to $(\vec{U}_{\mathrm{L}}, \ell)$ is the unique nonlinear solution $\vec{U}(t)$ to (cNLW), defined in a neighborhood of ℓ, and such that

$$\|\vec{U}(t) - \vec{U}_{\mathrm{L}}(t)\|_{\dot{H}^1\times L^2} \to 0 \quad \text{as} \quad t \to \ell.$$

The interest of a profile decomposition for a sequence of initial data, related to the linear equation (LW), comes from the fact one can evolve this decomposition by the *nonlinear* flow. Here is the precise statement:

Proposition 2.5 (Non linear evolution)**.** *Let* $\vec{u}_n$ *be a sequence of nonlinear solutions to* (cNLW)*. Assume that the sequence of initial data* $\vec{u}_n(0)$ *is bounded in* $\dot{H}^1 \times L^2$ *and admits a linear profile decomposition* $(\vec{U}_{\mathrm{L}}^j, (t_{j,n}, x_{j,n}, \lambda_{j,n})_n)_j$ *with remainder* w_n^J*, in the sense of Theorem 2.2. Let* $\vec{U}^j$ *be the nonlinear profiles associated to* $\left(\vec{U}_{\mathrm{L}}^j, \lim_n -\frac{t_{j,n}}{\lambda_{j,n}}\right)$.

Finally, let $s_n \geqslant 0$ *be a sequence of times such that for all* $j \geqslant 1$,

$$\forall n, \quad \frac{s_n - t_{j,n}}{\lambda_{j,n}} < T_+(\vec{U}^j) \quad and \quad \limsup_{n\to\infty} \|U^j\|_{S\left(\frac{-t_{j,n}}{\lambda_{j,n}}, \frac{s_n - t_{j,n}}{\lambda_{j,n}}\right)} < \infty.$$

Then $s_n \leqslant T^+(\vec{u}_n(t))$ *and*

$$\limsup_{n\to\infty} \|u_n\|_{S([0,s_n))} < \infty.$$

Furthermore, the following decomposition into nonlinear profiles holds: for all $J \geqslant 1$,

$$\forall t \in [0,s_n], \quad \vec{u}_n(t) = \sum_{j=1}^{J} \vec{U}^j[t_{j,n}, x_{j,n}, \lambda_{j,n}](t) + \vec{w}_n^J(t) + \vec{r}_n^J(t), \tag{2.6}$$

where the error $\vec{r}_n^J$ *satisfies*

$$\limsup_{n\to\infty} \left(\|r_n^J\|_{S([0,s_n))} + \|\vec{r}_n^J\|_{L_t^\infty([0,s_n];\dot{H}^1\times L^2)} \right) \to 0 \quad as \quad J \to +\infty.$$

Of course, we have a similar result for negative times s_n.

2 Linear energy outside the light cone

Before we state the soliton decomposition results, let us focus on an important feature of the linear flow. It plays a fundamental role in the proof of the main results of this chapter (essentially replacing monotonicity formulas), but has its own interest.
The question here is to understand where does the energy of a linear solution to the wave equation lie for large times. A first answer is that the energy concentrates around the light cone.

Theorem 2.6 ([7, 44])**.** *Let* $\vec{u} \in (\dot{H}^1 \times L^2)(\mathbb{R}^d)$ *be a radial linear solution to* (LW). *Then we have the following vanishing of the energy away from the forward light-cone* $\{|x| = t \geqslant 0\}$:

$$\limsup_{t\to+\infty} \|\nabla_{t,x} u(t,x)\|_{L^2(||x|-t|\geqslant T)} \to 0 \quad as \quad T \to +\infty. \tag{2.7}$$

Now, what will be of major importance is the refinement of this, namely does some energy remain *outside* the light cone, or does all the energy eventually go inside?
The answer lies in the following theorem. Surprisingly enough, the asymptotic behavior depends heavily on the parity of the dimension. To state it, we introduce the Hankel transform H and the Hilbert transform $\mathscr{H}$ on the half-line $(0,\infty)$:

$$(H\varphi)(\rho) := \int_0^\infty \frac{\varphi(\sigma)}{\rho+\sigma}\, d\sigma, \quad \text{and} \quad (\mathscr{H}\varphi)(\rho) := \int_0^\infty \frac{\varphi(\sigma)}{\rho-\sigma}\, d\sigma \tag{2.8}$$

where the second integral is to be taken in the principal value sense. Both these operators are bounded and self-adjoint (anti-selfadjoint, respectively) on $L^2((0,\infty), d\rho)$, with norm π. Furthermore, H is a positive operator since it is of the form $H = \mathscr{L}^2$ where $\mathscr{L}$ is the Laplace transform:

$$\mathscr{L}\phi(\rho) = \int_0^\infty \phi(\sigma) e^{-\rho\sigma} d\sigma.$$

(See for example Lax [80, Section 16.3.3] for details). In even dimensions, we find the following expression for the asymptotic exterior energy in terms of H and $\mathscr{H}$. In the following of this section, we slightly change our notation

$$\langle f, g \rangle := \int_0^\infty f(x)\overline{g(x)}\, dx$$

for two functions f, g on the half-line $(0,\infty)$.

Theorem 2.7 ([7])**.** *Let* $d \geqslant 2$ *be an integer, there exists an explicit constant* $C(d)$ *such that the following holds. Let* $(u_0, u_1) \in (\dot{H}^1 \times L^2)(\mathbb{R}^d)$ *be radial. Denote* $\widehat{u_0}$ *and* $\widehat{u_1}$ *their Fourier transform in* $\mathbb{R}^d$ *and* $\vec{u}(t)$ *the linear solution to* (LW) *with initial data* $\vec{u}(0) = (u_0, u_1)$.

1. *If d is odd, then*

$$\lim_{t\to\pm\infty} C(d)\|\nabla_{t,x} u(t,x)\|^2_{L^2(|x|\geqslant|t|)} = \frac{\pi}{2}\int (\rho^2|\widehat{u_0}(\rho)|^2 + |\widehat{u_1}(\rho)|^2)\rho^{d-1}\, d\rho$$
$$\pm \left((-1)^{\frac{d-1}{2}} \operatorname{Re}\langle H(\rho^{\frac{d+1}{2}}\widehat{u_0}), \rho^{\frac{d-1}{2}}\widehat{u_1}\rangle + \operatorname{Re}\langle \rho^{\frac{d+1}{2}}\widehat{u_0}, \mathscr{H}(\rho^{\frac{d-1}{2}}\widehat{u_1})\rangle \right). \tag{2.9}$$

2. *If d is even, then*

$$\lim_{t\to\pm\infty} C(d)\|\nabla_{t,x}u(t,x)\|^2_{L^2(|x|\geqslant|t|)} = \frac{\pi}{2}\int (\rho^2|\widehat{u_0}(\rho)|^2 + |\widehat{u_1}(\rho)|^2)\rho^{d-1}\,d\rho$$
$$+ \frac{(-1)^{\frac{d}{2}}}{2}\left(\langle H(\rho^{\frac{d+1}{2}}\widehat{u_0}), \rho^{\frac{d+1}{2}}\widehat{u_0}\rangle - \langle H(\rho^{\frac{d-1}{2}}\widehat{u_1}), \rho^{\frac{d-1}{2}}\widehat{u_1}\rangle\right) \pm \mathrm{Re}\langle \rho^{\frac{d+1}{2}}\widehat{u_0}, \mathscr{H}(\rho^{\frac{d-1}{2}}\widehat{u_1})\rangle. \tag{2.10}$$

We now use the positivity of the Hankel transform H to derive the corollaries that we will indeed use in the proof of the soliton decomposition. The first one is in the odd case, and allows to recover the exterior energy estimates obtained by Duyckaerts, Kenig, Merle [44] (see also [45, Proposition 2.7] for the non radial setting).

Corollary 2.8. *Let d be an odd integer. Let $\vec{u}(t) \in \mathscr{C}(\mathbb{R}, \dot{H}^1 \times L^2)$ be a radial linear solution to* (LW) *. Then one has either one of the following estimates:*

$$\begin{aligned} &\forall t \geqslant 0, \quad \|\nabla_{t,x}u(t,x)\|^2_{L^2(|x|\geqslant|t|)} \geqslant \frac{1}{2}\|\nabla_{t,x}u(0,x)\|^2_{L^2}, \\ \text{or} \quad &\forall t \leqslant 0, \quad \|\nabla_{t,x}u(t,x)\|^2_{L^2(|x|\geqslant|t|)} \geqslant \frac{1}{2}\|\nabla_{t,x}u(0,x)\|^2_{L^2}. \end{aligned} \tag{2.11}$$

In even dimension, results of the flavor of (2.11) can only hold if $\vec{u}(0)$ has a specific form, depending on $d \mod 4$.

Corollary 2.9. *Let $d \geqslant 2$ be even. Let $\vec{u}(t) \in \mathscr{C}(\mathbb{R}, \dot{H}^1 \times L^2)$ be a radial linear solution to* (LW)*.*

1. *If $d \equiv 0 \mod 4$, and $\partial_t u(0) = 0$, then*

$$\forall t \geqslant 0, \quad \|\nabla_{t,x}u(t,x)\|^2_{L^2(|x|\geqslant|t|)} \geqslant \frac{1}{2}\|\nabla_x u(0,x)\|^2_{L^2}. \tag{2.12}$$

2. *If $d \equiv 2 \mod 4$, and $u(0) = 0$, then*

$$\forall t \geqslant 0, \quad \|\nabla_{t,x}u(t,x)\|^2_{L^2(|x|\geqslant|t|)} \geqslant \frac{1}{2}\|\partial_t u(0,x)\|^2_{L^2}. \tag{2.13}$$

The estimate actually *fails* when not in the right setting.

Corollary 2.10. 1. *If $d \equiv 0 \mod 4$, there exists a sequence $\vec{u}_n \in \mathscr{C}(\mathbb{R}, \dot{H}^1 \times L^2)$ of (radial) linear solutions to* (LW) *such that $u_n(0) = 0$ and*

$$\lim_{t\to\pm\infty} \|\nabla_{t,x}u_n(t,x)\|^2_{L^2(|x|\geqslant|t|)} = o(\|\partial_t u_n(0)\|^2_{L^2}).$$

2. *If $d \equiv 2 \mod 4$, there exists a sequence $\vec{v}_n \in \mathscr{C}(\mathbb{R}, \dot{H}^1 \times L^2)$ of (radial) linear solutions to* (LW) *such that $\partial_t v_n(0) = 0$ and*

$$\lim_{t\to\pm\infty} \|\nabla_{t,x}v_n(t,x)\|^2_{L^2(|x|\geqslant|t|)} = o(\|\nabla_x v_n(0)\|^2_{L^2}).$$

Note that the exterior energy is decreasing in $|t|$, so that the limit exists, and we are reduced to compute them. In odd dimension, one can represent $\vec{u}$ explicit via the fundamental solution of the linear wave equation, and make use of the *strong* Huyghens principle to obtain (2.11). In even dimension, this tool is no longer available; in any case, it seems unlikely that one can compute the limits in (2.9)- (2.10) via a representation in the "physical space".
This is why we employ the Fourier transform in this computation: the starting point is a representation formula of $u(t)$ through Bessel functions. More precisely let $\hat{f}$ be the Fourier transform in $\mathbb{R}^d$:

$$\hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi} f(x)\,dx, \qquad f(x) = (2\pi)^{-d}\int_{\mathbb{R}^d} e^{ix\cdot\xi}\hat{f}(\xi)\,d\xi.$$

For radial functions, $\hat{f}$ is again radial. Recall that

$$\widehat{\sigma_{S^{d-1}}}(\xi) = (2\pi)^{\frac{d}{2}}|\xi|^{-\nu}J_\nu(|\xi|), \quad \nu := \frac{d-2}{2} \geqslant 0,$$

where J_ν is the Bessel function of the first type of order ν. It is characterized as being the solution of

$$x^2 J_\nu''(x) + xJ_\nu'(x) + (x^2 - \nu^2)J_\nu(x) = 0 \tag{2.14}$$

which is regular at $x = 0$ (unique up to a multiplicative constant). The inversion formula takes the form

$$f(r) = (2\pi)^{-\frac{d}{2}} \int_0^\infty \hat{f}(\rho)J_\nu(r\rho)(r\rho)^{-\nu}\rho^{d-1}\,d\rho.$$

The Plancherel identity takes the form $\|\hat{f}\|_2^2 = (2\pi)^d\|f\|_2^2$. The solution $u(t)$ to the linear equation (LW) is given by

$$u(t) = \cos(t|\nabla|)u(0) + \frac{\sin(t|\nabla|)}{|\nabla|}(\partial_t u)(0).$$

This means that

$$u(t,r) = (2\pi)^{-\frac{d}{2}} \int_0^\infty \left(\cos(t\rho)\hat{f}(\rho) + \frac{\sin(t\rho)}{\rho}\hat{g}(\rho)\right) J_\nu(r\rho)(r\rho)^{-\nu}\rho^{d-1}\,d\rho, \tag{2.15}$$

$$\partial_t u(t,r) = (2\pi)^{-\frac{d}{2}} \int_0^\infty \Big(-\sin(t\rho)\rho\hat{f}(\rho) + \cos(t\rho)\hat{g}(\rho)\Big) J_\nu(r\rho)(r\rho)^{-\nu}\rho^{d-1}\,d\rho. \tag{2.16}$$

We shall invoke the standard asymptotics for the Bessel functions, see [14],

$$\begin{aligned} J_\nu(x) &= \sqrt{\frac{2}{\pi x}}\left((1+\omega_2(x))\cos(x-\tau) + \omega_1(x)\sin(x-\tau)\right), \\ J_\nu'(x) &= \sqrt{\frac{2}{\pi x}}\left(\tilde{\omega}_1(x)\cos(x-\tau) - (1+\tilde{\omega}_2(x))\sin(x-\tau)\right), \end{aligned} \tag{2.17}$$

with phase-shift $\tau = (d-1)\frac{\pi}{4}$, and with the bounds (for $n \geqslant 0$, $x \geqslant 1$)

$$|\omega_1^{(n)}(x)| + |\tilde{\omega}_1^{(n)}(x)| \leqslant C_n\, x^{-1-n}, \quad |\omega_2^{(n)}(x)| + |\tilde{\omega}_2^{(n)}(x)| \leqslant C_n\, x^{-2-n}. \tag{2.18}$$

Using the representations (2.15) and (2.16), we have explicit formulas for $\|\nabla_{x,t}u(t)\|_{L^2(|x|\geqslant|t|)}$ or the delayed quantity. Then we expand these formulas using the asymptotics of the Bessel functions; for example, for even d, the leading term in the kinetic energy $\|\partial_t u(t)\|^2_{L^2(r\geqslant|t|)}$ is

$$\int_t^\infty\int_0^\infty\int_0^\infty (-\sin(t\rho_1)\rho_1\widehat{u_0}(\rho_1) + \cos(t\rho_1)\widehat{u_1}(\rho_1)) \cdot \Big(-\sin(t\rho_2)\rho_2\overline{\widehat{u_0}(\rho_2)} + \cos(t\rho_2)\overline{\widehat{u_1}(\rho_2)}\Big)$$
$$\cdot \cos(r\rho_1-\tau)\cos(r\rho_2-\tau)(\rho_1\rho_2)^{\nu+\frac{1}{2}}\,d\rho_1 d\rho_2\,dr. \tag{2.19}$$

In order to carry out the r integration, we use trigonometric formulas for expression of the type $\cos(r\rho_1-\tau)\cos(r\rho_2-\tau)$ and the relation

$$\int_t^\infty \cos(ar)dr = \pi\delta_0(a) - \frac{\sin(ta)}{a},$$

and we are with an integral in ρ_1 and ρ_2 made of terms of the type

$$\iint \frac{\sin(2t(\rho_1-\rho_2))}{\rho_1-\rho_2}\widehat{\partial_r u_0}(\rho_1)\overline{\widehat{u_1}(\rho_2)}d\rho_1 d\rho_2,$$

or

$$\iint \frac{\sin(2t(\rho_1+\rho_2))}{\rho_1+\rho_2}\widehat{\partial_r u_0}(\rho_1)\overline{\widehat{u_1}(\rho_2)}d\rho_1 d\rho_2.$$

It remains to take the limit of these terms as $t \to \pm\infty$ and we see the Hankel and Hilbert transform appear. By a delicate inspection of all terms involved (and also of the remainder terms that we neglected in the first place), we derive the desired expansion (2.10) as $t \to +\infty$.

3 The nonlinear wave equation in dimension 3

We will only consider radial data; in that case, there is no translation modulation and we denote r the (radial) space variable, and $[\lambda_0, t_0]$ the modulation action. For the wave equations, the profiles appearing in the decomposition are of two kinds: a regular part, essentially a linear scattering term, and suitably rescaled stationnary solutions $W \in \dot{H}^1$ satisfying

$$\Delta W = W^{\frac{4}{d-2}}, \quad W > 0.$$

W is the unique radial solution, up to scaling and translation. In fact, it is explicit:

$$W(r) = \frac{1}{\left(1 + \frac{r^2}{d(d-2)}\right)^{\frac{d}{2}-1}}. \tag{2.20}$$

Observe that $W \notin L^2$ due to lack of decay at spatial infinity.
In this subsection, we consider here *radial* solutions to (cNLW) in *3 dimensions*:

$$\partial_{tt} u - \Delta u + u^5 = 0.$$

For this specific equation, it is remarquable that a complete description is available.

Theorem 2.11 (Duyckaerts, Kenig, Merle [40]). *Let $\vec{u}$ be a* radial *solution to* (cNLW) *in dimension $d = 3$. Then one of the following alternatives hold:*

1. Type I blow up. $T_+(\vec{u}) < +\infty$ *and* $\lim_{t \to T_+(\vec{u})} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} = +\infty$.
2. Type II blow up. $T_+(\vec{u}) < +\infty$ *and there exist* $\vec{v} \in \dot{H}^1 \times L^2$*, an integer* $J \geqslant 1$*, and for all* $j \in [\![1, J]\!]$*, a signum* $\imath_j \in \{\pm 1\}$ *and continuous scaling functions* $\lambda_j : [0, T_+(\vec{u})) \to (0, +\infty)$ *such that, as* $t \to T_+(\vec{u})$*,*
$$\lambda_1(t) \ll \lambda_2(t) \ll \cdots \ll \lambda_J(t) \ll T_+(\vec{u}) - t, \tag{2.21}$$
$$\vec{u}(t) = \sum_{j=1}^{J} W[\lambda_j(t)] + \vec{v} + o_{\dot{H}^1 \times L^2}(1). \tag{2.22}$$
3. Type II global solutions. $T_+(\vec{u}) = +\infty$ *and there exist a linear solution* $\vec{v}_{\mathrm{L}}$ *to* (LW) *bounded in* $\dot{H}^1 \times L^2$*, an integer* $J \geqslant 1$*, and for all* $j \in [\![1, J]\!]$*, a signum* $i_j \in \{\pm 1\}$ *and continuous scaling functions* $\lambda_j : [0, T_+(\vec{u})) \to (0, +\infty)$ *such that, as* $t \to +\infty$*,*
$$\lambda_1(t) \ll \lambda_2(t) \ll \cdots \ll \lambda_J(t) \ll t, \tag{2.23}$$
$$\vec{u}(t) = \sum_{j=1}^{J} i_j W[\lambda_j(t)] + \vec{v}_{\mathrm{L}}(t) + o_{\dot{H}^1 \times L^2}(1). \tag{2.24}$$

Let us remark that in the case of type I blow up, the existence of the limit $\lim_{t \to T_+(\vec{u})} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2}$ (and not only of the lim sup) is non trivial; also, in that case a decomposition of the form (2.22) can not hold, as it implies a type II bound. In the global case, $\vec{u}$ necessarily bounded in $\dot{H}^1 \times L^2$, which is also a highly non trivial result.

The proof of such a result start with considering a profile decomposition for $\vec{u}(t_n)$ where $t_n \to T_+(\vec{u})$ approaches the maximal time of definition of $\vec{u}$, and to study the non linear profiles associated to them, making use in particular of Proposition 2.5. It turns out that the profiles appearing a solution enjoying specific compactness properties, and this motivates the following definition.

Definition 2.12 (Compact solutions). A nonlinear solution $\vec{u}$ to (cNLW) is said to be *compact up to modulation* if there exist functions $(x, \lambda) : (T_-(\vec{u}), T_+(\vec{u})) \to \mathbb{R}^d \times (0, +\infty)$ such that the envelop

$$K[x, \lambda](\vec{u}) = \{\vec{u}[0, x(t), \lambda(t)](t) \mid t \in (T_-(\vec{u}), T_+(\vec{u}))\}$$

is relatively compact in $\dot{H}^1 \times L^2$.

One of the main steps is therefore to classify compact solution. In the radial setting, only W (and its orbit under the invariance of the equation) is compact. Let us emphasize that compactness must hold at both ends of the interval of definition $(T_-(\vec{u}), T_+(\vec{u}))$; otherwise others solutions are allowed (for example the solution W_+, W_- constructed and studied in [47]).

Theorem 2.13 (Duyckaerts, Kenig, Merle [44]). *If $\vec{u}$ is a radial solution to* (cNLW) *in dimension $d = 3$ which is compact up to modulation, then either $\vec{u} = 0$, or there exists $\imath_0 \in \{\pm 1\}$ and $\lambda > 0$ such that*

$$\vec{u}(t) = \imath_0 W[\lambda_0].$$

Let us briefly we sketch in this paragraph the proof of Theorem 2.11, in the case of a global solution $\vec{u}$.

First, via a Virial type argument going back to Levine, it follows that the energy of $\vec{u}$ is nonnegative and

$$\liminf_{t\to+\infty} \|\nabla_{t,x} u(t)\|_{L^2}^2 \leqslant 3E(\vec{u}).$$

In particular, there exists a sequence $t_n \to +\infty$ such that $\vec{u}(t_n)$ is bounded in $\dot{H}^1 \times L^2$.

Second, we extract the dispersion linear term, which lives essentially on the light cone and this allows to describe the behaviour in this region and beyond:

Proposition 2.14. *There exists a radial solution $\vec{v}_{\mathrm{L}}$ of* (LW) *such that*

$$\forall R \in \mathbb{R}, \quad \lim_{t\to+\infty} \int_{t-R}^{+\infty} |\nabla_{t,r}(u - v_{\mathrm{L}})(t,r)|^2 r^2 dr = 0. \tag{2.25}$$

Observe that here the convergence hold as $t \to +\infty$, and not only on the subsequence t_n. This fact relies strongly on profile decomposition, finite speed of propagation, and the fact the the linear energy concentrates around the light cone (i.e moves at maximal speed, see Theorem 2.6), and it is a very robust property.

The key ingredient in the proof is the following nonlinear version of the channel of energy method, which we discussed in a linear setting in the previous paragraph.

Proposition 2.15. *Let u be a non-zero, radial solution of* (cNLW) *such that for all $\lambda > 0$ and all signs $\pm$, $\vec{u} \pm (W,0)[\lambda]$ is not compactly supported. Then there exist constants $R > 0$, $\eta > 0$ and a global, radial solution $\vec{w}$ of* (cNLW)*, scattering in both time directions such that*

$$\forall r \geqslant R, \quad \vec{w}(0,r) = \vec{u}(0,r),$$

and the following holds for all $t \geqslant 0$ or for all $t \leqslant 0$:

$$\int_{r\geqslant R+|t|} |\nabla_{t,r} w(t,r)|^2 r^2 dr \geqslant \eta. \tag{2.26}$$

From there, it is possible to extract all profiles along a subsequence of times, and to show that all profiles are stationnary, and that the error vanished in the energy space. It is the content of the follwing proposition.

Proposition 2.16. *Let $t_n \to +\infty$ be such that $\vec{u}(t_n)$ is bounded in $\dot{H}^1 \times L^2$, and $\vec{v}_{\mathrm{L}}$ be the linear solution given by Proposition 2.15. Then, after extraction of a subsequence in n, there exist an integer $J \geqslant 0$, J signum $i_1, \dots i_J \in \{\pm 1\}$, and J sequences $(\lambda_{j,n})_n$ with $0 < \lambda_{1,n} \ll \cdots \ll \lambda_{J,n} \ll t_n$ such that*

$$\vec{u}(t_n) = \vec{v}_{\mathrm{L}}(t_n) + \sum_{j=1}^{J} i_j (W,0)[\lambda_{j,n}, 0] + o_{\dot{H}^1\times L^2}(1).$$

We will now conclude of proof: the point is to show that for any sequence of times, the above decomposition above holds for at least one subsequence. First we choose the scaling parameter $\lambda_j(t)$: define

$$B_j = (j-1)\|\nabla W\|_{L^2}^2 + \int_0^1 |\nabla W|^2.$$

Then let $\lambda_j(t)$ be such that $\int_{\lambda_j(t)}^{+\infty} |\nabla_{t,r}(u - v_{\mathrm{L}})(t,r)|^2 r^2 dr = B_j$ if $\|\nabla_{t,r}(u - v_{\mathrm{L}})(t)\|_{L^2} \geqslant B_j$, and $\lambda_j(t) = 0$ otherwise. For any $I = (i_1, \dots, i_J) \in \{\pm 1\}^J$, consider the set

$$\mathscr{A}_{I,\delta} = \left\{ \vec{w} \in \dot{H}^1 \times L^2 \middle| \exists \lambda_1, \dots \lambda_J, \left\| \vec{w} - \vec{v}_{\mathrm{L}}(t) - \sum_{j=1}^{J} i_j (W, 0)[\lambda_j, 0] \right\|_{\dot{H}^1 \times L^2} \leqslant \delta \right\}.$$

One checks that there exists $\delta > 0$ such that for any $I \neq I'$, there holds $d_{\dot{H}^1 \times L^2}(\mathscr{A}_{I,\delta}, \mathscr{A}_{I',\delta}) \geqslant \delta$. From Proposition 2.16, there exists $I \in \{\pm 1\}^J$ and $\tau_n \to +\infty$ such that for all $n \in \mathbb{N}$, $\vec{u}(\tau_n) \in \mathscr{A}_{I,\delta}$. Hence, letting

$$\mathscr{T} = \{ t \geqslant 0 | \vec{u}(t) \in \mathscr{A}_{I,\delta} \},$$

a continuity argument shows that $\mathscr{T}$ contains an interval of the form $[t_0, +\infty)$, and from there, the convergence of the decomposition (2.23).

4 Equivariant wave maps to the sphere $\mathbb{S}^2$

The wave map equation is a model for geometric wave equations. Let (M, g) be a Riemannian manifold of dimension n, and $\mathbb{R}^{1+d}$ be endowed with the Minkowski metric $\eta = \mathrm{diag}(-1, 1, \dots, 1)$. Wave maps are defined as critical points of the Lagrangian

$$\mathscr{L}(U, \partial U) = \frac{1}{2} \int_{R^{1+d}} \eta^{\alpha\beta} \langle \partial_\alpha U, \partial_\beta U \rangle_g dx dt.$$

In local coordinates, they satisfy the Euler-Lagrange system

$$\forall k = 1, \dots, n, \quad \Box U^k = -\eta^{\alpha\beta} \Gamma_{ij}^k (U) \partial_\alpha U^i \partial_\beta U^j, \tag{2.27}$$

where Γ_{ij}^k are the Christoffel symbols on TM. By construction, wave maps preserve (at least formally) the energy

$$E(U) = \int \langle \partial_\alpha U, \partial_\beta U \rangle_g dx.$$

They also enjoy the following scaling and translation invariance (almost as (NLW)):

$$\vec{U}[\lambda_0, t_0, x_0](t, x) = \left(U \left(\frac{t - t_0}{\lambda_0}, \frac{x - x_0}{\lambda_0} \right), \frac{1}{\lambda_0} \partial_t U \left(\frac{t - t_0}{\lambda_0}, \frac{x - x_0}{\lambda_0} \right) \right).$$

We will mostly consider here wave maps with equivariant symmetry in dimension $d = 2$. More precisely, we assume that M is a 2 dimensional surface of revolution with metric

$$ds^2 = d\rho^2 + g(\rho)^2 d\theta^2,$$

where (ρ, θ) are the polar coordinates on M, and $g \in \mathscr{C}^3(\mathbb{R})$ (we keep the notation g as all the information on the riemannian metric is encoded in the function g). We say that $U : \mathbb{R}^{1+2} \to M$ has *equivariant symmetry* if denoting (r, ω) the polar coordinates on $\mathbb{R}^2$, it takes the form

$$U(t, r, \omega) = (\psi(t, r), \omega).$$

for some function ψ. System (2.27) then simplifies to the following equation on ψ :

$$\begin{cases} \partial_{tt}\psi - \partial_{rr}\psi - \frac{1}{r}\partial_r \psi + \frac{f(\psi)}{r^2} = 0 \\ (\psi, \partial_t \psi)|_{t=0} = (\psi_0, \psi_1) \end{cases} \quad \text{where } f = gg'. \tag{WM}$$

The energy then takes the form $E(\vec{\psi}) = E(\vec{\psi}; 0, +\infty)$ where we introduce for convenience the notation

$$E(\vec{\phi}; A, B) := \int_A^B \left(|\partial_t \phi(t, r)|^2 + |\partial_r \phi(t, r)|^2 + \frac{|g(\phi(t, r))|^2}{r^2} \right) r dr.$$

General wave map attracted a lot of attention since the early 90s. Of course, proofs in the general case are much more technical than when symmetry is assumed: observe that there

is no derivative in (WM) in contrast to (2.27). An important problem was to prove global well posedness for small data in the critical Sobolev space $\dot{H}^{d/2-1} \times \dot{H}^{d/2}$. In the case with symmetry, the result is due to Shatah and Tahvildar-Zadeh [120]: they rely on Strichartz estimates boosted by radial Sobolev embedding. The first breakthrough for general data is due to Tao [125, 126], when the target is the sphere $N = \mathbb{S}^{d-1}$, and due to T; it was then generalized by Tataru [128] to general targets.
Also it has been long understood that the geometry of the target M, plays a crucial role in the long time behavior of wave maps. One way to summarize this idea is that a wave map that blows up in finite time must bubble up a harmonic map at blow up time. This result was first proved by Struwe [124] for wave map with symmetry, and later extended to non symmetric data by Sterbenz and Tataru [122, 123] (see also the works by Tao [127] and Krieger and Schlag [72] when the target is the hyperbolic space).

We will now concentrate on equivariant wave maps (WM), and to fix ideas, we focus on the case of the sphere $M = \mathbb{S}^2$, that is

$$g = \sin, \quad f(\psi) = \frac{\sin(2\psi)}{2}.$$

In a series of papers in collaboration with Merle, Kenig, Lawrie and Schlag [1, 3, 4, 6], we ended in proving a soliton resolution along a sequence of times.
As in the (cNLW) context, the profiles appearing in the decomposition are of two kinds. The first one is linear scattering term, solution to

$$\begin{cases} \partial_{tt}\psi - \partial_{rr}\psi - \frac{1}{r}\partial_r\psi - \frac{\psi}{r^2} = 0, \\ (\psi, \partial_t\psi)(t=0, r) = (\psi_0(r), \psi_1(r)), \end{cases} \qquad (t,r) \in \mathbb{R} \times [0, +\infty). \tag{2.28}$$

The energy space for (2.28) is

$$H \times L^2 = \left\{ (\psi_0, \psi_1) \middle| \int_0^\infty \left(|\psi_1(r)^2 + (\partial_r\psi_0(r))^2 + \frac{\psi_0(r)^2}{r^2} \right) rdr \right\}.$$

Notice that (2.28) is the linear wave equation after a change of function: $\vec{\psi}$ solves (2.28) if and only if $\vec{\varphi}(t,r) := \frac{1}{r}\vec{\psi}(t,r)$ solves (LW) in 4 dimensions, and

$$\|\vec{\psi}\|_{H\times L^2} = \|\vec{\varphi}\|_{\dot{H}^1 \times L^2(r^3 dr)}.$$

The second kind of profiles is harmonic maps, i.e. stationnary solutions $Q(r)$ to (WM):

$$\Delta Q = \frac{f(Q)}{r^2},$$

and which are explicit in the case of target $\mathbb{S}^2$:

$$Q_{k,\pm}(r) = k\pi \pm 2\arctan r.$$

Observe that Q is monotonic, and joins two consecutive zeros of $g = \sin$ (this fact is general as shown in [35]). We are now in a position to state the result.

Theorem 2.17 ([1]). *Let $\vec{\psi}(t)$ be a finite energy equivariant wave map to $\mathbb{S}^2$. Then there exist a sequence of times $t_n \uparrow T^+(\vec{\psi})$, an integer $J \geqslant 0$, J harmonic maps $Q_1, \dots, Q_J$ such that*

$$Q_J(0) = \psi(0), \quad Q_{j+1}(\infty) = Q_j(0) \quad \text{for } j = 1, \dots, J-1,$$

and J sequences of scaling parameter $\lambda_{1,n}, \dots, \lambda_{J,n}$, such that one of the following hold.

1. Blow up wave map. $T^+(\vec{\psi}) < +\infty$. *Denote $\ell = \lim_{t\uparrow T^+(\vec{\psi})} \psi(t, T^+(\vec{\psi}) - t)$ (which is well defined). Then $J \geqslant 1$ and there exists a function $\vec{\phi}$ of finite energy, with $Q_1(\infty) = \phi(0) = \ell$ such that*

$$\lambda_{1,n} \ll \lambda_{2,n} \ll \cdots \ll \lambda_{J,n} \ll T_+(\vec{\psi}) - t_n, \tag{2.29}$$

$$\vec{\psi}(t_n) = \sum_{j=1}^{J} \left(Q_j\left(\cdot/\lambda_{j,n}\right) - Q_j(\infty), 0\right) + \vec{\phi} + o_{H\times L^2}(1). \tag{2.30}$$

2. Global wave map. $T^+(\vec{\psi}) = +\infty$. *Denote* $\ell = \psi(\infty) = Q_J(\infty)$. *Then there exists a solution* $\vec{\phi}_L(t) \in \mathscr{C}(\mathbb{R}, H \times L^2)$ *to the linear solution* (2.28) *such that*

$$\lambda_{1,n} \ll \lambda_{2,n} \ll \cdots \ll \lambda_{J,n} \ll t_n, \tag{2.31}$$

$$\vec{\psi}(t_n) = \sum_{j=1}^{J} \left(Q_j\left(\cdot/\lambda_{j,n}\right) - Q_j(\infty), 0\right) + (\ell, 0) + \vec{\phi}_L(t_n) + o_{H\times L^2}(1). \tag{2.32}$$

Let us emphasize that the decomposition holds even if no bound is assumed as in the case of the focusing wave equation (NLS) : indeed the energy here is made of positive terms and provides a bound. However, and in spite of this bound, a wave map sampled $(\vec{\psi}(t_n))_n$ has no reason a priori to admit a linear profile decomposition as described in Theorem 2.2: this sequence does not belong to the suitable functional space $H \times L^2$, actually no harmonic map Q (except 0) belongs to H.

In view of the soliton decomposition for the 3d (NLW) for radial data, it would be nice to extend the previous theorem in two directions. First, one should ask what happens for a different target: in particular the case of $g(r) = (1 - r^2)$ corresponds to the 4d (critical) radial Yang-Mills equation. This target however corresponds roughly (around the stationnary wave maps $\vec{\psi} = (\pm 1, 0)$) to the 6d wave equation: and the channel of energy for $d \equiv 2 \mod 4$ is not favorable, as we will see from the proof. The question whether this obstruction is technical only or deeper is unclear, and in any case, an answer would be nice.
Second, one would of course want to obtain a decomposition which hold for all times and not only a sequence of times. Again, the additionnal ingredient in for 3d (cNLW) is a refined channel of energy, which is not available in the even dimensional case.

Ideas of proof

As seen above, we can not use the linear profile decomposition directly, and this turns out to be where we concentrate our efforts: we will need to extract the harmonic maps Q_j by hands.
The first step is to carefully choose the sequence of times t_n. This is done so that in an averaged way $\partial_t \psi(t_n) \to 0$ in L^2. The crux of this argument is the vanishing of the energy in the self-similar region.

Proposition 2.18. *For all* $\lambda > 0$,

$$\limsup_{t\to+\infty} E(\vec{\psi}(t); \lambda t, t - A) \to 0 \quad as \quad A \to +\infty.$$

This ultimately relies on Virial identities, and was first derived by Christodoulou and Tahvildar-Zadeh [30, 31] in a slightly weaker form, and by Shatah and Tahvildar-Zadeh [121] in the blow up case; the above statement was actually proved in [4]. As a consequence,

Corollary 2.19. *There exists a sequence* $t_n \uparrow +\infty$ *such that*

$$\sup_{s, 0<s\leqslant t_n/2} \frac{1}{s} \int_{t_n-s}^{t_n+s} \int_0^{t/2} |\partial_t \psi(t,r)|^2 r dr dt \to 0 \quad as \quad n \to +\infty.$$

Morally speaking, it means that at all scales, $\partial_t \psi(t_n)$ vanishes in $L^2(rdrdt)$ in space and time in the self similar region. This control of $\partial_t \psi$ for wave type equations is one important reason that allows the whole scheme of proof to work. For example there is no such control for (NLS) or (gKdV).
As a consequence, at any scale, we can extract a weak limit out of $\vec{\psi}(t_n)$, which must be stationnary, i.e a harmonic map. Using an argument reminiscient of compactness of Paley-Smale sequence for elliptic equations, the convergence is in fact *strong* locally. This allows to extract from $\vec{\psi}(t_n)$ all harmonic maps Q_j, and will be left with an error vanishing at all scales. To state the precise bubble decomposition that we obtain, define the wave map

$$\vec{\psi}_n = \vec{\psi}[-t_n, t_n].$$

Proposition 2.20. *There exist an integer $J \geqslant 0$, J scales $(\lambda_{j,n})_n$ verifying*

$$0 < \lambda_{J,n} \ll \cdots \ll \lambda_{2,n} \ll \lambda_{1,n} \ll t_n,$$

and J finite energy harmonic maps Q_j such that, up to a subsequence σ,

$$\vec{\psi}_{\sigma(n)}(t) - (\ell, 0) = \sum_{j=1}^{J} \left(Q_j(\cdot/\lambda_{j,n}) - Q_j(\infty), 0\right) + \vec{b}_n(t),$$

where $\vec{b}_n \in \mathscr{C}([-t_n, t_n], H \times L^2)$ satisfies the following convergences. For all $A > 0$,

1. *(No energy at all scale) Let λ_n be a sequence such that $0 \leqslant \lambda_n \leqslant t_n/A$. Then*

$$\sup_{t \in [-A\lambda_n, A\lambda_n]} \|\vec{b}_n(t)\|_{H \times L^2(\lambda_n/A \leqslant r \leqslant A\lambda_n)} \to 0.$$

2. *(No energy up to the last scale) If $J \geqslant 1$, then*

$$\sup_{t \in [-A\lambda_{J,n}, A\lambda_{J,n}]} \|\vec{b}_n(t)\|_{H \times L^2(r \leqslant A\lambda_{J,n})} \to 0 \quad \textit{as } n \to +\infty.$$

   If $J = 0$, then $\sup_{t \in [-1/2, 1/2]} \|\vec{b}_n(t)\|_{H \times L^2(r \leqslant t_n/2)} \to 0$.

Also $\sum_{j=1}^{J} E(Q_j, 0) \leqslant E(\vec{\psi})$, and for all $1 \leqslant j < J$, $Q_{j+1}(\infty) = Q_j(0)$, and $Q_1(\infty) = \ell$.

The second step is to extract the linear term.

Proposition 2.21. *Let $\vec{\psi}$ be a finite energy wave map such that $T^+(\vec{\psi}) = +\infty$. Denote $k\pi = \psi(\infty)$ ($k \in \mathbb{Z}$). There exist a map $\vec{\phi}_L$ solution to linear problem* (2.28) *and an increasing non-negative continuous function $\alpha(t)$ such that $\alpha(t) = o(t)$ and*

$$\|\vec{\psi}(t) - (k\pi, 0) - \vec{\phi}_L(t)\|_{H \times L^2(r \geqslant \alpha(t))} \to 0 \quad \textit{as} \quad t \to +\infty.$$

This is proved in a similar fashion as for the nonlinear wave equation, relying heavily on profile decomposition techniques. However, we don't have an $H \times L^2$ bound yet, which is the very first step. The key observation is that $\psi(t_n) \to k\pi$ in $L^\infty(r \geqslant t_n/2)$ *irrespective of the size of* $\vec{\psi}$. Then we can use the following important scattering result assuming an L^∞ bound.

Theorem 2.22. *Let $\vec{\psi}$ be a wave map such that $\psi(\infty) = k\pi$, for some $k \in \mathbb{Z}$ and that for some $c < \pi$,*

$$\forall t \in [0, T^+(\vec{\psi})), \quad \|\psi(t) - k\pi\|_{L^\infty} \leqslant c < \pi. \tag{2.33}$$

Then $T^+(\vec{\psi}) = +\infty$ and $\vec{\psi}$ scatters at $+\infty$, in the sense sense

$$\|\psi - k\pi\|_{S([0, +\infty))} < +\infty.$$

($S = L^5_{t,r}(drdt/r^2)$ is the adequate Strichartz space in that context). It follows that there exists a (unique) solution $\vec{\phi}_L$ to (2.28) *such that*

$$\|\vec{\psi}(t) - (k\pi, 0) - \vec{\phi}_L(t)\|_{H \times L^2} \to 0 \quad \textit{as} \quad t \to +\infty.$$

As mentioned above, under (2.33), the energy provides a bound in $H \times L^2$. To prove Theorem 2.22, we are inspired by the concentration compactness method introduced by Merle, Kenig [67, 68]. We argue by contradiction: if the result does not hold we consider the minimal energy E_0 such that it fails above the E_0 energy threshold. Due to small data theory, $E_0 > 0$. Then we construct a *minimal* non scattering wave map $\vec{\psi}_0$, i.e $E(\vec{\psi}_0) = E_0$. For this, the main point is to check that the limit of a minimizing sequence also satisfies (2.33): this is done by refining the estimate on the remainder term of a linear profile decomposition. It this then standard that $\vec{\psi}_0$ is compact up to modulation, and so is a harmonic map. With the L^∞ bound (2.33), we see that $\vec{\psi}_0 = 0$, and reach a contradiction.

At this point, the error $\vec{b}_n(t)$ is controlled in $H \times L^2$ outside the self-similar region and beyond, and at all scale in the remaining cuspidal region. It remains to show that $\vec{b}_n(t)$ also vanishes there, which is the third and last step:

$$\|\vec{b}_n(0)\|_{H\times L^2(r\leqslant t_n/2)} \to 0 \quad \text{as} \quad n \to +\infty. \tag{2.34}$$

The argument goes by contradiction. The key idea is to use that the free wave $\vec{b}_{n,\mathrm{L}}(t)$ with initial data $(b_n(0), 0)$ actually maintains a fixed amount of energy outside the light cone; we emphasize that $\partial_t b_{n,L}(0) = 0$: this fact comes from the vanishing of the kinetic energy for $\vec{\psi}_n$ Corollary 2.19. This is where we use Corollary 2.9).

Indeed remark that if $\vec{\phi}$ is a solution to the linear equation (2.28), then $\vec{\varphi}(t,r) = \dfrac{1}{r}\vec{\phi}(t,r)$ satisfies the 4d linear wave equation. This is because $g'(\ell) \in \{\pm 1\}$ for all $\ell \in \mathbb{R}$ such that $g(\ell) = 0$ (recall that we consider the case $g = \sin$ and $\ell \in \pi\mathbb{Z}$). More generally, solutions to (2.28) are linked to solutions to the linear wave equation in dimension $d = 2 + 2g'(\ell)$: we see that if $g'(\ell)$ were *even* integer, then $d \equiv 2 \mod 4$, and the linear exterior energy (2.12) fails. This is the only point in the proof which restrict us to $g = \sin$.
Now, the linear exterior estimate (2.12), which is weaker that its odd counterpart (2.11), it is still sufficient for our purpose, because we already know that $\partial_t b_{n,L}(0) = 0$. we can prove that (2.12) forces $\vec{\psi}(t_n)$ to concentrate energy on the boundary of the cone. For this, we proceed in two steps for each profile, both requiring evolving a nonlinear profile decomposition backwards in time. First, we show that the evolutions of $\vec{b}_n(t)$ and $\vec{\psi}_n(t)$ remain close on an exterior region during a time-scale on which we can control the first profile (by means of Proposition 2.5).
Then, we focus the analysis outside the light cone: we need to evolve the decomposition past the time-scale on which we can control the first profile, but fortunately this large profile does not contribute in this exterior region. In fact, we evolve the profile decomposition with the first profile removed, exterior to the cone, up to the time scale of the *second* profile, and infer that some energy remains outside he light cone. Arguing similarly for every profile, we conclude that some energy remains outside the light cone for *all* times (in fact it concentrates on the boundary). Unscaling this information, we see that $\vec{\psi}(t_n)$ must concentrate some energy in the self similar region, a contradiction with Proposition 2.18.
The proof of Theorem 2.17 is complete.

5 Back to the $\dot{H}^1 \times L^2$ critical wave equation: dimension 4

For the 4d energy critical wave equation, we do not quite go as far as for its 3d counterpart, but obtain an analog of what we achieved for wave maps. This means that we need to assume an extra $\dot{H}^1 \times L^2$ bound, i.e we consider type II solutions (which can be global or blow up), and we obtain a decomposition along a subsequence of times.
Here is the precise statement. Observe that it is similar to an earlier result in 3d by Duyckarets, Kenig, Merle [42].

Theorem 2.23 (C., Lawrie, Kenig, Schlag [5])**.** *Let $\vec{u}$ be a* type II radial *solution to* (cNLW) *in dimension $d = 4$, in the sense that*

$$\sup_{t\in[0,T_+(\vec{u}))} \|\vec{u}(t)\|_{\dot{H}^1\times L^2} < +\infty. \tag{2.35}$$

There exist an integer $J \geqslant 0$, and for all $j \in [\![1, J]\!]$, a signum $i_j \in \{\pm 1\}$, J sequences of scaling parameters $\lambda_{1,n}, \dots, \lambda_{J,n}$, and a sequence of times $t_n \to T_+(\vec{u})$ such that one of the alternative holds:

1. Type II blow up. *$T_+(\vec{u}) < +\infty$. There exists $\vec{v} \in \dot{H}^1 \times L^2$, such that, as $n \to +\infty$, we have*

$$\lambda_{1,n} \ll \lambda_{2,n} \ll \cdots \ll \lambda_{J,n} \ll T_+(\vec{u}) - t_n, \tag{2.36}$$

$$\vec{u}(t_n) = \sum_{j=1}^{J} i_j W[\lambda_{j,n}] + \vec{v} + o_{\dot{H}^1\times L^2}(1). \tag{2.37}$$

2. Type II global solutions. $T_+(\vec{u}) = +\infty$. *There exists a linear solution* $\vec{v}_L \in \mathscr{C}(\mathbb{R}, \dot{H}^1 \times L^2)$ *to* (LW) *such that, as* $n \to +\infty$, *we have*

$$\lambda_{1,n} \ll \lambda_{2,n} \ll \cdots \ll \lambda_{J,n} \ll t_n, \tag{2.38}$$

$$\vec{u}(t_n) = \sum_{j=1}^{J} i_j W[\lambda_{j,n}] + \vec{v}_L(t_n) + o_{\dot{H}^1 \times L^2}(1). \tag{2.39}$$

While many of the techniques introduced by Duyckaerts, Kenig, Merle[42, 44, 45] in the 3d case carry over to the even dimensional setting, several key elements of the argument are quite different. In particular, the missing ingredients in even dimensions were:

(1) Exterior energy estimates for the underlying free radial wave equation.

(2) A proof that that the energy of a smooth solution cannot concentrate in the self-similar region of the light-cone.

The first of these ingredients is discussed in section 2. We recall that that the crucial exterior energy estimates established in [44, 45] are false in even dimensions, thus rendering the use of the channel of energy method of [40, 42, 44, 45] in doubt for the case of even dimensions. A priori, the 4d exterior energy estimate (2.12) is not enough for our purpose, because, in contrast with the wave map case, we don't know yet that the kinetic part of the error $\partial_t b_{n,L}(0)$ vanishes.
The second point, in the 3d case, also follows from dispersion property for a compactly supported solution of (cNLW), without any smallness assumption on the solution, combined with a profile decomposition. Again, all this relies on the channel of energy method and ultimately on exterior energy estimates which are false in even dimension. In the case of equivariant wave maps, (2) is well known (in the global case, this is Proposition (2.18)): the classical arguments rely crucially on multiplier identities, the monotonicity of the local energy, and on the positivity of the flux – both of which appear to be absent in the semilinear wave equation set-up.

The main new ingredient in Theorem 2.23 is the proof of (2) for solutions to the 4*d* cNLW. We need to proceed differently: in fact, we use a reduction to a 2*d* equation that bears many similarities to a wave map type equation. This is the opposite of what is usually done (for example in small data theory [6, 120] or the soliton resolution for wave maps Theorem 2.17), when equivariant wave maps are transformed to look like an energy critical nonlinear wave equation.
We first proceed to extract the linear scattering term (in the global case) or the regular part of the solution (in the blow up case). Then, on the singular part of the solution, the crucial monotonicity of the localized energy and the positivity of the flux are established in the relevant self-similar region. One can then follow the classical techniques for wave maps to prove (2) for radial solutions to 4d (cNLW). There is an interplay between this two properties, which are joined together in a bootstrap argument. This proves (2) for the 4d $\dot{H}^1 \times L^2$ critical wave equation.
Once we have the vanishing of the energy in the self-similar region in hand, we can argue as in the proof of Corollary 2.19 (proved as a consequence of Proposition 2.18, and deduce a vanishing of the L^2 norm of the time derivative of the singular part of the solution along a sequence of times. Following the arguments the previous section; in particular, we recover that the kinetic part of the error term vanishes. The 4d exterior energy estimate (2.12) is therefore sufficient for our purposes, and we can conclude the proof in the wave map case as in the previous section.

Again, as for wave maps, the result for 6d (cNLW) remains open, and one would like the decomposition obtained to hold for all times. An extension which seems out of reach for now is to get rid of the symmetry assumption. This would need in particular a better comprehension of the flow around any stationary solution: recall that these are not even classified yet. However some progress has been made in this direction at least regarding the first, most concentrated profile (see Duyckaerts, Kenig, Merle [43]).

Another direction of research is the *converse* question: given a decomposition near the maximal time of existence, can one construct a solution which behaves in this prescribed way. This question was studied rather extensively in the case of *one* nonlinear profile W (i.e. $J = 1$): we refer to Krieger, Schlag, Tataru [74] (see also [73] in the wave map setting) and Donninger, Krieger [39]. For several profiles, the question is widely open; [34, 36] tackled this question for (gKdV).

CHAPTER 3

Blowup for the 1D semi linear wave equation

WE CONSIDER in this chapter the one-dimensional semilinear wave equation

$$\begin{cases} \partial_{tt}^2 u = \partial_{xx}^2 u + |u|^{p-1}u, \\ \vec{u}(0) = (u, \partial_t u)(0) = (u_0, u_1), \end{cases} \quad (t,x) \in \mathbb{R}^{1+1}, \quad u(x,t) \in \mathbb{R} \tag{NLW}$$

where $p > 1$. Our goal here is to study the blow up solutions of this equation and describe their behavior at blow-up time.
The Cauchy problem for (NLW) is well posed locally in time in the uniformly local space $H^1_{\mathrm{loc,u}} \times L^2_{\mathrm{loc,u}}$ defined by:

$$\|v\|^2_{L^2_{\mathrm{loc,u}}} = \sup_{a\in\mathbb{R}} \int_{|x-a|<1} |v(x)|^2 dx \quad \text{and} \quad \|v\|^2_{H^1_{\mathrm{loc,u}}} = \|v\|^2_{L^2_{\mathrm{loc,u}}} + \|\nabla v\|^2_{L^2_{\mathrm{loc,u}}}.$$

(we refer to Ginibre, Soffer, Velo [56], Lindblad, Sogge [83]). Also, these solutions preserve the energy

$$\int \left(\frac{1}{2}|\partial_t u(t,x)|^2 + \frac{1}{2}|\partial_x u(t,x)|^2 - \frac{1}{p+1}|u(t,x)|^{p+1} \right) dx.$$

1 Blow up and blow up curve

As the equation has finite speed of propagation, a solution defined at some point (t_0, x_0) is also defined on a backward light cone. Due to the local well posedness, (t_0, x_0) is a blow up point if $\vec{u}$ concentrates some $H^1 \times L^1$ norm inside the backward light cone

$$\mathscr{C}(t_0, x_0, 1) = \{(t,x) \mid t \leqslant t_0, |x - x_0| \leqslant t_0 - t\}$$

stemming from (t_0, x_0). From there, one can define the space time blow up set: this is a 1-Lispchitz curve (here light has speed 1).

Definition 3.1 (Alinhac [15])**.** Let $\vec{u}$ be an arbitrary blow up solution of (NLW). Then $\vec{u}$ is defined on a maximal domain $D(\vec{u}) \subset [0, +\infty) \times \mathbb{R}$, called the domain of influence of $\vec{u}$, and

$$D(\vec{u}) = \{(t,x) \mid 0 < t < T(x)\},$$

where T is a 1-Lispchitz function. The curve $\Gamma := \{(T(x), x) \mid x \in \mathbb{R}\}$ is called the blow up curve, and $T_+(\vec{u}) = \inf_{x\in\mathbb{R}} T(x)$ is the blow up time.

The first blow up criterion was derived by Levine [81]: if $(u_0, u_1) \in H^1 \times L^2(\mathbb{R})$ satisfies

$$\int_{\mathbb{R}} \left(\frac{1}{2}|u_1(x)|^2 + \frac{1}{2}|\partial_x u_0(x)|^2 - \frac{1}{p+1}|u_0(x)|^{p+1} \right) dx < 0, \tag{3.1}$$

then the solution of (NLW) cannot be global in time. This is a standard Virial type argument, where no description of the blow up is given. Our goal here is specifically to this.

In order to study the asymptotic behavior of u near a given $(x_0, T(x_0)) \in \Gamma$, it is convenient to introduce similarity variables defined for all $x_0 \in \mathbb{R}$ and $T_0 \in \mathbb{R}$ by

$$w_{x_0,T_0}(y,s) = (T_0 - t)^{\frac{2}{p-1}} u(x,t), \quad y = \frac{x - x_0}{T_0 - t}, \quad s = -\log(T_0 - t). \tag{3.2}$$

If $T_0 = T(x_0)$, we will simply write w_{x_0} instead of $w_{x_0,T(x_0)}$. We emphasize that this change of variable can be done (and is useful) even if $T_0 < T(x_0)$.
The function $w = w_{x_0,T_0}$ satisfies the following equation for all $y \in (-1,1)$ and $s \geqslant -\log T(x_0)$:

$$\partial_s^2 w = \mathscr{L} w - \frac{2(p+1)}{(p-1)^2} w + |w|^{p-1} w - \frac{p+3}{p-1} \partial_s w - 2y \partial^2_{y,s} w, \tag{3.3}$$

$$\text{where} \quad \mathscr{L} w = \frac{1}{\rho} \partial_y (\rho (1-y^2) \partial_y w) \quad \text{and} \quad \rho = (1-y^2)^{\frac{2}{p-1}}. \tag{3.4}$$

We only look at w on the cylinder $(s,y) \in -\log T(x_0) \times (-1,1)$, because this contains the information in the backward light cone $\mathscr{C}(T_0, x_0)$, which is the only relevant. This equation can be put in the following first order form:

$$\partial_s \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} w_2 \\ \mathscr{L} w_1 - \frac{2(p+1)}{(p-1)^2} w_1 + |w_1|^{p-1} w_1 - \frac{p-3}{p-1} w_2 - 2y \partial_y w_2 \end{pmatrix}. \tag{3.5}$$

2 Description of the blow up

Let us fix from now on a blow up solution $\vec{u}$ to (NLW).
It turns out that the behavior of $\vec{u}$ (or $\vec{w}$) at a blow up point is very different depending on a *geometric property* of the blow up curve Γ near that point. This motivates the following definition:

Definition 3.2. Let $\vec{u}$ be blow up solution to (NLW). A point $x_0 \in \mathbb{R}$ is called regular or non-characteristic if there exist $\delta_0 \in (0,1)$ and $t_0 < T(x_0)$ such that $\vec{u}$ is defined on a splaying cone $\mathscr{C}(T(x_0), x_0, 1+\delta_0) \cap \{t \geqslant t_0\}$, where

$$\mathscr{C}(T(x_0), x_0, 1+\delta_0) = \{(t,x) \mid t < T(x_0),\ |x - x_0| \leqslant (1+\delta_0)|T(x_0) - t|\}$$

is a cone with slope $1/(1+\delta_0)$ smaller than 1 (and so contains the light cone stemming from $T(x_0), x_0$).
If this is not the case, we say that x_0 is characteristic or singular.
We denote by $\mathscr{R}$ the set of regular points, and by $\mathscr{S}$ the set of singular points.

In the remarquable sequence of papers [106–108, 110], Merle and Zaag gave an exhaustive description of the geometry of the blow-up set on the one hand, and the asymptotic behavior of solutions near the blow-up set on the other hand. With Zaag, we refined some of the asymptotics in [13].

We now summarize these results. $\vec{u}$ has a very specific behaviour at all singular points $(T(x_0), x_0)$, x_0 being regular or characteristic. The profiles appearing are explicit: for all $|d| < 1$, we introduce the following stationary solutions of (3.3) (or solitons) defined by

$$\kappa(d,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy)^{\frac{2}{p-1}}} \quad \text{where} \quad \kappa_0 = \left(\frac{2(p+1)}{(p-1)^2} \right)^{\frac{1}{p-1}} \quad \text{and} \quad |y| < 1. \tag{3.6}$$

Theorem 3.3 (Merle, Zaag [107])**.** *Let x_0 be a regular point.*
There exist $\mu_0 > 0$ and $C_0 > 0$ such that for all $x_0 \in \mathscr{R}$, there exist $\theta(x_0) = \pm 1$, $d(x_0) \in (-1,1)$ and $s_0(x_0) \geqslant -\log T(x_0)$ such that for all $s \geqslant s_0$:

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \theta(x_0) \begin{pmatrix} \kappa(d(x_0)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \leqslant C_0 e^{-\mu_0 (s - s_0)}. \tag{3.7}$$

Moreover, $E(w_{x_0}(s)) \to E(\kappa_0)$ as $s \to \infty$.

Observe that the profile is universal for regular points. For characteristic point, a similar profile exist, except that it is now made of several $\kappa(d)$.
Let us introduce

$$\bar{\zeta}_i(s) = \gamma_i \log s + \bar{\alpha}_i(p,k), \quad \gamma_i = \left(i - \frac{(k+1)}{2}\right)\frac{(p-1)}{2} \tag{3.8}$$

where the sequence $(\bar{\alpha}_i)_{i=1,\dots,k}$ is uniquely determined by the fact that $(\bar{\zeta}_i(s))_{i=1,\dots,k}$ is an explicit solution with zero center of mass for the following ODE system:

$$\frac{1}{c_1}\dot{\zeta}_i = e^{-\frac{2}{p-1}(\zeta_i-\zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1}-\zeta_i)}, \tag{3.9}$$

where $c_1 = c_1(p) > 0$ and $\zeta_0(s) = \zeta_{k+1}(s) := 0$ (the only freedom of this ODE is the space translation $(\xi_1(s),\dots,\xi_k(s)) \mapsto \xi_1(s)-a,\dots,\xi_k(s)-a)$). $c_1 = c_1(p) > 0$ is a constant.

Theorem 3.4 (Merle, Zaag [106], refined in [13]). *Let x_0 be a characteristic point. Then there exist $\zeta_0(x_0) \in \mathbb{R}$, an integer $k(x_0) \geqslant 2$, and a sign $\vartheta \in \{\pm 1\}$ such htat*

$$\left\| \begin{pmatrix} w_{x_0}(s) \\ \partial_s w_{x_0}(s) \end{pmatrix} - \theta_1 \begin{pmatrix} \sum_{i=1}^{k(x_0)} (-1)^{i+1}\kappa(d_i(s)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \to 0 \text{ and } E(w_{x_0}(s)) \to k(x_0)E(\kappa_0) \tag{3.10}$$

as $s \to \infty$, where

$$d_i(s) = -\tanh \zeta_i(s) \quad \text{and} \quad \zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0, \tag{3.11}$$

where $\bar{\zeta}_i(s)$ is introduced above in (3.8).

As one can see from (3.11) and (3.8), ζ_0 is the center of mass of the $\zeta_i(s)$ for any $s \geqslant -\log T(x_0)$. As any point in $\mathbb{R}$ belongs to $\mathscr{R}$ or $\mathscr{S}$, we infer that $x_0 \in \mathscr{S}$ if and only if $k(x_0) \geqslant 2$; and $x_0 \in \mathscr{R}$ if and only if $k(x_0) = 1$.
Also, the rate of convergence in (3.10) is polynomial in s, in sharp contrast with (3.7) where the rate is exponential.
Observe that the soliton's signs are alternating; the left half of the soliton move left of the center of mass, the right one goes to the right – if there is an odd number of them, one remains in the middle.

As usual in blow-up problems, the geometrical features of the blow-up curve Γ are linked to the parameters of the asymptotic behavior of the solution.
From these theorems (and actually from the proof of it), one obtains some geometric features on the blow up curve Γ.

Theorem 3.5 (Merle, Zaag [110]). *The set of regular points $\mathscr{R}$ is open, and the blow up time $x \mapsto T(x)$ is a $\mathscr{C}^1$ function on $\mathscr{R}$.*
If $x_0 \in \mathscr{R}$, then $T'(x_0) = d(x_0)$ where $d(x_0)$ was introduced in Theorem 3.3.

We emphasize this stricking point that the derivative of T at $x_0 \in \mathscr{R}$ is intimately linked to the the blow up profile at x_0.
This result was refined by Nouailli [115] who proved that T has even $\mathscr{C}^{1,\alpha}$ regularity on $\mathscr{R}$ for some $\alpha > 0$. At characteristic points, the curve in fact makes a corner.

Theorem 3.6 (Merle, Zaag [108], refined in [13]). *The set of characteristic points $\mathscr{S}$ is discrete. Let $x_0 \in \mathscr{S}$. Then as $x \to x_0$, $x \neq x_0$, there hold*

$$T(x) = T(x_0) - |x-x_0| + \frac{\gamma e^{2\,\mathrm{sgn}(x_0-x)\zeta_0(x_0)}|x-x_0|(1+o(1))}{|\log|x-x_0||^{\frac{(k(x_0)-1)(p-1)}{2}}}, \tag{3.12}$$

$$T'(x) = \mathrm{sgn}(x_0-x)\left(1 - \frac{\gamma e^{2\,\mathrm{sgn}(x_0-x)\zeta_0(x_0)}(1+o(1))}{|\log|x-x_0||^{\frac{(k(x_0)-1)(p-1)}{2}}}\right), \tag{3.13}$$

as $x \to x_0$, where $\gamma = \gamma(p) > 0$.

Observe that the expansion on T' (3.13) is simply the differentiation of the expansion in T (3.12).
Unlike what one may think from less accurate estimates, we surprisingly see from the expansion (3.12) that the blow-up set is *never* symmetric with respect to a characteristic point x_0, except maybe when $\zeta_0(x_0) = 0$.

The analysis developed here is rather robust, and can be extended to more general semi-linear nonlinearities (with different result: for $|u|^p$, there is no singular points). It can also be extended to certain point to higher dimension: as long as the w remains near a $\kappa(d)$ (see Merle, Zaag [105, 109]). For radial data, the analysis goes through at least outside 0. For general data, the situation is less clear, essentially because the stationary solutions (generalizing the $\kappa(d)$) are not understood well enough.

Ideas of proofs

Although we presented the results describing the blow up on one side and the geometry of of blow up curve on the other side, the proofs are much more intertwined: the description of the blow up actually *implies* consequences on the blow up curve. Most of the analysis is done in the similarity variable $w(s,y)$, and then translated in the $u(t,x)$ formulation when stated in the theorems.

The energy space takes the following expression in the similarity variables:

$$\mathscr{H} = \left\{ (q_1, q_2) \mid \|(q_1,q_2)\|^2_{\mathscr{H}} := \int_{-1}^{1} \left(q_1^2 + \left(q_1'\right)^2 (1-y^2) + q_2^2 \right) \rho dy < +\infty \right\}. \tag{3.14}$$

And from the conservation of the energy, we obviously obtain a first bound:

$$\forall s \geqslant 0, \quad \|(w, \partial_s w)(s)\|_{\mathscr{H}} \leqslant C. \tag{3.15}$$

The starting point of the analysis is that the equation for w (3.3) admits a Lyapunov functional.

Proposition 3.7 (Antonini, Merle [17]). *Let $\vec{w} = (w, \partial_s w) \in \mathscr{C}([S_0, +\infty), \mathscr{H})$ be a solution to* (3.3). *Then the quantity*

$$E(w(s)) = \int \left(\frac{1}{2} (\partial_s w)^2 + \frac{1}{2} \left(\partial_y w\right)^2 (1-y^2) + \frac{(p+1)}{(p-1)^2} w^2 - \frac{1}{p+1} |w|^{p+1} \right) \rho(y) dy \tag{3.16}$$

is non-increasing in time. For $s_2 \geqslant s_1$,

$$E(w(s_2)) - E(w(s_1)) = -\frac{4}{p-1} \int_{s_1}^{s_2} \int |\partial_s w(s,y)|^2 (1-y^2)^{\frac{2}{p-1}-1} dy ds \leqslant 0.$$

This provides both a bound on E and on $\partial_s w$, and allows to derive a first coarse decomposition in solitons.

Proposition 3.8. *Let $x_0 \in \mathbb{R}$. There exist an integer $k(x_0)$, $k(x_0)$ signum $\varepsilon_i \in \{\pm 1\}$, and $k(x_0)$ continuous functions $d_i(s)$ ($i = 1, \ldots, k(x_0)$) such that*

$$\left\| \begin{pmatrix} w_{x_0,T(x_0)}(s) \\ \partial_s w_{x_0,T(x_0)}(s) \end{pmatrix} - \sum_{i=1}^{k(x_0)} \varepsilon_i \begin{pmatrix} \kappa(d_i(s), \cdot) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \to 0 \quad as \quad s \to +\infty, \tag{3.17}$$

and, with $\zeta_i = \arg\tanh(d_i)$, $\zeta_{i+1}(s) - \zeta_i(s) \to +\infty$.

To prove this, we use the bounds from the energy and Lyapunov functional to find local limits: for some sequences $s_n \to +\infty$, in the $\xi = \arg\tanh(y)$ variable,

$$w_{x_0,T(x_0)}(\xi + \xi_n, s + s_n) \to w^* \quad \textit{stationary} \text{ solution, in } H^1_{\text{loc}}.$$

Now, stationary solutions can easily be classified: they are exactly the $\kappa(d,\cdot)$. In the $(\zeta,\xi) = (\arg\tanh d, \arg\tanh y)$ variables, κ is actually the usual soliton of (NLS):

$$\kappa(\zeta,\xi) = \frac{1}{\cosh(\xi-\zeta)^{\frac{2}{p-1}}}.$$

The fact that $\zeta_{i+1}(s) - \zeta_i(s) \to +\infty$ comes from the definition of a local limit. Notice that at this point, we have no control on the signum ε_i or the number of solitons $k(x_0)$.

Let us now analyse non-characteristic points. If $x_0 \in \mathscr{R}$, using the splaying cone property and a covering argument, one can get rid of the weight in the energy bound (3.15), so that in fact:

$$\|(w_{x_0,T(x_0)}, \partial_s w_{x_0,T(x_0)})\|_{H^1\times L^2} \leqslant C.$$

This implies that we in fact have a global limit instead of local limits: $k(x_0) = 1$. Via modulation, a linear version of the Lyapunov functional yields exponential convergence

$$\forall s \geqslant s(x_0), \quad \left\| \begin{pmatrix} w_{x_0,T(x_0)}(s) \\ \partial_s w_{x_0,T(x_0)}(s) \end{pmatrix} - \varepsilon(x_0) \begin{pmatrix} \kappa(T'(x_0)) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \leqslant C_0 e^{-\mu_0(s-s_0)}. \tag{3.18}$$

This is because in the similarity variables, the problem is now *parabolic* in nature (as it can been seen also through the existence of the Lyapunov functional (3.16)). It turns out that the analysis here is stable: in fact, the soliton is an attractor and we have the following trapping result.

Proposition 3.9 (Merle, Zaag [107, Theorem 3])**.** *There exist $\delta_0 > 0$ and $C_0 > 0$ such that if for some $x_0 \in \mathbb{R}$, $s_0 \geqslant -\log T(x_0)$, $\varepsilon \in \{\pm 1\}$, $d \in (-1,1)$ and $\delta \in (0,\delta_0]$, we have*

$$\left\| \begin{pmatrix} w_{x_0,T(x_0)}(s_0) \\ \partial_s w_{x_0,T(x_0)}(s_0) \end{pmatrix} - \varepsilon \begin{pmatrix} \kappa(d) \\ 0 \end{pmatrix} \right\|_{\mathscr{H}} \leqslant \delta,$$

then $x_0 \in \mathscr{R}$, T is differentiable at x_0, and $w_{x_0,T(x_0)}(s) \to \varepsilon\kappa(T'(x_0))$ as $s \to \infty$. (The convergence is in fact exponential).

As a consequence, the set $\mathscr{R}$ of non-characteristic points is open, and one can prove that T is $\mathscr{C}^1$ on $\mathscr{R}$.

We now consider characteristic points. To make the decomposition (3.17) more precise, we first need to introduce a self-similar version of the soliton $\kappa(d)$. Define for $\nu > -1 + |d|$, $\kappa^*(d,\nu,y) = (\kappa_1^*(d,\nu,y), \kappa_2^*(d,\nu,y))$, where

$$\kappa_1^*(d,\nu,y) = \kappa_0 \frac{(1-d^2)^{\frac{1}{p-1}}}{(1+dy+\nu)^{\frac{2}{p-1}}} \quad \text{and} \quad \kappa_2^*(d,\nu,y) = \nu\partial_\nu \kappa_1^*(d,\nu,y). \tag{3.19}$$

Then for $\mu \in \mathbb{R}$, $\kappa^*(d,\mu e^s, y)$ is solution of (3.5), and for $\mu = 0$, we recover the usual $\kappa(d)$. For $\mu > 0$, we have an heteroclinic solution, linking $\kappa(d)$ (at $s \to -\infty$) to 0 (at $s \to +\infty$). If $\mu < 0$, it blows up at time $s = \ln\left(\frac{|d|-1}{\mu}\right)$. The point of introducing the κ^* is that the extra parameter ν can be modulated to allow an extra orthogonality condition. More precisely, we can write a decomposition

$$\begin{pmatrix} w_{x_0,T(x_0)}(s) \\ \partial_s w_{x_0,T(x_0)}(s) \end{pmatrix} = q(s) + \sum_{i=1}^{k(x_0)} (-1)^j \kappa^*(d_i(s), \nu_i(s)), \quad \|q(s)\|_{\mathscr{H}} \to 0$$

with the coercivity property:

$$\|q(s)\|_{\mathscr{H}}^2 \leqslant C \int_{-1}^{1} ((\partial_s q)(s,y)^2 + (\partial_y q)(s,y)^2(1-y^2) - K(s,y)q(s,y)^2)\rho dy.$$

This makes it possible to close the estimates on $\nu_i(s)$, $d_i(s) = -\tanh(\zeta_i(s))$ and $q(s)$ as follows

$$\frac{|\dot{\nu}_i - \nu_i|}{1-d_i^2} \leqslant C(\|q\|_{\mathscr{H}}^2 + J + \|q\|_{\mathscr{H}}\bar{J}) \tag{3.20}$$

$$\left|\frac{1}{c_1(p)}\dot{\zeta}_i - \left(e^{-\frac{2}{p-1}(\zeta_i-\zeta_{i-1})} - e^{-\frac{2}{p-1}(\zeta_{i+1}-\zeta_i)}\right)\right| \leqslant C(\|q\|_{\mathscr{H}}^2 + (J+\|q\|_{\mathscr{H}})\bar{J} + J^{1+\delta}) \tag{3.21}$$

$$\|q(s)\|_{\mathscr{H}}^2 \leqslant Ce^{-\delta(s-s_0)}\|q(s_0)\|_{\mathscr{H}}^2 + C\hat{J}(s)^2 \tag{3.22}$$

with $\zeta_i(s) = -\arg\tanh(d_i(s))$, and the error terms:

$$J = \sum_{i=2}^{k(x_0)} e^{-\frac{2}{p-1}(\zeta_i-\zeta_{i-1})}, \quad \bar{J} = \sum_{i=1}^{k(x_0)} \frac{|\nu_i|}{1-d_i^2}, \quad \hat{J} = \sum_{i=2}^{k(x_0)} e^{-\frac{\bar{p}}{p-1}(\zeta_i-\zeta_{i-1})},$$

where $\bar{p} = \min(p, 2^-)$ (and 2^- is any number less that 2). The 3 estimates above are the crucial bounds driving the dynamic.
The estimate (3.22) is extremely stable for the infinite dimensional parameter q. It turns out that the unperturbed ODE derived from (3.20)-(3.21) actually has a global attractor, which is $\nu_i = 0$ and $\zeta_i = \bar{\zeta}_i + \xi_0$ where $\bar{\zeta}_i$ was defined in (3.8) and ξ_0 is the barycenter of the $\zeta_i(0)$. Therefore, one can also prove convergence for the pertubed ODE (3.20)-(3.21) (with floating barycenter ζ_∞): as $s \to +\infty$, we have the expansion

$$\zeta_i = \left(i - \frac{k+1}{2}\right)\frac{(p-1)}{2}\ln s + \zeta_\infty + O(s^{1-\bar{p}}), \quad J = O(s^{-2}), \quad \text{and} \quad \|q\|_{\mathscr{H}}, \ \frac{|\nu_i|}{1-d_i^2}, \ \hat{J}, \ \bar{J} = O(s^{-\bar{p}}). \tag{3.23}$$

This concludes the dynamical analysis at a characteristic point, and shows the convergence (3.10) and (3.11). However at this point we could have $k(x_0) < 2$.

It remains to derive the geometric properties on the blow up curve. The first observation is that a decomposition on $w_{x_0,T(x_0)}$ translates into a decomposition for x nearby x_0 (at least while the incoming light cones intersect): for example, assuming $T(0) = 0$ (by time translation), we have

$$w_{x,T(x)}(y,s) = (1-(1-B)xe^s)^{-\frac{2}{p-1}} w_{0,0}(Y,S),$$
$$Y = \frac{y + xe^s}{1-(1-B)xe^s}, \quad S = s - \ln(1-(1-B)xe^s). \tag{3.24}$$

Using this remark and the precise asymptotics (3.23), one can prove that the blow-up curve is corner shaped at any characteristic point, that is the expansions (3.12) and (3.13) hold.
From there, $\mathscr{S}$ necessarily has empty interior. Indeed, if $\mathscr{S}$ contains compact interval, the corner shape property of T at each $x \in S$ implies that T cannot reach its minimum on this interval, a contradiction with the continuity of T (T is 1-Lipshitz).

We are now in a position to prove that $k(x_0) \geqslant 2$ for any $x_0 \in \mathscr{S}$. The key fact is that any charateristic point x_0 actually lies on the bounday $\partial\mathscr{S}$ (as $\mathscr{S} = \partial\mathscr{S}$). For such a x_0 on the boundary, one can not have $k(x_0) = 0$, due to energy trapping. But $k(x_0) = 1$ is also not possible, because otherwise x_0 would not be characteristic. Hence $k(x_0) \geqslant 2$. We emphasize that this analysis can only be carried out because x_0 lie on the boundary of $\mathscr{S}$: it heavily relies on stability property of regular points.

Finally if $x_0 \in \mathscr{S}$, one can again relate by (3.24) the decomposition of $w_{x,T(x)}$ to that of $w_{x_0,T(x_0)}$ for x nearby x_0. It turns out that, say if $x < x_0$, no solitons for $w_{x,T(x)}$ can travel to the left: but this implies that there is no more than one soliton, i.e $k(x) \leqslant 1$. From the previous analysis, x is not characteristic, that is $x \in \mathscr{R}$. Therefore, x_0 is an isolated characteristic point. This completes the proof of Theorems 3.4 and 3.6.

3 Construction of characteristic points

Once the description at blow up of both regular and singular points is obtained, a natural question is whether all cases of the classification can be achieved, or whether one can further reduce the possible behaviors.

As far as non-characteristic points are concerned, the answer is easy. Indeed, any blow-up solution has non-characteristic points ($\mathscr{R}$ is never empty), for example those constructed by Levine's criterion (3.1).
Regarding the asymptotic behavior, any profile given in (3.7) does occur. Indeed, note first that for any $d \in (-1,1)$, the function

$$u(x,t) = (1-t)^{-\frac{2}{p-1}} \kappa\left(d, \frac{x}{1-t}\right) = \frac{\kappa_0 (1-d^2)^{\frac{1}{p-1}}}{(1-t+dx)^{\frac{2}{p-1}}} \tag{3.25}$$

is a particular solution to equation (NLW) defined for all $(x,t) \in \mathbb{R}^2$ such that $1-t+dx > 0$, blowing up on the curve $T(x) = 1 + dx$ and such that for any $x_0 \in \mathbb{R}$, $T'(x_0) = d$ and $w_{x_0}(y,s) = \kappa(d,y) = \kappa(T'(x_0), y)$, and (3.7) is trivially true. However, the problem with this solution is that it is not a solution of the Cauchy problem at $t = 0$, in the sense that it is not even defined for all $x \in \mathbb{R}$ when $t = 0$. This is in fact not a problem thanks to the finite speed of propagation. Indeed, performing a truncation of (3.25) at $t = 0$, the new solution will coincide with (3.25) for all $|x_0| \leqslant R$ and $t \in [0, T(x_0))$ for some $R > 0$, and (3.7) holds for the new solution as well, for all $|x_0| < R$.

Now, considering characteristic points, the answer is much more delicate. Unlike what was commonly believed after the work of Caffarelli and Friedman [25, 26], Merle and Zaag proved in [106] the *existence of solutions* of (NLW) admitting at least one characteristic point, i.e. such that $\mathscr{S} \neq \emptyset$. The idea was to construct an odd blow up solution. In that case, the number of solitons appearing in the decomposition (3.10) at $x_0 = 0$ has to be even: therefore $k(0) \geqslant 2$, and $0 \in \mathscr{S}$. No other information was available. After this result, the following question remained open :

> Given an integer $k \geqslant 2$, is there a blow-up solution of equation (NLW) with a characteristic point x_0 such that the decomposition (3.10) holds with k solitons?

This is actually true, and is the content of the following theorem.

Theorem 3.10 (C, Zaag [13])**.** *Let $k \geqslant 2$ be an integer and $\zeta_0 \in \mathbb{R}$, there exists a blow-up solution $u(x,t)$ to equation* (NLW) *in $H^1_{\text{loc,u}} \times L^2_{\text{loc,u}}(\mathbb{R})$ with $0 \in \mathscr{S}$ such that*

$$\left\| \begin{pmatrix} w_0(s) \\ \partial_s w_0(s) \end{pmatrix} - \begin{pmatrix} \sum_{i=1}^{k} (-1)^{i+1} \kappa(d_i(s)) \\ 0 \end{pmatrix} \right\|_{\mathcal{H}} \to 0 \text{ as } s \to \infty, \tag{3.26}$$

with

$$d_i(s) = -\tanh \zeta_i(s), \quad \zeta_i(s) = \bar{\zeta}_i(s) + \zeta_0 \tag{3.27}$$

and $\bar{\zeta}_i(s)$ defined in (3.8).

Note from (3.27) and (3.8) that the barycenter of $\zeta_i(s)$ is fixed, in the sense that

$$\forall s \geqslant -\log T(0), \quad \frac{\zeta_1(s) + \cdots + \zeta_k(s)}{k} = \frac{\bar{\zeta}_1(s) + \cdots + \bar{\zeta}_k(s)}{k} + \zeta_0 = \zeta_0. \tag{3.28}$$

Observe that in this construction, we have much essentially no degree of freedom (once given the number $k \geqslant 2$ of solitons), except for the barycenter ζ_0, which acknowledges for the translation invariance in the ODE system (3.9). This is in sharp contrast with the multi-solitons constructed in Chapter 1, where each soliton was essentially independent of the others (and even generated a 1 parameter family of multi-solitons in the unstable case); this constraint in the "asymptotic state" is of course due to the convergence (3.10), which reminds us of the parabolic nature of the problem in similarity variable mentioned earlier (page 31).

One interesting question remaining after this is to determine a solution with a given discrete characteristic set and given blow up times. If these points are sufficiently separated (or if the blow up time is not prescribed), one can glue together the solutions constructed above and make use of finite speed of propagation to conclude. In general however, the result is open. In the same cloud of questions, one could ask about the construction of a solution to (NLW) with a prescribed blowup curve.

Ideas of proof

The proof is reminiscent of the construction of multi-solitons seen in Chapter 1. However, there is major difference here: we can not construct a solution backwards in the similarity variable w. To fix notations, let 0 be the desired characteristic point, with blow up time $T(0) = 0$. In view of constructing the characteristic blowup, assume that we are given a final data $w_n(S_n)$ for some large S_n. This means in the original variable we have a final data $u_n(t_n)$ defined on the slice $x \in [-|t_n|, |t_n|]$ of the incoming light cone at 0 ($t_n \uparrow 0$). But this data is only sufficient to define a solution to (NLW) on $|x| \leqslant |t_n - t|$, which is much smaller than the entire incoming light cone where we would like to define u_n (and the domain of influence keeps decreasing as $S_n \to +\infty$, i.e. $t_n \to 0$).
Therefore, we can not argue backwards in time, we have to construct the solution *foward* in time. This irreversibility constraint can be seen as another manifestation of the parabolic nature of the problem in the similarity variables. It is this same parabolic flavored feature which will save the day, under the form of an asymptotic stability property, as in the convergence (3.10). Observe that as we work forward in time, we will not construct approximate solutions, but directly a suitable one.

The main point of the proof is to construct a solution $w \in \mathscr{C}([S_0, +\infty), \mathscr{H})$ to (3.5) with k solitons, irrespective of the barycenter. Up to choosing S_0 large enough, we can assume w to be very close for all times to the sum of k solitons, and in this case, the modulation ensures that there is a one-to-one correspondence between $w(s)$ and $(q(s), (d_i(s))_i, (\nu_i(s))_i)$. Therefore, our goal is to find initial conditions $(q(s_0), (d_i(s_0))_i, (\nu_i(s_0))_i)$ such that w is globally defined on $[S_0, +\infty)$ and

$$q(s) \to 0, \quad d_i(s) \sim \bar{d}_i(s) \quad \text{and} \quad \nu_i(s) \to 0 \quad \text{as} \quad s \to +\infty.$$

Let us recall the equations the dynamic (3.20)-(3.21)-(3.22) once again.
As mentioned above, q has an extremely strong decaying property. The perturbed system for the ζ_i is also stable, although not asymptotically stable, due to the translation invariance. To deal with this, consider the error $\zeta_i(s) := \xi_i(s) - \bar{\bar{\xi}}_i(s)$, and the system of equation (3.21) linearized around $(\bar{\bar{\xi}}_1, \dots, \bar{\bar{\xi}}_k)$. Then up to a linear change of variable (independent of s) represented by a square $k \times k$ matrix P and denoting $\phi(s) = \begin{pmatrix} \phi_1 \\ \vdots \\ \phi_k \end{pmatrix} = P \begin{pmatrix} \zeta_1 \\ \vdots \\ \zeta_k \end{pmatrix}$, this linearized system writes

$$\dot{\phi} = M\phi, \quad \text{where } M \text{ satisfies} \quad (M\phi, \phi) \leqslant -\sum_{i=2}^{k} \phi_i^2, \quad \text{and} \quad M(1, 0, \dots, 0)^T = 0. \tag{3.29}$$

M has signature $(0, k-1)$, and the system in ϕ is stable. However the equation for ν_i (at leading order $\dot{\nu}_i \sim \nu_i$) is *transversally* unstable. For these directions, we will use a topological argument which has the same flavor as when we constructed multi-solitons in the L^2 supercritical case in chapter 1.

Define the rescaling $\Gamma_s : (\nu_1, \dots, \nu_k) \mapsto (s^{-1/2-|\gamma_1|}\nu_1, \dots, s^{-1/2-|\gamma_k|}\nu_k)$.
We consider solutions to (3.5), or more precisely, their modulation, defined as follows. Let ν_0 in the unit ball $\mathbb{B}$ of $\mathbb{R}^k$. Define the rescaling $\nu(s_0) = \Gamma_{s_0}(\nu_0)$, where γ_i is defined in (3.8);

and initial data whose modulation is

$$(0, (\bar{d}_i(s_0))_i, (\nu_i(s_0)_i) \quad \text{that is} \quad (w, \partial_s w)(s_0) = 0 + \sum_{i=1}^{k} \kappa^*(\bar{d}_i(s_0), \nu_i(s_0)).$$

Then denote $(q(s), ((d_i(s))_i, (\nu_i(s))_i)$ the modulation of the evolution $(w, \partial_s w)(s)$ with the initial conditions above at $s = s_0$. The objects under consideration are

- the rescaled flow $\Phi : (s, \nu_0) \mapsto \Gamma_s^{-1}((\nu_1(s), \dots, \nu_k(s))$, and
- the exit time $s^*(\nu_0) = \sup\{s \geqslant s_0 | \, \forall \tau \in [s_0, s], \ \Phi(\tau, \nu_0) \in \mathbb{B}\}$.

($\Phi(\cdot, \nu)$ is defined at least on the interval $[s_0, s^*(\nu))$. Our goal is to find $\nu_0 \in \mathbb{B}$ such that $s^*(\nu_0) = +\infty$. For this, we argue by contradiction and assume that $s^*(\nu) < +\infty$ for all $\nu \in \mathbb{B}$. Then the exit point is well defined: let

$$\Psi : \mathbb{B} \to \mathbb{B}, \quad \nu \mapsto \Phi(s^(\nu), \nu).$$

By maximality of of the exit time, we have Ψ takes its value on the sphere $\mathbb{S}^{k-1}$. Due to the stability properties mentioned above, there exist C and $\eta > 0$ such that for all $\nu \in \mathbb{B}$ and for all $s \in [s_0, s^*(\nu)]$,

$$s^{1/2+\eta} \|q(s)\|_{\mathscr{H}} + s^\eta \sum_{j=2}^{k} |\phi_i(s)| + s^{1+\eta} |\dot{\phi}_1(s)| \leqslant C. \tag{3.30}$$

(Recall ϕ_i was introduced in (3.29), and is directly linked to ζ_i and d_i). Observe that we have no control on ϕ_1 (as it accounts for space translations), only on its derivative; it however ensures that $s_0^\eta \phi_1(s)$ remains bounded. This is enough for our purposes: we already have the leading order for $\zeta_i = \bar{\zeta}_i + O(1)$, the ϕ_i only account for the second order.
By the transversality mentioned above, we see that Ψ is a continuous map on $\mathbb{B}$, and that the exit is instantaneous on the sphere: if $\nu \in \mathbb{S}^{k-1}$, then $s^*(\nu) = s_0$ and $\Psi(\nu) = \nu$.
We just constructed a continuous map $\Psi : \mathbb{B} \to \mathbb{S}^{k-1}$ such that $\Psi|_{\mathbb{S}^{k-1}} = \mathrm{Id}_{\mathbb{S}^{k-1}}$: this contradicts Brouwer's theorem. Therefore, our assumption was wrong and there exists $\nu_0 \in \mathbb{B}$ such that $s^*(\nu_0) = +\infty$.

Now the estimates (3.30) still hold for $\Phi(s, \nu_0)$: this ensures that $\nu_i \to 0$, $q \to 0$, ϕ_1 converges and for $i \geqslant 2$, $\phi_i \to 0$. This is enough to conclude to the existence of a solution $w_\sharp \in \mathscr{C}([s_0, +\infty), \mathscr{H})$ satisfying the alternating k-soliton decomposition (3.26) and (3.27), and barycenter $|\zeta_0| \leqslant C s_0^{-\eta}$. From $w_\sharp$, we define on $(-1, 1)$:

$$u_\sharp(0, x)|_{(-1,1)} = w_\sharp(x, s), \quad \partial_t u_\sharp(x, 0)|_{(-1,1)} = \partial_s w_\sharp(x, s_0) + \frac{2}{p-1} w^\sharp(x, s_0) + x \partial_y w^\sharp(x, s_0).$$

(An we extend it in a smooth way outside $(-1, 1)$). Then we check that the solution $u_\sharp$ to (NLW) is defined on the truncated cone $\{(t, x) \mid |x| \leqslant 1 - t, t \geqslant 0\}$, and that in this region,

$$u_\sharp(x, t) = (1-t)^{-\frac{2}{p-1}} w_\sharp \left(\frac{x}{1-t}, s_0 - \ln(1-t) \right).$$

Therefore, 0 is a characteristic point of $u_\sharp$. To fix the barycenter ζ_0, it suffices to perform a suitable Lorentz transform on $u_\sharp$. This completes the construction of a blow up solution with prescribed characteristic point.

CHAPTER 4

The Zakharov-Kuznetsov flow around solitons

IN THIS CHAPTER we are interested in studied some features of solutions to the (generalized) Zakharov-Kuznetsov equation:

$$\begin{cases} \partial_t u + \partial_{x_1}(\Delta u + |u|^{p-1}u) = 0, \\ u(t=0,x) = u_0(x), \end{cases} \qquad (t,x) \in \mathbb{R} \times \mathbb{R}^d, \quad x = (x_1, x_2) \in \mathbb{R} \times \mathbb{R}^{d-1}, \tag{ZK}$$

where $p \in \mathbb{N}$ is such that $2 \leqslant p < \infty$ if $d = 1, 2$ and $2 \leqslant p < 1 + \dfrac{4}{d-2}$ if $d \geqslant 3$ (the equation is $\dot{H}^1$ subcritical). The original (ZK) equation, with $p = 2$ was introduced by Zakharov, Kuznetsov [76] to describe the propagation of ionic-acoustic waves in uniformly magnetized plasma in the two dimensional and three dimensional cases. Lannes, Linares, Saut [79] carried out the derivation of (ZK) from the Euler-Poisson system with magnetic field in the long wave limit; and Han-Kwan [62] derived the (ZK) equation from the Vlasov-Poisson system in a combined cold ions and long wave limit.

Observe that when $d = 1$, (ZK) becomes simply the (gKdV) equation. In the x_1 direction, (ZK) bears indeed many similarities with (gKdV). On the other part, the symbol of the linear part is $i\tau - i\xi_1|\xi'|^2$, so that for fixed ξ_1 (or fixed x_1) we recover the one of (NLS). We can summarize this by saying that the x_1 direction is privileged, and in the $d-1$ other directions x_2, the equation resembles (NLS).

The mass and energy, which are the same as for (NLS), are conserved at least formally by the flow of (ZK):

$$M(u) = \|u(t)\|_{L^2}^2, \tag{4.1}$$

$$E(u) = \int \left(\frac{1}{2}|\nabla u(t,x)|^2 - \frac{1}{p+1} u(t,x)^{p+1} \right) dx. \tag{4.2}$$

The well-posedness theory for (ZK) has been extensively studied in the recent years. Let us solely recall the best result available yet regarding local well posedness of the Cauchy problem:

- In the two dimensional case, the problem is well posed in $H^s(\mathbb{R}^2)$ for $s > \dfrac{1}{2}$, see Grünrock, Herr [61] and Molinet, Pilod [113] (we also refer to Faminskii [52] and Linares, Pastor [82]).
- In the three dimensional case, the problem is well posed in $H^s(\mathbb{R}^3)$ for $s \geqslant 1$, see Ribaud and Vento [117]. Those solutions were extended globally in time in [113].

Note however that it is still an open problem to obtain well-posedness for the original (ZK) equation (with $p = 2$) in $L^2(\mathbb{R}^2)$ and in 3d, we miss the energy space $H^1(\mathbb{R}^3)$ (the energy space).

The solitons of (ZK) are based on the same ground state Q as (NLS), ie the unique positive solution in $H^1(\mathbb{R}^d)$ to (0.5):

$$-\Delta Q + Q - Q^p = 0,$$

and by scaling we have the expression ground state for any c_0:

$$Q_{c_0}(x) := c_0^{\frac{1}{p-1}} Q(\sqrt{c_0}x).$$

Then given $c_0 > 0$ and $x_0 \in \mathbb{R}^d$

$$Q[c_0, x_0](t, x) = Q_{c_0}(x_1 - c_0 t - x_{0,1}, x_2 - x_{0,2})$$

is the soliton solution to (ZK).

(ZK) is L^2 critical for $p = 1 + \frac{4}{d}$, L^2 subcritical for $p < 1 + \frac{4}{d}$ and L^2 supercritical for $p > 1 + \frac{4}{d}$. Now, in view of conservation of mass and energy, the analysis done in the preamble applies, and de Bouard [23] proved that solitons are orbitally stable in the L^2 subcritical case, and unstaible in the L^2 supercritical case.

Our goal here is to prove results in the spirit of Martel, Merle [86–88, 90] for the (gKdV) equation, namely rigidity theorems (of Liouville type) for non dispersive solution around a solitons, asymptotic stability for a soliton, and stability of a sum of solitons. These ideas where later implemented by Béthuel, Gravejat, Smets [21, 22] for the Gross-Pitaevskii equation in 1 space dimension

$$i\partial_t \psi + \Delta\psi + \psi(1 - |\psi|^2) = 0, \quad |\psi(t, x)| \to 1 \quad \text{as } |x| \to 1. \tag{GP}$$

In the case of (GP), the conditions at infinity (and the induced change of functionnal setting) make the equation share many similarities with (gKdV), although the initial equation is (NLS).
We believe that (ZK) is the first two and higher dimensional model where asymptotic stability of solitons is proved in the energy space, and with no nonstandard spectral assumptions. It would be nice to extend these result to other dispersive equations, the first of which being of course (NLS).

It turns out that a spectral property is central in all the discussion below, and involves the spectrum of the linearized operator

$$L = -\Delta + 1 - pQ^{p-1},$$

and the direction ΛQ, where Λ is the scaling operator

$$\Lambda v = \frac{d}{dc} c^{\frac{1}{p-1}} v(\sqrt{c}\cdot)|_{c=1} = x \cdot \nabla v + \frac{1}{p-1} v.$$

Spectral assumption: there exists $\lambda > 0$ such that

$$\forall v \in H^1, \quad \langle Lv, v\rangle \geqslant \lambda \|v\|_{H^1}^2 - \frac{1}{\lambda}\left(\langle v, \lambda Q\rangle^2 + \sum_{i=1}^{d} \langle v, \partial_{x_i} Q\rangle^2\right). \tag{4.3}$$

This spectral assumption is true in dimension $d = 1$, and has been numerically checked in a strong way in dimension $d = 2$ for $p = 2$; it is unclear in dimension $d = 3$ for $p = 2$.

Our main assumption in this Chapter is the following is twofold:

(A) (d, p) is such that the spectral assumption (4.3) holds, and that the Cauchy problem for (ZK) is well posed in H^1 (locally in time).

From the discussion above, assumption (A) holds in particular for $d = 2$ and $p = 2$ i.e. for the 2 dimensional original (ZK) equation.

1 Liouville theorems

The main result of this section is a classification of solutions to (ZK) which remain close to a soliton for all time, and does not disperse in the x_1 direction in a sense made precise below.

Theorem 4.1 (Nonlinear Liouville property around Q)**.** *Assume that assumption (A) is true. There exists $\varepsilon_0 > 0$ such that the following holds. Let $u \in \mathscr{C}(\mathbb{R}, H^1(\mathbb{R}^d))$ be a solution of* (ZK) *such that*

1. *$u(t)$ remains close for all time to the soliton in the sense that for some function $\rho(t) = (\rho_1(t), \rho_2(t))$*
$$\forall t \in \mathbb{R}, \quad \|u(\cdot + \rho(t)) - Q\|_{H^1} \leqslant \epsilon_0, \tag{4.4}$$
2. *u is L^2 compact in the sense that*
$$\forall \varepsilon > 0,\ \exists A > 0 \quad \text{such that} \quad \sup_{t\in\mathbb{R}} \int_{|x_1|>A} u^2(x+\rho(t), t)dx \leqslant \varepsilon. \tag{4.5}$$

Then u is a soliton, i.e there exist $c_0 > 0$ (close to 1) and $(\rho_{0,1}, \rho_{0,2}) \in \mathbb{R}^2$ such that

$$\forall (t, x_1, x_2) \in \mathbb{R}^3, \quad u(t, x_1, x_2) = Q_{c_0}(x_1 - c_0 t - \rho_{0,1}, x_2 - \rho_{0,2}). \tag{4.6}$$

In the case of the original (ZK) equation $p = 2$, in dimension 2, the solitons are H^1 orbitally stable due to the result of de Bouard [23]. Therefore in this case, Theorem 4.1 still holds true if we relax assumption (4.4) to
$$\|u_0 - Q\|_{H^1} \leqslant \varepsilon_0. \tag{4.7}$$
We see that the L^2 compactness asssumption (4.5) once again singularized the x_1 direction, which is to be expected. Also there is one other compact object in the picture: the line solitons of (gKdV) $\bar{Q}(t,x) = Q_{1d}(x_1 - t)$, where $Q_{1d} \in H^1(\mathbb{R})$ solves $-Q'' + Q - Q^p = 0$. Of course, this line soliton is not in $H^1(\mathbb{R})$, but we see that (4.5) also allows to set it aside.

Due to continuity of the flow, (4.5) holds when the supremum is taken over a compact interval of time. In view of the local well posedness result, it is clear that a solution which satisfies (4.4) on all its interval of definition $(T_-(u), T_+(u))$ is in fact defined globally for all $t \in \mathbb{R}$. Therefore, what only matter is the behaviour at $t \to \pm\infty$.
However, it is important to assume L^2 compactness at *both ends $t \to +\infty$ and $t \to -\infty$*. Indeed, in the L^2 supercritical case, recall that there exists a *family* of solutions $(U_a)_{a\in\mathbb{R}}$ to, say, (gKdV) such that $\|U_a(t) - Q[1,0](t)\|_{H^1} \to 0$ as $t \to +\infty$. Any U_a thus satisfies the 1 dimensional equivalent of (4.5), but is *not* a soliton for $a \neq 0$.

Ideas of proof

The first step is to prove regularity and decay properties on compact solutions u.
As u remains close to Q, it enjoys some monotonicity properties: for example

Lemma 4.2. *Let $L > 0$ and define the smooth cut-off $\psi_M(y) = \arg\tanh(e^{y/L}$ (for $y \in \mathbb{R}$) and*

$$I_{t_0,y_0}(t) = \int u(t,x)^2 \psi_M\left(x_1 - \rho_1(t_0) + \frac{1}{2}(t - t_0) - y_0\right) dx.$$

Then for $\varepsilon_0 > 0$ small enough and M large enough,

$$\forall (t_0, y_0),\ \forall t \leqslant t_0, \quad I_{t_0,y_0}(t_0) - I_{t_0,y_0}(t) \leqslant Ce^{-y_0/M}. \tag{4.8}$$

(The proof consists in actually controlling the terms in the computation of the time derivative of I_{t_0,y_0}). Pluging in the information of L^2 compactness (4.5), one derives from there some exponential decay

$$\int u(t_0,x)^2 \psi_M(x_1 - \rho_1(t_0) - y_0)dx + \int_{-\infty}^{t_0} (|\nabla u|^2 + u^2)(t,x)\psi'_m\left(x_1 - \rho_1(t_0) + \frac{1}{2}(t - t_0) - y_0\right) dxdt \leqslant Ce^{-y_0/M}.$$

Observe the derivative on u in the space time integral: from there an induction argument (using also (4.5)) actually shows that u is smooth (bounded in $\mathscr{C}(\mathbb{R}, H^k)$ for all k) and decaying in the sense that for any multi-index $\alpha \in \mathbb{N}^d$,

$$\int (\partial^\alpha u)(t_0, x)^2 \psi_M(x_1 - \rho_1(t_0) - y_0) dx \\ + \int_{-\infty}^{t_0} (|\nabla \partial^\alpha u|^2 + (\partial^\alpha u)^2)(t,x) \psi_m' \left(x_1 - \rho_1(t_0) + \frac{1}{2}(t - t_0) - y_0 \right) dxdt \leqslant C_\alpha e^{-y_0/M}.$$

Then one can reformulate this decay in a more manageable way as follows: for any multi-index $\alpha \in \mathbb{N}^d$,

$$\forall (t, x_1) \in \mathbb{R}^2, \quad \int_{x_2} (\partial^\alpha u)^2(t, x_1 + \rho_1(t), x_2 + \rho_2(t)) dx_2 \leqslant C_\alpha e^{-\sigma|x_1|}. \tag{4.9}$$

The proofs of the results above are done in an analoguous way as for (gKdV), with the important technial proviso in the induction argument that H^1 does not embed in L^∞ anymore in dimension $d \geqslant 2$: we have to rely on suitably localized Sobolev embedding (and $\dot{H}^1$ subcriticallity of the nonlinearity) to control the higher order and nonlinear terms.

Now that we gained regularity and decay on u we can start the dynamical analysis. As $u(t)$ remains close to the soliton, we work on the error term, in a perturbative setting: let us write $u(t) = Q[1, \rho(t)](t) + \eta(t)$, so that the equation on η writes

$$\partial_t \eta = \partial_{x_1} L\eta + O(\|\eta\|^2 + |\dot{\rho} - (1, 0 \ldots, 0)|).$$

Most efforts are concentrated to understand the Liouville rigidity property at the linear level. This is the content of the following statement.

Theorem 4.3 (Linear Liouville property around Q, C., Muñoz, Pilod, Simpson [11])**.** *Assume that the spectral assumption* (4.3) *holds. Let $\sigma > 0$ and $\eta \in \mathscr{C}^\infty(\mathbb{R}^{1+d})$ be a solution to*

$$\partial_t \eta = \partial_{x_1} L\eta \quad \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}^d, \tag{4.10}$$

such that

$$\forall (t, x_1) \in \mathbb{R}^2, \quad \int_{x_2} \eta^2(t, x_1, x_2) dx_2 \leqslant Ce^{-\sigma|x_1|}. \tag{4.11}$$

Then, there exists $a \in \mathbb{R}^d$ such that

$$\forall (t, x) \in \mathbb{R}^{1+d}, \quad \eta(t, x) = a \cdot \nabla Q(x). \tag{4.12}$$

Observe that the non dispersion bound (4.11) and the regularity of η is guaranteed by (4.9). However one could weaken these assumptions and prove smoothness and decay as in the nonlinear case.
The starting point to prove Theorem 4.3 is to consider a dual problem by defining

$$v = L\eta - \alpha_0 Q, \quad \text{where} \quad \alpha_0 = \frac{\langle \eta(0) | \Lambda Q \rangle}{\langle Q | \Lambda Q \rangle}.$$

Then v satisfies *more* orthogonality conditions: for all $t \in \mathbb{R}$,

$$\forall i = 1, \ldots, d, \quad \langle v(t) \mid \partial_{x_i} Q \rangle = 0, \quad \langle v \mid \Lambda Q \rangle = 0. \tag{4.13}$$

Also v satisfies (4.11). The crux of the argument derives from a Virial identity which we describe now. Let $\phi \in C^2(\mathbb{R})$ be an even positive function such that $\phi' \leqslant 0$ on $[0, +\infty)$,

$$\phi(x_1) = \begin{cases} 1 & \text{if } |x_1| \leqslant 1, \\ e^{-|x_1|} & \text{if } |x_1| \geqslant 2 \end{cases}, \quad \text{and for all } x_1 \in \mathbb{R}, \tag{4.14}$$

$$e^{-|x_1|} \leqslant \phi(x_1) \leqslant 3e^{-|x_1|}, \quad |\phi'(x_1)| \leqslant C\phi(x_1) \quad \text{and} \quad |\phi''(x_1)| \leqslant C\phi(x_1). \tag{4.15}$$

Then we define $\varphi(x_1) = \int_0^{x_1} \phi$ and the rescaling for $A > 0$

$$\varphi_A(x_1) := A\varphi\left(\frac{x}{A}\right), \quad \phi_A(x_1) = \phi\left(\frac{x}{A}\right) = \varphi_A'(x_1).$$

The crucial Virial estimate is the following bound from below:

$$-\frac{1}{2}\frac{d}{dt}\int \varphi_A(x_1)v^2(t,x)dx \\ \geqslant \int \phi_A(x_1)\left((\partial_{x_1}v(t,x))^2 + |\nabla v(t,x)|^2 + v(t,x)^2 - pQ(x)^{p-1}v(t,x)^2\right)dx\,. \quad (4.16)$$

From the spectral assumption (4.3), standard localization arguments (and the orthogonality conditions (4.13)) show that, if A is large enough

$$\forall t, \quad \int \phi_A(x_1)\left(|\nabla v(t,x)|^2 + v(t,x)^2 - pQ(x)^{p-1}v(t,x)^2\right)dx \geqslant \frac{\lambda}{2}\|v(t)\|_{H^1}^2.$$

For this coercivity to hold, the introduction of the dual problem, which gives the orthogonality conditions with $\partial_i Q$, and the choice of the modulation parameter α_0 are crucial.
In particular, after integrating (4.16), we get for any $T_-, T_+ \in \mathbb{R}$, $T_- < T_+$,

$$\int_{T_-}^{T_+}\int \phi_A(x)v(t,x)^2dxdt \leqslant C(A)(\|v(T_+)\|_{L^2}^2 + \|v(T_-)\|_{L^2}^2 \leqslant C(A). \quad (4.17)$$

Therefore, letting $T_- \to -\infty$ and $T_+ \to +\infty$,

$$\int_{-\infty}^{\infty}\int \phi_A(x)v(t,x)^2dxdt < +\infty,$$

and in particular there exist two sequences $(t_{n,A}^{\pm})_n$ such that $t_{n,A}^{\pm} \to \pm\infty$ and

$$\int \phi_A(x)v(t_{n,A}^{\pm},x)^2dx \to 0 \quad \text{as} \quad n \to +\infty.$$

Here we used that v is defined globally on $\mathbb{R}$. This is true for all A large enough: as v satisfies the same exponential localization as η (see (4.11)), we infer that

$$\|v(t_{n,A}^{\pm})\|_{L^2} \to 0 \quad \text{as} \quad n \to +\infty.$$

Using again (4.17) with $T_- = t_{n,A}^-$ and $T_+ = t_{n,A}^+$, we conclude that

$$\int_{-\infty}^{\infty} \phi_A(x)v(t,x)^2dxdt = 0.$$

As a consequence, $v \equiv 0$, and then (4.12) follows easily.

To prove the nonlinear version of the Liouville theorem, we can not use its linear counterpart as a black box, due to the rigid nature of the theorem. However, the method of proof is robust, and carries over very well to the nonlinear case. The additional ingredient is the modulation of the scaling to make up for α_0 and recover all the orthogonality conditions (4.13). By a careful inspection, we control all the nonlinear terms appearing in the Virial identity (4.16), using in particular (4.9).

2 Asymptotic stability

With the nonlinear Liouville theorem in hand, we can prove that the solitary waves of the ZK equation (ZK) are asymptotically stable.

Theorem 4.4 (Asymptotic stability)**.** *Assume that assumption (A) is true. There exists $\varepsilon_0 > 0$ such that if $u \in C(\mathbb{R}, H^1(\mathbb{R}^d))$ is a solution of* (ZK) *satisfying*

$$\forall t \geqslant 0, \quad \inf_{y_0 \in R^d} \|u(t) - Q(\cdot - y_0)\|_{H^1} \leqslant \varepsilon_0, \quad (4.18)$$

then the following holds true. There exist two functions $\rho \in C^1(\mathbb{R}, \mathbb{R}^d)$ and $c \in C^1(\mathbb{R}, \mathbb{R})$ such that

$$u(t) - Q_{c(t)}(\cdot - \rho(t)) \to 0 \quad \text{in } H^1(x_1 > \frac{1}{2}t) \quad \text{as} \quad t \to +\infty. \tag{4.19}$$

Let us start with a few remarks. First, the convergence in the asymptotic statement can not hold in the whole space $H^1(\mathbb{R}^d)$, because of conservation of mass and energy. There must be some loss, which can be due to small (and slow) solitons, or to dispersion. One way to get rid of them is to use weighted space; our analysis is here sharper, as we use local spaces *without weights*.
Observe also that the orbital stability statement is somehow unrelated to the asymptotic stability result stated above. Of course, in the L^2 subcritical case, one could replace (4.18) with a similar hypothesis for the solution at initial time $t = 0$ only (invoking orbital stability in this case). However, theorem 4.4 holds true even in the L^2 supercritical case (up to assumption (A)), because it only considers solutions which we assume remain close to the soliton for all times. The picture in the L^2 supercritical is that there should exist a center stable manifold of finite codimension (probably 1), on which one would observe asymptotic stability.

The proof of the asymptotic stability derives from the Liouvillle theorems, as in the case of (gKdV) (for that equation, see [88, 90] for the asymptotic stability, and [85, 86] for the Liouville theorems). The idea is to consider any sequence $t_n \to +\infty$, a weak limit u_∞ of $u(t_n)$, and the nonlinear solution $\tilde{u}$ with data $\tilde{u}(0) = u_\infty$. The main point of looking at objects at infinity is that these enjoy more properties: in some sense this procedure allows to eliminate dispersion. More precisely, as a consequence of monotonicity properties, $\tilde{u}$ in fact satisfies (4.5). From the Liouville theorems, $\tilde{u}$ is a perfect soliton, and as the limit does not depend on the sequence t_n, we obtain a weak limit for all times.
Finally we need to recover strong local convergence. For this, we actually need a stronger version of the monotonicity, with the cut-off function moving directions in a slighted opened cone around the x_1 direction: more precisely the quantity

$$I_{y_0,t_0,\theta_0}(t) = \int u(t,x)^2 \psi_M \left(x_1 + \theta_0 \cdot x_2 - \rho_1(t_0) + \frac{1}{2}(t - t_0) - y_0 \right) dx$$

(where $\theta_0 \in \mathbb{R}^{d-1}$ is small) satisfies the same monotonicity bounds as (4.8), and one can conclude from there.

No monotonicity property seems to hold for the x_2 direction, mainly because of the conjectured existence of trains of small solitons moving to the left in x_1 but without restrictions on the x_2 coordinate. From the point of view associated to the x_2 variable, such solutions represent movement of mass along the x_2 direction without a privileged dynamics. In particular, no asymptotic stability result is expected for a half-plane involving the x_2 variable only. This is the standard situation in many $2d$ models like (KP-I) and (NLS) equations. However, here we are able to prove the asymptotic stability of (ZK) solitons because the (gKdV) dynamics is exactly enough to control the movement of mass along the x_2 direction.

3 Multi-solitons

Finally, as a consequence of the monotonicity properties associated to the linear part of the dynamics (in particular Lemma 4.2), we are able to prove that decoupled solitons are stable in the sense below. First let us make precise what decoupled mean in this context.

Definition 4.5. Let $N \geqslant 2$ be an integer and $L \geqslant 0$. Consider N solitons with scalings $c_1^0, \dots, c_N^0 > 0$ and centers $\rho^{1,0}, \dots, \rho^{N,0} \in \mathbb{R}^d$, where $\rho^{j,0} = (\rho_1^{j,0}, \rho_2^{j,0})$. We say that these N solitons are L-decoupled if

$$\inf \left\{ |((c_k^0 - c_j^0)t, 0, \dots, 0) + \rho_k^0 - \rho_j^0| \mid j \neq k, \ t \geqslant 0 \right\} \geqslant L, \tag{4.20}$$

that is, the solitons centers remains separated by a distance of at least L for positive times.

L decoupled solitons can be characterized by a condition on the initial data only, at least up to a constant in L: indeed, one can check that if, for all $j \neq k$, we have either:

- $|\rho_2^{j,0} - \rho_2^{k,0}| \geqslant L$, or
- $c_k^0 > c_j^0$ and $\rho_1^{k,0} - \rho_1^{j,0} \geqslant L$,

then the N solitons are L decoupled.

Theorem 4.6 (Stability of the sum of N decoupled solitons)**.** *Assume $d = 2$. Consider a set of N solitons of the form*

$$Q_{c_1^0}(x - \rho^{1,0}),\ Q_{c_2^0}(x - \rho^{2,0}), \ldots,\ Q_{c_N^0}(x - \rho^{N,0}),$$

where each c_j^0 is a fixed positive scaling, $c_j^0 \neq c_k^0$ for all $j \neq k$, and $\rho^{j,0} = (\rho_1^{j,0}, \rho_2^{j,0}) \in \mathbb{R}^d$. Assume that the N solitons are L-decoupled, in the sense of Definition 4.5.
Then there are $\varepsilon_0 > 0$, $C_0 > 0$ and $L_0 > 0$ depending on the previous parameters such that, for all $\varepsilon \in (0, \varepsilon_0)$, and for every $L > L_0$, the following holds. Suppose that $u_0 \in H^1(\mathbb{R}^d)$ satisfies

$$\|u_0 - \sum_{j=1}^{N} Q_{c_j^0}(x - \rho^{j,0})\|_{H^1} < \varepsilon. \tag{4.21}$$

Then there are $\gamma_1 > 0$ fixed and $\rho^j(t) \in \mathbb{R}^d$ defined for all $t \geqslant 0$ such that $u(t)$, solution of (ZK) *with initial data $u(0) = u_0$ satisfies*

$$\sup_{t \geqslant 0} \|u(t) - \sum_{j=1}^{N} Q_{c_j^0}(x - \rho^j(t))\|_{H^1(\mathbb{R}^d)} < C_0(\varepsilon + e^{-\gamma_1 L}). \tag{4.22}$$

The proof of this result is obtained by adapting the ideas by Martel, Merle and Tsai [96] for(gKdV) (which is 1D). Note that we do not need strictly well-prepared initial data as in [96]: all we need is to ensure that the solitons do not collide for positive times, and the x_2 variables give us some more room than in the (gKdV) case.

Let us give a very short account of the proof. First by continuity of the flow, we can in fact assume that all solitons are well prepared, i.e. decoupled in the x_1 variable and arranged by increasing speed as x_1 grows. Then the arguments are wrapped in a bootstrap: we consider the maximal time $[0, T_*)$ on which (4.22) holds. For any time t on this interval, we can modulate the solution, decomposing

$$u(t) = \sum_{j=1}^{N} Q[c_j(t), \rho_j(t)](0) + z(t), \quad \text{with} \quad \langle z, Q[c_j(t), \rho_j(t)](0)\rangle = \langle z, \partial_{x_i} Q[c_j(t), \rho_j(t)](0)\rangle = 0,$$

for all $i = 1, \ldots, d$, $j = 1, \ldots, N$ (recall we are in the L^2 subcritical setting).
Define the localized mass

$$M_j(t) = \int u(t,x)^2 \psi_M \left(x_1 - \frac{c_j^0 + c_{j-1}^0}{2} t \right) dx, \quad \text{where} \quad \psi_j(t, x_1) = \psi_M \left(x_1 - \frac{c_j^0 + c_{j-1}^0}{2} t \right).$$

A monotonicity result in the spirit of Lemma 4.2 yields

$$M_j(t) - M_j(0) \leqslant C e^{-\gamma_1 L}.$$

Also, we can expand M_j:

$$M_j(t) = \|Q\|_{L^2}^2 \sum_{k=j}^{N} c_k(t)^{\frac{2}{p-1} - \frac{d}{2}} + \int z(t,x)^2 \psi_j(t, x_1) dx + O(e^{-2\gamma_1 t}).$$

Therefore, as we are in the L^2 subcritical case, $\frac{2}{p-1} - \frac{d}{2} > 1$, so that we can linearize around $c_k(0)$ and get for all $j = 1, \dots, N$

$$\max\left(0, \sum_{k=j}^{N} c_k(t) - c_k(0)\right) \leqslant C\|z(0)\|_{H^1}^2 + Ce^{-\gamma_1 L}.$$

(Here the implicit constant depend on the c_j^0). We can expand the energy in the same way: using again that $\langle z(t), Q[c_j(t), \rho_j(t)](0)\rangle = 0$, we have

$$E(u(t)) = E(Q)\sum_{j=1}^{N} c_j(t)^{\frac{2}{p-1}-\frac{d}{2}-1} + O(\|z\|_{H^1}^2).$$

Using conservation of the energy, we have obtained $N+1$ constraints on the N scaling parameters c_j. By a convexity argument, we infer that the variation of each c_j is *quadratic* in z. This allows to close the bootstrap assumption and conclude that $T_* = +\infty$.

CHAPTER 5

Formation of Néel walls

THIS CHAPTER studies some aspects of micromagnetism, more precisely, the formation of two-dimensional ferromagnetic thin films allowing the occurrence and persistence of special transition layers called Néel walls.

1 A two-dimensional model for thin-film micromagnetism

The micromagnetic energy

We consider here magnetization, which is a vector field

$$m : \Omega \to \mathbb{S}^2, \quad \text{where} \quad \Omega = (-1,1) \times \mathbb{T}.$$

The coordinates on Ω are denoted $x = (x_1, x_2)$, and we impose periodicity in the x_2-direction in order to rule out lateral surface charges. We denote the coordinates of $m = (m_1, m_2, m_3)$ and it will be convenient to use the notation $m' = (m_1, m_2)$.
We will consider the following micromagnetic energy approximation in a thin-film regime that is written in the absence of crystalline anisotropy and external magnetic fields (see e.g. [37], [71]):

$$E_{\delta,\varepsilon}(m) = \int_\Omega \left(|\nabla m|^2 + \frac{1}{\varepsilon^2} m_3^2 + \frac{1}{\delta} ||\nabla|^{-1/2} \nabla \cdot m'|^2 \right) dx, \tag{5.1}$$

where $\delta > 0$ and $\varepsilon = \varepsilon(\delta) > 0$ are two small parameters. To force the emergence of singular patterns as $\varepsilon, \delta \to 0$, we consider magnetization m that connects two macroscopic directions forming an angle, i.e., for a fixed $m_{1,\infty} \in [0,1)$,

$$m(x_1, x_2) = \begin{cases} m_{-\infty} & \text{for } x_1 = -1, \\ m_{+\infty} & \text{for } x_1 = 1, \end{cases} \quad \text{where} \quad m_{\pm\infty} = \begin{pmatrix} m_{1,\infty} \\ \pm\sqrt{1 - m_{1,\infty}^2} \\ 0 \end{pmatrix}. \tag{5.2}$$

The first term in (5.1) is called the exchange energy, while the other two terms stand for the stray field energy created by the surface charges m_3 at the top and bottom of the sample and by the volume charges $\nabla \cdot m'$ in the interior of the sample.

Let us focus on the last term in (5.1) $||\nabla|^{-1/2} \nabla \cdot m'|^2$. First let us give a precise definition of it. For this, we introduce the functional calculus derived from the Laplace operator on Ω with Dirichlet boundary conditions, as it fits the boundary conditions (5.2). More precisely, for $f \in H^{-1}(\Omega)$, we define $g := (-\Delta)^{-1} f$ as the solution of

$$\begin{cases} -\Delta g = f & \text{in } \Omega, \\ g(x_1, x_2) = 0 & \text{on } \partial\Omega, \text{ i.e., for } |x_1| = 1, x_2 \in \mathbb{T}. \end{cases} \tag{5.3}$$

Then $(-\Delta)^{-1}$ is a bounded operator $H^{-1}(\Omega) \to H^1_0(\Omega)$ and a compact self-adjoint operator $L^2(\Omega) \to L^2(\Omega)$. We can therefore construct a functional calculus based on it, and denote as usual $|\nabla|^{-2s} := \left[(-\Delta)^{-1}\right]^s$ for $s \in \mathbb{R}$: this is makes the definition of the energy $E_{\delta,\varepsilon}(m)$ (5.1) meaningful.
Let us now explain where the term $||\nabla|^{-1/2}\nabla \cdot m'|^2$ comes from. We introduce the stray field $h(m') : \mathbb{R} \times \mathbb{T} \times \mathbb{R} \to \mathbb{R}^3$ generated by the volume charges only, and defined as the unique $L^2(\mathbb{R} \times \mathbb{T} \times \mathbb{R}, \mathbb{R}^3)$ gradient field

$$h(m') = \nabla_{x,z} U(m') \tag{5.4}$$

that is x_2-periodic (the third variable is denoted $z \in \mathbb{R}$), where $U(m')$ is the harmonic extension on $\Omega \times \mathbb{R}$ with Neumann data $\nabla m'$ on $\Omega \times \{0\}$ and Dirichlet boundary condition on $\partial\Omega \times \mathbb{R}$:

$$\begin{cases} -\Delta_{x,z} U(m') = 0 & \text{in } \Omega \times \mathbb{R} \\ \partial_z U(m') = \nabla \cdot m' & \text{on } \Omega \times \{0\} \\ U(m') = 0 & \text{on } \partial\Omega \times \mathbb{R}. \end{cases} \tag{5.5}$$

Equivalently, $U(m')$ satisfies Maxwell's static equation in the weak sense:

$$\forall \zeta \in \mathscr{D}(\mathbb{R} \times \mathbb{T} \times \mathbb{R}), \quad \int_{\Omega\times\mathbb{R}} \nabla_{x,z} U(m') \cdot \nabla_{x,z}\zeta dxdz = \int_\Omega m'(x) \cdot \nabla\zeta(x,0) dx. \tag{5.6}$$

In other words, $h(m')$ is the Helmholtz projection of the vector measure $m'\mathscr{H}^2 \llcorner \Omega \times \{0\}$ onto the $L^2(\Omega \times \mathbb{R})$-space of gradient fields.
One can see (5.5) as an evolution equation in the z variable, involving the operator $(-\Delta)$ of the functional calculus introduced above (due to the boundary conditions). From the initial condition at $z = 0$ and $U(m') \in H^1(\Omega \times \mathbb{R})$, we obtain an explicit formulation for $U(m')$:

$$U(m')(\cdot, z) = -\frac{\exp(-z|\nabla|)}{|\nabla|}(\nabla \cdot m').$$

Therefore

$$U(m')(\cdot, 0) = -|\nabla|^{-1}(\nabla \cdot m').$$

We can now relate $h(m')$ to $E_{\delta,\varepsilon}(m)$. Indeed, by (5.6) and density,

$$\begin{aligned} \int_{\Omega\times\mathbb{R}} |h(m')|^2 dxdz &= \int_{\Omega\times\mathbb{R}} \nabla_{x,z} U(m') \nabla_{x,z} U(m') dxdz = \int_\Omega m'(x) \cdot \nabla U(m')(x,0) dx \\ &= -\int_\Omega (\nabla \cdot m'(x)) U(m')(x,0) dx = \int_\Omega (\nabla \cdot m')|\nabla|^{-1}(\nabla \cdot m') dx \\ &= \int_\Omega ||\nabla|^{-1/2}\nabla \cdot m'|^2 dx. \end{aligned}$$

Hence, the last term in the energy (5.1) is simply the potential energy generated by the stray field $h(m')$. We also refer to [64] for another presentation via Fourier Transform in the case $\Omega = \mathbb{R} \times \mathbb{T}$.

Finally we introduce the operator

$$\mathscr{P}(m') := -\nabla|\nabla|^{-1}\nabla \cdot m', \tag{5.7}$$

which is bounded $H^1(\Omega) \to L^2(\Omega)$, and the gradient of the energy $E_{\delta,\varepsilon}(m)$ is therefore given by

$$\nabla E_{\delta,\varepsilon}(m) = -2\Delta m + \left(-\frac{1}{\delta}\mathscr{P}(m'), \frac{2m_3}{\varepsilon^2}\right). \tag{5.8}$$

Observe that $m : \Omega \to \mathbb{S}^2$ has finite energy $E_{\delta,\varepsilon}(m) < +\infty$ if and only if $m \in H^1(\Omega, \mathbb{S}^2)$. Of course, an important (and interesting) mathematical feature is that $E_{\delta,\varepsilon}$ is a nonlocal energy, due to the operator $\mathscr{P}$.

Néel walls and vortices

In this model, we expect two types of singular pattern: Néel walls and vortices (so-called Bloch lines in the micromagnetic jargon). These patterns result from the competition between the different contributions in the total energy $E_\delta(m)$ and the nonconvex constraint $|m| = 1$.

The Néel wall is a dominant transition layer in thin ferromagnetic films. It is characterized by a one-dimensional in-plane rotation connecting two directions (5.2) of the magnetization. More precisely, it is a one-dimensional transition $m = (m_1, m_2) : [-1, 1] \to \mathbb{S}^1$ that minimizes the energy under the boundary constraint (5.2):

$$E_\delta(m) = \int_{\mathbb{R}} \left| \frac{dm}{dx_1} \right|^2 dx_1 + \frac{1}{2\delta} \int_{\mathbb{R}} \left| \left| \frac{d}{dx_1} \right|^{1/2} m_1 \right|^2 dx_1.$$

It follows that the minimizer is a two length scale object: it has a small core with fast varying rotation (transition of size $O(\delta)$) and two logarithmically decaying tails (of length size $O(1)$). The energetic cost (by unit length) of a Néel wall is given by

$$E_\delta(\text{Néel wall}) = \frac{\pi}{2\delta |\log \delta|} (1 - m_{1,\infty})^2 (1 + o(1)) \quad \text{as} \quad \delta \to 0. \tag{5.9}$$

We refer to DeSimone, Kohn, Müller and Otto [38]) and Ignat [63] for more details.

A vortex point corresponds in our model to a topological singularity at the microscopic level where the magnetization points out-of-plane. The prototype of a vortex configuration is given by a vector field $m : \mathbb{D} \to \mathbb{S}^2$ defined in a unit disk $\mathbb{D}$ that satisfies:

$$\nabla \cdot m' = 0 \quad \text{in } \mathbb{D} \quad \text{and} \quad m'(x) = x^\perp := (-x_2, x_1) \quad \text{on } \partial\mathbb{D},$$

and minimizes the energy (5.1):

$$E_\varepsilon(m) = \int_{\mathbb{D}} |\nabla m|^2 \, dx + \frac{1}{\varepsilon^2} \int_{\mathbb{D}} m_3^2 \, dx.$$

Since the magnetization turns in-plane at the boundary of the disk $\mathbb{D}$ (so that $\deg(m', \partial\mathbb{D}) = 1$), a localized region is created, that is the core of the vortex of size $O(\varepsilon)$, where the magnetization becomes indeed perpendicular to the horizontal plane. Observe that the reduced energy E_δ controls the Ginzburg-Landau energy, i.e.,

$$E_\varepsilon(m) \geqslant E_{GL,\varepsilon} := \int_{\mathbb{D}} \left(|\nabla m'|^2 + \frac{1}{\varepsilon^2} (1 - |m'|^2)^2 \right) dx \tag{5.10}$$

since $|\nabla(m', m_3)|^2 \geqslant |\nabla m'|^2$ and $m_3^2 \geqslant m_3^4 = (1 - |m'|^2)^2$. Due to the similarity with vortex points in Ginzburg-Landau type functionals, the energetic cost of a micromagnetic vortex is given by

$$E_\varepsilon(\text{Vortex}) = 2\pi |\log \varepsilon| + O(1).$$

(See e.g. the seminal book by Bethuel, Brezis and Hélein [20]).

Physically relevant regime requires $\delta \to 0$; also we will focus on an energetic regime allowing Néel walls, but excluding vortices. More precisely, we will assume that ε is related to δ as follows:

$$\delta \to 0 \quad \text{and} \quad \varepsilon = \varepsilon(\delta) \to 0 \quad \text{such that} \quad \frac{1}{\delta |\log \delta|} = o\left(|\log \varepsilon|\right) \tag{5.11}$$

and we will consider families of magnetization $\{m_\delta\}_{0<\delta<1/2}$ satisfying the energy bound

$$\sup_{\delta \to 0} \delta |\log \delta| E_{\delta,\varepsilon}(m_\delta) < +\infty. \tag{5.12}$$

In particular, (5.11) implies that the size ε of the vortex core is exponentially smaller than the size of the Néel wall core δ, i.e., $\varepsilon = O(e^{-\frac{1}{\delta |\log \delta|}})$.

2 Compactness and optimality of Néel walls

This section is devoted to the analysis of static magnetizations as the parameters $\delta, \varepsilon \to 0$.

Compactness of Néel walls

The first result is that the energetic regime (5.12) is indeed favorable to the formation of Néel walls: more precisely, we have the following compactness result.

Theorem 5.1 ([2, Theorem 1]). *Let $\delta > 0$ and $\varepsilon(\delta) > 0$ satisfy the regime* (5.11). *Let $m_\delta \in H^1(\Omega, \mathbb{S}^2)$ satisfies* (5.2) *and* (5.12). *Then $\{m_\delta\}_{\delta\to 0}$ is relatively compact in $L^2(\Omega)$ and any limit $m : \Omega \to \mathbb{S}^2$ satisfies the constraints* (5.2) *and*

$$|m'| = 1, \quad m_3 = 0, \quad \nabla \cdot m' = 0 \ \ in \ \mathscr{D}'(\Omega).$$

This result and its proof are in the spirit of the compactness results of Ignat and Otto [65, 66]. The proof of compactness is based on an argument of approximating $\mathbb{S}^2$-valued magnetizations by $\mathbb{S}^1$-valued magnetizations having the same level of energy:

Proposition 5.2 ([2, Theorem 5]). *Let $\beta \in (0,1)$. Let $\delta > 0$ and $\varepsilon(\delta) > 0$ satisfy the regime* (5.11) *and let $m_\delta = (m'_\delta, m_{3,\delta}) \in H^1_{\mathrm{loc}}(\Omega, \mathbb{S}^2)$ satisfy* (5.2) *and* (5.12). *Then there exists an $\mathbb{S}^1$ valued magnetization $M_\delta \in H^1_{\mathrm{loc}}(\Omega, \mathbb{S}^1)$ that satisfies the boundary conditions* (5.2) *and*

1. $\|M_\delta - m'_\delta\|^2_{L^2(\Omega)} \leqslant C\varepsilon^{2\beta} E_{\delta,\varepsilon}(m_\delta)$ *and* $\|\nabla(M_\delta - m'_\delta)\|^2_{L^2(\Omega)} \leqslant C E_{\delta,\varepsilon}(m_\delta)$,
2. $\||\nabla|^{-1/2}\nabla \cdot M - |\nabla|^{-1/2}\nabla \cdot m'\|^2_{L^2(\Omega)} \leqslant C\varepsilon^\beta E_{\delta,\varepsilon}(m_\delta)$,
3. $E_{\delta,\varepsilon}(M_\delta) \leqslant E_{\delta,\varepsilon}(m_\delta) \left(1 + \left(\dfrac{C}{\delta|\log\delta||\log\varepsilon|}\right)^{\frac{1}{6}^-}\right).$

($\frac{1}{6}^-$ is any fixed positive number less than $\frac{1}{6}$). Moreover, for every full square $T(x,r)$ centered at x of side of length $2r$ with $\varepsilon^\beta/r \to 0$ as $\delta \to 0$, we have

$$\int_{T(x,r-2\varepsilon^\beta)} |\nabla M_\delta|^2\,dx \leqslant \left(1 + \left(\frac{C}{\delta|\log\delta||\log\varepsilon|}\right)^{\frac{1}{6}^-}\right) \int_{T(x,r)} \left(|\nabla m'_\delta|^2 + \frac{1}{\varepsilon^2} m_{3,\delta}^2\right) dx. \quad (5.13)$$

Proposition 5.2 is reminiscent of the argument developed by Ignat and Otto [65] with the improvement given by 3., i.e., the approximating $\mathbb{S}^1$-map M_δ has lower energy than the $\mathbb{S}^2$-map m_δ (up to $o(1)$ error).
Let us emphasize that such an approximation is possible due to our regime (5.11) and (5.12) which excludes the existence of topological point defects.

To prove Proposition 5.2, we divide the rectangle $[-1,1] \times [0,1]$ by a square grid with cell of size ε^β. A key observation is that for any cell C of the grid, $\deg(m', \partial C) = 0$, and so we can consider the minimizer u of the Ginzburg-Landau energy on C such that $u = m'$ on ∂C. Due to the vanishing of the degree of m' on ∂C, $|u| \geqslant 1/2$ on C, and we are entitled to define $M' = u/|u|$ on C. Gluing this construction for all cells of the grid, we define a magnetization M': we check that it satisfies the conditions of Proposition 5.2.

Once this step is done, one can invoke the result [66, Theorem 4] by Ignat and Otto, which gives the desired compactness on M'_δ and therefore on m_δ. The main point there is to show that the constraint $|M'_\delta| = 1$ passes to the limit.

Optimality of the Néel wall

The second result is the optimality of the Néel wall, namely that the Néel wall is the unique asymptotic minimizer of $E_{\delta,\varepsilon}$ over $\mathbb{S}^2$-magnetizations satisfying the boundary condition (5.2).

For every magnetization $m : \Omega \to \mathbb{S}^2$, we associate the energy density $\mu_\delta(m)$ as a non-negative x_2-periodic measure on $\Omega \times \mathbb{R}$ via

$$\forall \zeta = \zeta(x,z) \in \mathscr{D}(\Omega \times \mathbb{R}), \tag{5.14}$$
$$\int_{\Omega\times\mathbb{R}} \zeta d\mu_\delta(m) := \frac{2}{\pi}\delta|\log\delta|\Big(\int_\Omega \zeta(x,0)\big(|\nabla m|^2 + \frac{1}{\epsilon^2}m_3^2\big)\,dx + \frac{1}{\delta}\int_{\Omega\times\mathscr{R}} \zeta|h(m')|^2\,dxdz\Big).$$

(We recall that the stray field $h(m')$ was defined in (5.4) and (5.5)-(5.6)). We now show that the straight walls (5.16) are the unique minimizers of E_δ as $\delta \to 0$ in which case the energy density μ_δ is concentrated on a straight line in x_2-direction.

Theorem 5.3 ([2]). *Let $\delta > 0$ and $\epsilon(\delta) > 0$ satisfy the regime* (5.11). *Let $m_\delta \in H^1_{\text{loc}}(\Omega, \mathbb{S}^2)$ satisfy* (5.2) *and*

$$\limsup_{\delta\to 0} \delta|\log\delta|E_\delta(m_\delta) \le \frac{\pi}{2}(1 - m_{1,\infty})^2. \tag{5.15}$$

Then there exists a subsequence $\delta_n \to 0$ such that $m_{\delta_n} \to m^*$ in $L^2(\Omega)$ where m^* is a straight wall given by

$$m^*(x_1, x_2) = \begin{cases} m_{-\infty} & \text{for } x_1 < x_1^*, \\ m_{+\infty} & \text{for } x_1 > x_1^*, \end{cases} \quad \text{for some } x_1^* \in [-1,1]. \tag{5.16}$$

In this case, the measures defined at (5.14) *concentrate on the line:*

$$\mu_{\delta_n}(m_{\delta_n}) \rightharpoonup (1 - m_{1,\infty})^2 \mathscr{H}^1 \llcorner \{x_1^*\} \times \mathbb{T} \times \{0\} \quad * -\text{weakly in } \mathscr{M}(\Omega \times \mathbb{R}).$$

The energy bound (5.15) is relevant for Néel walls in view of (5.9). Theorem 5.3 extends to $\mathbb{S}^2$-valued magnetizations the similar result previously proved by Ignat and Otto [66] in the case of $\mathbb{S}^1$-valued magnetizations (and so the proof is similar, with the additional approximation argument allowed by Proposition 5.2 in hand). One way to reformulate it is as follows: let $m_\delta \in H^1(\Omega, \mathbb{S}^2)$ satisfy (5.2), then

$$\liminf_{\delta\to 0}\ \delta|\log\delta|E_{\delta,\epsilon}(m_\delta) \geqslant \frac{\pi}{2}(1 - m_{1,\infty})^2. \tag{5.17}$$

3 Formation of static Néel walls

The Landau-Lifshitz-Gilbert equation

The dynamics in ferromagnetism is governed by a torque balance which gives rise to a damped gyromagnetic precession of the magnetization around the effective field defined through the micromagnetic energy. The resulting system is the Landau-Lifshitz-Gilbert (LLG) equation written below, which is neither a Hamiltonian system nor a gradient flow.

More precisely, the dynamics of the state of a thin ferromagnetic sample is described by the time-dependent magnetization

$$m = m(t,x) : [0, +\infty) \times \Omega \to \mathbb{S}^2,$$

that solves the following equation:

$$\partial_t m + \alpha m \wedge \partial_t m + \beta m \wedge \nabla E_{\delta,\epsilon}(m) + (v\cdot\nabla)m = m \wedge (v\cdot\nabla)m \quad \text{on } [0,+\infty)\times\Omega. \tag{LLG}$$

Here, $\wedge$ denotes the cross product in $\mathbb{R}^3$, while $\alpha > 0$ is the Gilbert damping factor characterizing the dissipation form of (LLG) and $\beta > 0$ is the gyromagnetic ratio characterizing the precession. $v : [0, +\infty) \times \Omega \to \mathbb{R}^2$ represents the direction of an applied spin-polarized current (by definition $(v \cdot \nabla)m = v_1\partial_1 m + v_2\partial_2 m$). This equation has been derived in a related setting by Zhang, Li [132] and Thiaville, Nakatani, Miltat, Suzuki [129]; we refer to Gilbert [55] and Landau, Lifshitz [78] for the original, simpler form of the equation, which does not take into account the additional drift term v.

We highlight that (5.11) and $\delta \to 0$ result in a *loss of any uniform H^1 bound*. Although the asymptotics of magnetization m_δ solutions to various forms of (LLG) were studied (see Capella, Melcher, Otto [27], Kurzke, Melcher, Moser [75], Melcher [97] for example) we must stress out that the analysis carried out in those papers were simplified in two ways when compared to our settings:

1. δ is kept fixed (and $\varepsilon \to 0$), or even $\delta \to +\infty$: in both cases the bound on the energy does give an H^1 bound. Let us also recall that physically relevant regimes require $\delta \to 0$.

2. The energy considered there does not have any nonlocal terms.

Our strategy relies on the fine qualitative behavior of the magnetization presented in Theorems 5.1 and 5.3 above.

Global weak solutions to (LLG)

Before studying the asymptotics $\delta \to 0$, we must first construct global magnetizations m_δ solutions to (LLG). We consider initial data with finite energy at $\delta > 0$ fixed. Naturally, we understand that here the boundary condition (5.2) reads as

$$m_\delta(t, x_1, x_2) = m_{\pm\infty} \quad \text{for} \quad x_1 = \pm 1,\ x_2 \in \mathbb{T}.$$

Moreover these solutions have finite energy for all time $t \geqslant 0$. We insist on the fact that the energy can possibly increase in time, unlike the case $v = 0$ which is dissipative.

Definition 5.4. We say that m is a global weak solution to (LLG) if

$$m \in L^\infty([0, +\infty), H^1(\Omega, \mathbb{S}^2)) \cap \dot{H}^1([0, +\infty), L^2(\Omega)), \tag{5.18}$$

and m solves the equation (LLG) in the distributional sense $\mathscr{D}'((0, +\infty) \times \Omega)$.

Observe that the regularity assumption (5.18) of this definition allows to make all terms in the (LLG) meaningful in the distributionnal sense: this gives its relevance to the definition. We construct global weak solutions for (LLG) in the following theorem.

Theorem 5.5 ([2, Theorem 3])**.** *Let $v \in L^\infty([0, +\infty) \times \Omega, \mathbb{R}^2)$. Let $\delta \in (0, 1/2)$ be fixed and $m^0 \in H^1(\Omega, \mathbb{S}^2)$ be an initial data.*
Then there exists a global weak solution m to (LLG) *(in the sense of the above definition), which satisfies the boundary conditions*

$$m(t, \cdot) \rightharpoonup m^0 \quad \text{in } H^1(\Omega) \quad \text{as } t \to 0, \tag{5.19}$$

$$m(t, x_1, x_2) = m^0(x_1, x_2) \quad \text{if } x_1 = \pm 1 \text{ and for every } \ x_2 \in \mathbb{T},\ t \geqslant 0. \tag{5.20}$$

Furthermore m satisfies the following energy bound: for all $t \geqslant 0$,

$$E_{\delta,\varepsilon}(m(t)) + \frac{\alpha}{2\beta} \int_0^t \|\partial_t m(s)\|^2_{L^2(\Omega)} ds \leqslant E_{\delta,\varepsilon}(m^0) \exp\left(\frac{4}{\alpha\beta} \int_0^t \|v(s)\|^2_{L^\infty(\Omega)} ds \right). \tag{5.21}$$

The proof of Theorem 5.5 takes its roots in [16] via a space discretization. To the best of our knowledge however, there is no such result taking into account the non-local term $\mathscr{P}$ in $\nabla E_{\delta,\varepsilon}$ (see (5.8)). One needs to carry on the computations carefully, specially as it comes together with the constraint of $\mathbb{S}^2$-valued map; the way we defined the operator $\mathscr{P}$ in (5.7), done in a very symmetric way is crucial here.

Stationnary Néel walls

Let us now specify our set of assumptions for the dynamics in the asymptotics $\delta, \epsilon(\delta) \to 0$:

(A1) The regime (5.11) holds as $\delta \to 0$, that is $\dfrac{1}{\delta|\log\delta|} = o(|\log\varepsilon|)$, and the parameters α and β satisfy

$$\alpha = \mu\varepsilon \quad \text{and} \quad \beta = o(\varepsilon\sqrt{\delta|\log\delta|}) \tag{5.22}$$

where $\mu > 0$ is a fixed constant.

(A2) The initial data $m_\delta^0 \in H^1(\Omega, \mathbb{S}^2)$ satisfy the boundary condition (5.2) and the energy bound (5.12) that is:

$$\sup_{\delta\to 0} \delta|\log\delta| E_{\delta,\varepsilon}(m_\delta^0) < +\infty.$$

(A3) The spin-polarized current satisfies

$$\|v_\delta\|^2_{L^\infty([0,+\infty)\times\Omega)} \leqslant \alpha\beta. \tag{5.23}$$

In particular, we have $v_\delta \to 0$ in $L^\infty([0,+\infty)\times\Omega)$.

Due to the energy estimate (5.21), the regime (A2) holds for all times $t \geqslant 0$ (with no uniformity in t though). In particular, Theorem 5.1 implies that for all $t > 0$, the magnetization $m_\delta(t)$ admits a subsequence converging in $L^2(\Omega)$ to a limiting magnetization $(m'(t), 0)$. Our main result is that the subsequence does not depend on t, and that the limiting configuration is stationary.

Theorem 5.6 ([2, Theorem 4])**.** *Let $\{m_\delta^0\}_{0<\delta<1/2}$ be a family of initial data in $H^1(\Omega,\mathbb{S}^2)$. Suppose that the assumptions* (A1), (A2) *and* (A3) *above are satisfied. Let $\{m_\delta\}_{0<\delta<1/2}$ denote any family of global weak solutions to* (LLG) *satisfying* (5.19), (5.20) *and the energy estimate* (5.21).
Then there exist a subsequence $\delta_n \to 0$ and $m = (m', 0) \in L^\infty([0,+\infty), L^2(\Omega))$ with

$$\forall t \geqslant 0, \quad |m'(t)| = 1 \quad \text{and} \quad \nabla\cdot m'(t) = 0 \quad \text{in } \mathscr{D}'(\Omega),$$

and such that for all $t \geqslant 0$, $m_{\delta_n}(t) \to m(t)$ in $L^2(\Omega)$ as $n \to \infty$. Moreover, we have

$$\partial_t m' = 0 \quad \text{in } \mathscr{D}'([0,+\infty)\times\Omega).$$

The proof combines a bound on $\partial_t m$ in $L^2_{\text{loc}}([0,+\infty), H^{-1}(\Omega)$ (which is a consequence of the energy bound and our choice of regime of the various parameters (A1)-(A2)-(A3)), and the compactness result Theorem 5.1.
It follows immediately from Theorems 5.3 and 5.6 that for well-prepared initial data the asymptotic magnetization is a static straight wall for all $t \geqslant 0$.

Corollary 5.7. *We make the same assumptions and use the same notations as in Theorem 5.6, and assume moreover that the initial data are well-prepared:*

$$\limsup_{\delta\to 0} \delta|\log\delta| E_\delta(m_\delta^0) \leqslant \frac{\pi}{2}(1 - m_{1,\infty})^2.$$

Let $\delta_n \to 0$ and let $x_1^* \in [-1,1]$ be such that $m_{\delta_n}^0 \to m^*$ in $L^2(\Omega)$, where m^* is a straight wall defined by (5.16). *Then, for all $t \geqslant 0$, $m_{\delta_n}(t) \to m^*$ in $L^2(\Omega)$.*

After these results, a natural question concerns the interaction of *several* Néel walls, and deriving a motion law for these. This requires to control the (LLG) on longer interval of times (and to change suitably the regime (A1)), and therefore to understand its structure in a deeper way: one should make use of physically relevant quantities such as the vorticity $\omega = \langle m, \partial_{x_1} m \wedge \partial_{x_2} m\rangle$.

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