Solutions to the generalized Korteweg-de Vries equations with a prescribed asymptotic behavior

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Given $p > 1$, we consider the generalized Korteweg-de Vries equations

$$\begin{cases} u_t + (u_{xx} + |u|^p)_x = 0, & t, x \in \mathbb{R}, \\ u(t=0) = u_0, \end{cases} \tag{gKdV}$$

From [4], these equations are locally well-posed in H^1, and even in L^2 in L^2 critical case $p = 5$ (which we denote (cKdV)). A fundamental feature of (gKdV) is the existence of explicit travelling wave solutions : the solitons. Let Q be the unique profile (up to translation) to

$$Q > 0, \quad Q \in H^1(\mathbb{R}), \text{ and } Q_{xx} + Q^p = Q.$$

Then the soliton $R_{c,x_0} = Q_c(x - x_0 - ct) = c^{\frac{1}{p-1}} Q(\sqrt{c}(x - x_0 - ct))$ is a solution to (gKdV). Notice that solitons are exponentially decaying and travel to the right.

Denote $U(t)$ the linear group, that is $U(t)\phi$ is the unique solution to $u_t + u_{xxx} = 0$,, $u(t = 0) = \phi$, or explicitly $\widehat{U(t)\phi} = e^{it\xi^3}\hat{\phi}$.

We construct solutions to (gKdV), defined for large enough times, which, as times goes to $+\infty$, behave (in H^1) as the sum of a linear solution and of N solitons.

Theorem 1 (Non-linear wave operator, subcritical case [2])**.** *Let $p = 4$. Let $V \in H^{5,1} \cap H^{2,2}$ be such that $x_+^{4/3} D_x^5 V \in L^2$ and $x_+^8 V \in H^1$. Let $N \in \mathbb{N}$, $0 < c_1 < \ldots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$, we introduce the N solitons $R_j(t, x) = Q_{c_j}(x - x_j - c_j t)$. Then there exists $u^* \in C([T_0, +\infty[, H^4)$, for some $T_0 \in \mathbb{R}$, solution to (gKdV) (with $p = 4$), such that :*

$$\left\| u^*(t) - U(t)V - \sum_{j=1}^N R_j(t) \right\|_{H^4} \le Ct^{-1/3}.$$

Theorem 2 (Non-linear wave operator, critical case [1])**.** *Let $p = 5$. Let $V \in H^1$ be such that $x_+^{2+\delta_0} V \in L^2$ for some $\delta_0 > 0$. Let $N \in \mathbb{N}$, $0 < c_1 < \ldots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$, we introduce the N solitons $R_j(t, x) = Q_{c_j}(x - x_j - c_j t)$. Then there exists $u^* \in C([T_0, +\infty[, H^1)$, for some $T_0 \in \mathbb{R}$, solution to (cKdV), such that :*

$$\left\| u^*(t) - U(t)V - \sum_{j=1}^N R_j(t) \right\|_{H^1} \to 0 \quad \textit{as} \quad t \to \infty.$$

The decay on the right condition on V corresponds to small interaction of the linear term $U(t)V$ with the solitons. In the critical case, the requirement seems almost optimal for the method.

The proof of these results follows the following scheme. We introduce a sequence of time $S_n \to \infty$ as $n \to \infty$, and the solutions u_n, which have exactly the desired profile at time $S_n : u_n(S_n) = U(S_n)V + \sum_{j=1}^N R_j(S_n)$. Our goal is to obtain uniform estimates on the error term $u_n(t) - U(t)V + \sum_{j=1}^N R_j(t)$ on some interval with fixed lower bound $[T_0, S_n]$.

The proof of the uniform estimates goes in two main steps. First, we rely on the work of Martel, Merle and Tsai [6] regarding the stability of a sum of solitons, and we obtain a control on the right. Introduce the cut-off function between the solitons and the linear term : $\psi(x) = 1 - \frac{2}{\pi}\arctan(\exp x)$ and

$$\psi_0(t,x) = \psi(x - \sigma_0 t), \quad \text{where} \quad \sigma_0 = \min\{c_1, c_2 - c_1, \dots, c_N - c_{N_1}\}/2.$$

Then we have

$$\|w_n(t)\|_{L^2(1-\psi_0(t))} \leq \int_t^{S_n} \|U(t)V\|_{L^2(1-\psi_0(t)} dt + ... \tag{1}$$

The term on the right hand side should be understood as interaction between the linear term and the solitons. Using our decay assumptions on V, we get a polynomial decay with arbitrary order.

The second step is to obtain global estimates. For this, we relie on two result of linear scattering for small data : the work Hayashi and Naumkin [3] in the case $p = 4$, and the work of Kenig Ponce and Vega [4] in the critical case $p = 5$.

In the critical case, the linear estimates of [4], along with (1) allows to conclude that for some fixed function $\eta(t) \to 0$ as $t \to \infty$,

$$\|w_n(t)\|_{L^2} \leq \eta(t),$$

which is the desired uniform decay estimates.

In the case $p = 4$, the solitons prevent a nice cancellation which was at the heart of the estimates in [3]. Hence we need to strenghten the settings : we derived H^4 uniform decay bounds, using "almost conservation" laws. We can then bootstrap the estimates, and obtain :

$$\|w_n(t)\|_{H^4} \leq C/t^{1/3}.$$

The proof of the Theorems then follows form a compactness argument on u_n.

The uniqueness of $u^8(t)$ is unclear, although one has uniqueness in the cases of a pure solitons behavior (see [5]) or pure linear behavior. A second question is the restriction to $p = 4$: from [3], one could expect a cosntruction of a non linear wave operator in the whole range $p \in (3, 5)$.

Références

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