SOLITON RESOLUTION FOR EQUIVARIANT WAVE MAPS TO THE SPHERE

RAPHAËL CÔTE

We consider finite energy corotationnal wave maps $\psi : \mathbb{R}_t \times [0, +\infty)_r \to \mathbb{R}$ solution to

$$\text{(WM)} \qquad \begin{cases} \partial_{tt}\psi - \partial_{rr}\psi - \dfrac{1}{r}\partial_r \psi + \dfrac{f(\psi)}{r^2} = 0 \\ (\psi, \partial_t \psi)|_{t=0} = (\psi_0, \psi_1) \end{cases} \qquad \text{with } f = gg',$$

where $g : \mathbb{R} \to \mathbb{R}$ is a $\mathscr{C}^3$ function. For a function $\vec{\phi} = (\phi_0, \phi_1)$, define the energy

$$E(\vec{\phi}) := \int_0^\infty \left(|\phi_1(r)|^2 + |\partial_r \phi_0(r)|^2 + \frac{|g(\phi_0(r))|^2}{r^2} \right) r dr.$$

At least formally, a wave map $\vec{\psi} = (\psi, \partial_t \psi)$ preserves the energy. If $\vec{\phi}$ has finite energy, then ϕ continuous and bounded, and has well defined limits at 0 and $+\infty$, which cancel g: we denote them $\phi(0)$ and $\phi(\infty)$. If $\vec{\phi}$ is a wave map, these limits do not depend on time. This motivates the introduction of the set where g vanishes

$$\mathscr{V} := \{\ell \in \mathbb{R} \mid g(\ell) = 0\}.$$

Also, let

$$G(x) := \int_0^x |g(y)| dy.$$

Our goal in this paper is to obtain a similar classification for wave maps of arbitrarily large energy, that is to relax the bound on the energy, inspired by the works of Duyckaerts Kenig and Merle [7, 8, 9, 10] in the context of the 3D $\dot{H}^1$-critical wave equation. It extends previous works [14, 2, 4, 5].

We provide a description of a wave map into decoupled profiles, a so called soliton resolution. It turns out that these profiles are harmonic maps and linear scattering terms. Recall that a harmonic map is a solution Q of finite energy of

$$\partial_{rr} Q + \frac{1}{r}\partial_r Q = \frac{f(Q)}{r^2}.$$

(Hence $(Q, 0)$ is a finite energy stationary wave map).

On the other hand, given $\ell \in \mathscr{V}$, we define the linearized wave map flow around ℓ:

$$(\text{LW}_\ell) \qquad \partial_{tt}\phi - \partial_{rr}\phi - \frac{1}{r}\partial_r \phi + \frac{g'(\ell)^2}{r^2}\phi = 0.$$

We make the following assumptions on the metric g:

- (A1) $G(x) \to \pm\infty$ as $x \to \pm\infty$.
- (A2) $\mathscr{V}$ is discrete,
- (A3) For all $\ell \in \mathscr{V}$, $g'(\ell) \in \{-1, 1\}$.

Date: Oberwolfach, August 12th 2013.

2010 *Mathematics Subject Classification.* 35L05,35L71.

Key words and phrases. wave maps, equivariant, classification, profile, soliton resolution.

The author gratefully acknowledges the support of the European Research Council under the project "Blow up, Dispersion and Solitons".

(A1) and (A2) are natural non degeneracy assumptions. (A3) captures the case of $g = \sin$ which correspond to wave maps with target manifold $\mathbb{S}^2$. We also have a weaker result under the relaxed assumption

(A3') For all $\ell \in \mathscr{V}$, $g'(\ell) \in \{-2, -1, 1, 2\}$,

which encompasses the 4D radial Yang-Mills equation as well.
We work in the functional space $H \times L^2$ defined by the norm

$$\|\phi_0\|_H^2 := \int_0^\infty \left(|\partial_r \phi_0(r)|^2 + \frac{|\phi_0(r)|^2}{r^2} \right) r dr.$$

Theorem 1. *We make assumptions (A1)-(A2)-(A3').*
Let $\vec{\psi}(t)$ be a finite energy wave map. Then there exist a sequence of time $t_n \uparrow T^+(\vec{\psi})$, an integer $J \geqslant 0$, J sequences of scales $\lambda_{J,n} \ll \cdots \ll \lambda_{2,n} \ll \lambda_{1,n}$ and J harmonic maps $Q_1, \ldots, Q_J$

$$\vec{\psi}(t_n) = \sum_{j=1}^{J} (Q_j(\cdot/\lambda_{j,n}) - Q_j(\infty), 0) + \vec{\phi}_n + \vec{b}_n. \tag{0.1}$$

where $\vec{b}_n$ vanishes in the following sense: for any sequence $\lambda_n > 0$, and $A > 0$

$$\|b_{n,0}\|_{H(t_n/A \leqslant r \leqslant A\lambda_n)} \to 0,$$

(so that $\|b_{n,0}\|_{L^\infty} \to 0$) and $\|b_{n,1}\|_{L^2} \to 0$; and ϕ_n is as follows:

(1) If $T^+(\vec{\psi}) < +\infty$, then $J \geqslant 1$, $\lambda_{1,n} \ll T^+(\vec{\psi}) - t_n$ and there exists a fixed function $\vec{\phi}$ of finite energy such that

$$\vec{\phi}_n = \vec{\phi}.$$

(2) If $T^+(\vec{\psi}) = +\infty$, denote $\ell = \psi(\infty)$, there exists a solution $\vec{\phi}_L(t) \in \mathscr{C}(\mathbb{R}, H \times L^2)$ to the linear wave equation (LW_ℓ) such that

$$\vec{\phi}_n = (\ell, 0) + \vec{\phi}_L(t_n).$$

If we furthermore assume (A3), then (in both cases)

$$\|\vec{b}_n\|_{H \times L^2} \to 0 \quad \text{as} \quad n \to +\infty.$$

The first step in the proof is to choose a sequence of time $t_n \to T^+(\vec{\psi})$ on which the space-time kinetic energy inside the light cone vanishes. This is a reformulation that the averaged kinetic energy inside the light cone vanishes, which is well known.

The second step is concerned with sequences of wave maps whose space-time kinetic energy vanishes. Up to a subsequence, one can construct a bubble decomposition i.e extract the harmonic maps, up to an error which tends to 0 in L^∞. This result does not make use of assumption (A3) or (A3'), but only (A1) and (A2).

The bound on the error is insufficient to capture the linear scattering term. However, when this bubble decomposition is combined with the concentration-compactness procedure developed in [12, 13] via linear profile decompositions introduced [1], we manage to derive a sharp scattering theorem *below the threshold in L^∞*. As linear scattering is involved, we do need assumption (A3') here. This result has its own interest, and as a consequence, we can extract the scattering term (for *all times*, not merely a sequence) in the global case. In an analogous way, we can define the regular part $\vec{\phi}$ in the blow up case.

Finally, we revisit the bubble decomposition to prove that the remainder vanishes in the energy space. We rely on channels of energy of the linear problem, a method first introduced by Duyckaerts Kenig and Merle [7, 8, 9, 10] in the context of the 3D nonlinear wave equation. Here we strongly rely on assumption (A3) which makes the linear problem conjugated to the 4D linear wave equation, and on [6] which proves the existence of channels of energy in that case.

Many open problems remain: the first one being to obtain a soliton resolution for all times and not merely a sequence of times. Also one should consider the Yang-Mills case. Finally, the possibilities for the behavior of $\lambda_{j,n}$ are not well understood, and in fact, constructing a solution where $J \geqslant 2$ is a challenging open question.

References

[1] Hajer Bahouri and Patrick Gérard. "High frequency approximation of solutions to critical nonlinear wave equations". *Amer. J. Math.* 121 (1999), 131–175.

[2] Raphaël Côte, Carlos E. Kenig, and Frank Merle. "Scattering below critical energy for the radial 4D Yang-Mills equation and for the $2D$ corotational wave map system". *Comm. Math. Phys.* 284 (2008), no. 1, 203–225.

[3] Raphaël Côte. "Instability of nonconstant harmonic maps for the $(1 + 2)$-dimensional equivariant wave map system". *Int. Math. Res. Not.* 2005, no. 57, 3525–3549.

[4] Raphaël Côte, Carlos E. Kenig, Andrew Lawrie, and Wilhelm Schlag. "Characterization of large energy solutions of the equivariant wave map problem: I". *Preprint 2012.* arXiv:1209.3682.

[5] Raphaël Côte, Carlos E. Kenig, Andrew Lawrie, and Wilhelm Schlag. "Characterization of large energy solutions of the equivariant wave map problem: II". *Preprint 2012.* arXiv:1209.3684.

[6] Raphaël Côte, Carlos E. Kenig, and Wilhelm Schlag. "Energy partition for the linear radial wave equation". *Preprint 2012.* arXiv:1209.3678.

[7] Thomas Duyckaerts, Carlos Kenig, and Frank Merle. "Universality of blow-up profile for small radial type II blow-up solutions of the energy-critical wave equation". *J. Eur. Math. Soc. (JEMS)* 13 (2011), no. 3, 533–599.

[8] Thomas Duyckaerts, Carlos Kenig, and Frank Merle. "Universality of the blow-up profile for small type II blow-up solutions of the energy-critical wave equation: the nonradial case". *J. Eur. Math. Soc. (JEMS)* 14 (2012), no. 5, 1389–1454.

[9] Thomas Duyckaerts, Carlos Kenig, and Frank Merle. "Profiles of bounded radial solutions of the focusing, energy-critical wave equation". *Geom. Funct. Anal.* 22 (2012), no. 3, 639–698. arXiv:1201.4986.

[10] Thomas Duyckaerts, Carlos Kenig, and Frank Merle. "Classification of radial solutions of the focusing, energy-critical wave equation". *Preprint 2012.* arXiv:1204.0031.

[11] Carlos E. Kenig, Andrew Lawrie, and Wilhelm Schlag. "Relaxation of wave maps exterior to a ball to harmonic maps for all data". *Preprint 2013.* arXiv:1301.0817.

[12] Carlos E. Kenig and Frank Merle. "Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case". *Invent. Math.* 166 (2006), no. 3, 645–675.

[13] Carlos E. Kenig and Frank Merle. "Global well-posedness, scattering and blow-up for the energy-critical focusing non-linear wave equation". *Acta Math.* 201 (2008), no. 2, 147–212.

[14] Michael Struwe. "Equivariant wave maps in two space dimensions". *Comm. Pure Appl. Math.* 56 (2003), no. 7, 815–823.