Around self-similar solutions to the modified Korteweg-de Vries equation

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(joint work with Simão Correia and Luis Vega)

We are interested in the dynamics near self-similar solutions for the modified Korteweg-de Vries equation:

$$\text{(mKdV)} \qquad \partial_t u + \partial^3_{xxx} u + \epsilon \partial_x (u^3) = 0, \quad u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}.$$

The signum $\epsilon \in \{\pm 1\}$ indicates whether the equation is focusing or defocusing. In our framework, ϵ will play no major role.
The (mKdV) equation enjoy a natural scaling: if u is a solution then

$$u_\lambda(t,x) := \lambda^{1/3} u(\lambda t, \lambda^{1/3} x)$$

is also a solution to (mKdV). As a consequence, the self-similar solutions, which preserve their shape under scaling

$$S(t,x) = t^{-1/3} V(t^{-1/3} x)$$

are of special interest. Self-similar solutions play an important role for the (mKdV) flow: they exhibit an explicit blow up behavior, and are also related with the long time description of solutions. Even for small and smooth initial data, solutions display a modified scattering where self-similar solutions naturally appear: we refer to Hayashi and Naumkin [6, 5], which was revisited by Germain, Pusateri and Rousset [3] and Harrop-Griffiths [4].
Self-similar solutions and the (mKdV) flow are also relevant as a model for the behavior of vortex filament in fluid dynamics. More precisely, Goldstein and Petrich proposed the following geometric flow for the description of the evolution of the boundary of a vortex patch in the plane under the Euler equations:

$$\partial_t z = -\partial_{sss} z + \partial_s \bar{z} (\partial_{ss} z)^2, \quad |\partial_s z|^2 = 1,$$

where $z = z(t,s)$ is complex valued and parametrize by its arctlength s a plane curve which evolves in time t. A direct computation shows that its curvature solves the focusing (mKdV) (with $\epsilon = 1$), and self-similar solutions with initial data

$$(1) \qquad U(t) \rightharpoonup c\delta_0 + \alpha \,\text{v.p.} \left(\frac{1}{x} \right) \quad \text{as } t \to 0^+, \quad \alpha, c \in \mathbb{R},$$

corresponds to logarithmic spirals making a corner: this kind of spirals are observed in a number of fluid dynamics phenomena (we refer [8] for more details). We were also motivated by the works by Banica and Vega (see for example [1]) on nonlinear Schrödinger type equations.

Our goal in this paper is to study the (mKdV) flow around self similar. Our first result is the description of self-similar solutions in Fourier space.

Theorem 1. *Given $c, \alpha \in \mathbb{R}$ small enough, there exists unique $a \in \mathbb{R}$, $A, B \in \mathbb{C}$ and a self-similar solution $S(t,x) = t^{-1/3} V(t^{-1/3}x)$, where V satisfies*

$$\text{for } p \geq 2, \quad e^{-itp^3} \hat{V}(p) = A e^{ia \ln |p|} + B \frac{e^{3ia \ln |p| - i \frac{8}{9} p^3}}{p^3} + z(p), \tag{2}$$

$$\text{for } |p| \leq 1, \quad e^{-itp^3} \hat{V}(p) = c + \frac{3i\alpha}{2\pi} \operatorname{sgn}(p) + z(p), \tag{3}$$

where $z \in W^{1,\infty}(\mathbb{R})$, $z(0) = 0$ and for any $k < \frac{4}{7}$, $|z(p)| + |pz'(p)| = O(|p|^{-k})$ as $|p| \to +\infty$.

Hence self-similar solutions exhibit logarithmic oscillations for large frequencies (which are related to the critical nature of the problem), and if $\alpha \neq 0$, a jump at frequency $p = 0$. We are also able to related the constants involved (a, A, B) with those appearing in the description in physical space.
The proof consists in writing the problem as a fixed point. We compute expansions for the first three Picard iterates, for the function and is derivative; and we are able to control the remainder term in the weighted L^∞ based space indicated. The techniques involves essentially stationnary phase analysis with a careful control on the error.

Our second result is concerned with local well posedness in a critical space which contains the self-similar solutions constructed above. We work with the norm (for space time functions):

$$\|u\|_{\mathscr{E}(I)} := \sup_{t \in I} \|u(t)\|_{\mathscr{E}(t)}, \quad \|v\|_{\mathscr{E}(t)} := \|\widehat{\mathcal{G}(-t)v}\|_{L^\infty(\mathbb{R})} + t^{-1/6} \|\partial_p \widehat{\mathcal{G}(-t)v}\|_{L^2((0,+\infty))}, \tag{4}$$

where $\mathcal{G}(t)$ denote the linear KdV group and $I \subset (0, +\infty)$ is a time interval. The above norm is scaling invariant, in the sense that $\|u_\lambda(t)\|_{\mathscr{E}(t)} = \|u(\lambda t)\|_{\mathscr{E}(\lambda t)}$; also it is finite for the self-similar solutions of Theorem 1. Our result is as follows.

Theorem 2. *Let $u_1 \in \mathscr{E}(1)$. Then there exist $T > 1$ and a solution $u \in \mathscr{E}([1/T, T])$ to* (mKdV) *such that $u(1) = u_1$.*
Furthermore, one has forward uniqueness. More precisely, let $0 < t_0 < t_1$ and u and v be two solutions to (mKdV) *such that $\hat{u}, \hat{v} \in \mathscr{C}([t_0, t_1], L^\infty)$ and*

$$\|u\|_{\mathscr{E}([t_0,t_1])}, \|v\|_{\mathscr{E}([t_0,t_1])} < +\infty.$$

If $u(t_0) = v(t_0)$, then for all $t \in [t_0, t_1]$, $u(t) = v(t)$.

For small data in $\mathscr{E}(1)$, the solution is actually defined for large times, and one can describe the asymptotic behavior. This is the content of our last main result.

Theorem 3. *There exists $\delta > 0$ small enough such that the following holds.*
If $\|u_1\|_{\mathscr{E}(1)} \leq \delta$, the corresponding solution satisfies $u \in \mathscr{E}([1, +\infty))$. Furthermore, let S be the self-similar solution such that

$$\hat{S}(1, 0^+) = \hat{u}_1(0^+) \in \mathbb{C}.$$

Then $\|u(t) - S(t)\|_{L^\infty} \lesssim \|u_1\|_{\mathscr{E}(1)} t^{-5/6^-}$ *and there exists a profile* $U_\infty \in \mathscr{C}_b(\mathbb{R} \setminus \{0\}, \mathbb{C})$*, with* $|U_\infty(0^+)| = \lim_{p\to+\infty} |\hat{S}(1,p)|$ *is well-defined, and*

$$\left|\tilde{u}(t,p) - U_\infty(p) \exp\left(\frac{i}{4\pi}|U_\infty(p)|^2 \log t\right)\right| \lesssim \frac{\delta}{\langle p^3 t\rangle^{\frac{1}{12}}} \|u_1\|_{\mathcal{E}(1)}.$$

As a consequence, one has the asymptotic in the physical space. In the setting of the above Theorem 3, if we let

$$y = \begin{cases} \sqrt{-x/3t}, & \text{if } x < 0, \\ 0, & \text{if } x > 0. \end{cases}$$

one has, for all $t \geq 1$ and $x \in \mathbb{R}$,

$$\left| u(t,x) - \frac{1}{t^{1/3}} Ai\left(\frac{x}{t^{1/3}}\right) U_\infty(y) \exp\left(\frac{i}{6}|U_\infty(y)|^2 \log t\right)\right| \lesssim \frac{\delta}{t^{1/3}\langle x/t^{1/3}\rangle^{3/10}}.$$

The main challenge we faced in proving Theorem 2 is that we cannot work with smooth data, due to the jump at frequency 0. Also, the multiplier estimates suitable for the space $\mathcal{E}(I)$ require computations on a non linear solution, and are not amenable to a fixed point scheme. Therefore, we had to rely on the resolution of an approximate problem first, followed by a compactness argument. It turns out that the approximation has to obey several constraints, which could ultimately be met by following a Friedrichs scheme with a suitably twisted cut-off in frequency. Again, the approximate problem is solved by fixed point in a space where smooth function are not dense. The compactness argument is then fairly standard. Theorem 3 follows a similar path, the expansion as $t \to +\infty$ being merely a by product of the analysis.
For the forward uniqueness result of Theorem 2, we consider the variation of localized L^2 norm of the difference w of two solutions. Our solutions do not belong to L^2, but we make use of an improved decay of functions in $\mathscr{E}(I)$ to make sense of it. If the cut-off is furthermore chosen to be non decreasing, we can make use of a monotonicty property to control the variations of this L^2 quantity and conclude via a Gronwall-type argument. This kind of monotonicity property was first observed and used by Kato, and is a key feature in the study of the dynamics of solitons by Martel and Merle [7]. To our knowledge, it is however the first time that is used in the context of self-similar solutions.

We are now concerned with the behavior near blow up time $t = 0$: in particular, whether the self-similar blow up is stable, and the understanding of perturbations of self-similar solutions.

References

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