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THÈSE

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Spécialité : Mathématiques

Construction et propriétés de solutions pour des équations dispersives focalisantes

par

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Résumé

Dans cette thèse, nous étudions quelques propriétés de solutions d'équations aux dérivées partielles dispersives focalisantes. On étudie deux types d'équations.

Dans un premier temps, nous étudions les équations de Korteweg-de Vries généralisées (gKdV). Étant donnée une solution de l'équation de Korteweg-de Vries linéaire, nous construisons une solution de (gKdV) qui se comporte ainsi pour des temps grands. Étant également donnés N solitons (ondes solitaires solutions de (gKdV)), nous construisons, dans les cas L^2-critique et sous-critique, une solution de (gKdV) se comportant comme la somme de ces N solitons et de la solution linéaire.

Dans un deuxième temps, nous nous intéressons au système des wave maps en dimension critique (1+2) : c'est un modèle simple d'équation des ondes dans un cadre géométrique. Nous montrons que les fonctions harmoniques (wave maps stationnaires) sont instables dans l'espace d'énergie, en un sens fort, pour ce système.

Mots clés : Équations de Korteweg-de Vries, opérateur d'onde, comportement asymptotique, solitons, wave map, instabilité, phénomène critique.

Abstract

In this work we study some properties of solutions to dispersive focalizing partial differential equations. We study two types of equations.

In chapters 2 to 4, we study the generalized Korteweg-de Vries equations (gKdV). Given a solution to the linear Korteweg-de Vries equation, we construct a solution to (gKdV) which behaves like this for large times. Given N solitons solutions (stationnary wave solutions to (gKdV)), we construct in the L^2-critical and sub-critical cases, a solution to (gKdV) which behaves like the sum of these solitons and of the linear solution.

In chapter 5, we are interested in the wave map system in critical dimension (1+2) : this is a simple model for the wave equation in a geometrical background. We prove that harmonic functions (stationnary wave maps) are instable in the energy space, in a strong sense, for this system.

Keywords : Korteweg-de Vries equations, wave operator, asymptotic behavior, solitons, wave map, instability, critical phenomena

AMS Classification : 35Q53, 35B40, 35Q51, 35L15.

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Chapitre 1

Introduction

L'objet de cette thèse est de comprendre certains aspects du comportement, pour des temps grands, des solutions d'équations aux dérivées partielles dispersives : notamment les wave maps et les équations de Korteweg-de Vries généralisées. Il s'agit en particulier d'étudier des solutions qui ont un comportement fondamentalement non-linéaire.

Chacune de ces équations admet une solution particulière explicite, qui est une onde solitaire, ou une solution stationnaire, c'est-à-dire un objet qui conserve une forme constante au cours de temps. En un sens, les ondes solitaires réalisent un équilibre entre la partie linéaire des équations, qui est dispersive (c'est-à-dire qui tend à faire se diluer la solution), et la partie non-linéaire, qui est un terme qui au contraire tend à concentrer, à faire focaliser la solution.

L'équation de Korteweg-de Vries décrit la propagation d'ondes dans des eaux peu profondes. Cette équation, ainsi que des généralisations, a fait l'objet de nombreuses recherches et on appréhende aujourd'hui assez bien les comportements possibles des solutions de ces équations. Une partie des travaux de thèse a porté sur la construction de solutions ayant un comportement asymptotique (pour des temps grands) prescrit, et notamment un comportement mixte avec une partie "linéaire" et une partie "non-linéaire".

L'équation des wave maps est un modèle simple d'équation des ondes dans un cadre géométrique : en dimension critique ($n = 2$), on conjecture que certaines données initiales sont susceptibles de développer des singularités en temps fini (ce que l'on appelle explosion). L'autre partie des travaux de thèse a été consacrée à la compréhension de ce phénomène.

1.1 Construction de solutions des équations de Korteweg-de Vries généralisées

1.1.1 L'équation de Korteweg-de Vries et les solitons

Généralités

On considère les équations suivantes :

$$\begin{cases} u_t + (u_{xx} + |u|^p)_x = 0, & t, x \in \mathbb{R}, \\ u(t=0) = u_0, \end{cases} \tag{gKdV}$$

pour $p > 1$. Dans le cas de $p \in \mathbb{N}$, on peut considérer indifféremment les équations

$$u_t + (u_{xx} + u^p)_x = 0.$$

Ces équations modélisent la formation d'ondes solitaires dans le contexte d'eau peu profonde. Pour $p=2$ et $p=3$ (équations de Korteweg-de Vries (KdV) et de Korteweg-de Vries modifiée (mKdV) respectivement), ces équations ont de nombreuses applications à la Physique.

(gKdV) est un sytème hamiltonien. En particulier, trois quantités sont conservées, au moins formellement :

$$\begin{aligned} \int u(t,x)dx &= \int u_0(x)dx, \\ \int u^2(t,x)dx &= \int u_0^2(x)dx & \text{(masse } L^2\text{)}, \\ E(u(t)) = \frac{1}{2}\int u_x^2(t,x)dx - \frac{1}{p+1}\int |u|^{p+1}(t,x)dx &= E(u_0) & \text{(énergie)}. \end{aligned}$$

L'espace d'énergie naturel pour l'étude de cette équation est donc H^1. Notons cependant que la première loi de conservation est peu utilisée, du fait qu'il ne s'agit pas d'une quantité signée, et que de plus elle ne se situe pas dans l'espace d'énergie.

Par ailleurs, l'équation admet une invariance d'échelle : si u est solution de (gKdV), $u_\lambda(t,x) = \lambda^{2/(p-1)}u(\lambda^3 t, \lambda x)$ l'est également. Remarquons que

$$\int u_\lambda = \lambda^{\frac{3-p}{p-1}}\int u, \qquad \int u_\lambda^2 = \lambda^{\frac{5-p}{p-1}}\int u^2, \qquad E(u_\lambda) = \lambda^{\frac{p+3}{p-1}}E(u).$$

En particulier, pour $p=5$, (gKdV) est une équation L^2-critique pour l'invariance d'échelle, c'est (cKdV) :

$$u_t + (u_{xx} + u^5)_x = 0. \tag{cKdV}$$

Solitons

Une caractéristique essentielle de (gKdV) est l'existence de solitons explicites qui sont des ondes solitaires, appelées solitons. Historiquement, (KdV) a été construite pour admettre ces solutions, de la forme $v(x-ct)$, où $v : \mathbb{R} \to \mathbb{R}$ a un profil de courbe en cloche.

Plus précisément, soit Q l'unique solution (aux translations près) de :

$$Q > 0, \quad Q \in H^1(\mathbb{R}), \quad Q_{xx} + Q^p = Q, \quad \text{i.e.} \quad Q(x) = \left(\frac{p+1}{2\cosh^2(\frac{p-1}{2}x)}\right)^{\frac{1}{p-1}}. \tag{1.1}$$

Notons, pour $c>0$, $Q_c(x) = c^{\frac{1}{p-1}}Q(\sqrt{c}x)$. Alors le soliton

$$R_{c,x_0} = Q_c(x - x_0 - ct) = c^{\frac{1}{p-1}}Q(\sqrt{c}(x - x_0 - ct)) \text{ est une solution de (gKdV).} \tag{1.2}$$

L'existence de solitons est intimement liée au fait que l'on considère la version focalisante de (gKdV), c'est-à-dire que l'on a choisi la non-linéarité $+|u|^{p-1}u_x$. En choisissant le signe opposé, on obtient l'équation dite défocalisante, qui n'admet pas de solitons. On peut également voir que l'on affaire à une équation focalisante au signe $-$ dans l'énergie $E(u)$.

Dans le cas critique $p=5$, notons que pour tout $c>0$, $x_0 \in \mathbb{R}$,

$$\int R_{c,x_0}^2 = \int Q^2, \qquad E(R_{c,x_0}) = 0. \tag{1.3}$$

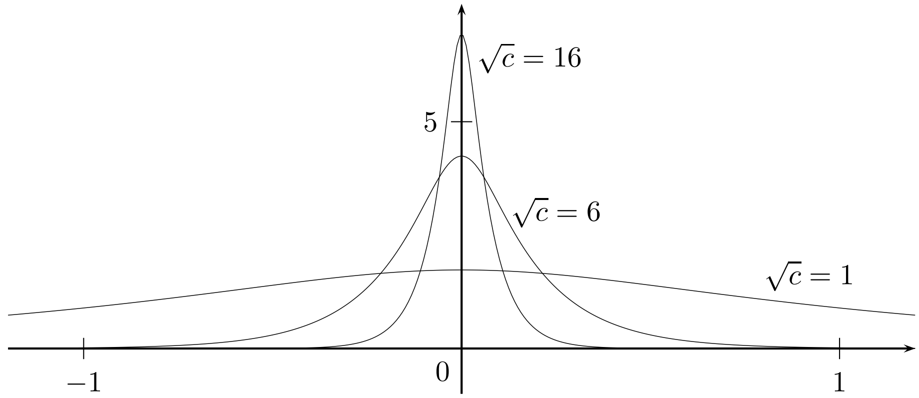

FIG. 1.1 – Quelques solitons avec paramètres $\sqrt{c} = 1,\ 6,\ 16$, pour $p = 5$.

Scattering Inverse

La dynamique des solutions de (gKdV) est bien comprise dans les cas $p = 2$ (KdV) et $p = 3$ (mKdV), grâce à la méthode du scattering inverse. Par exemple, si u_0 est une donnée initiale de $\mathcal{S}$, alors la solution $u(t)$ associée est globale en temps et se décompose en une somme de solitons (cf. [43], Eckhaus and Schuur [9], Miura [39]) :

$$\left\| u(t) - \sum_{j=1}^{N} R_{c_j,x_j}(t) \right\|_{L^\infty(x \geq -t^{1/3})} \leq \frac{C}{t^{1/3}} \qquad (\text{quand } t \to \infty).$$

Cependant, la méthode du scattering inverse a trois défauts. Tout d'abord, elle demande une grande régularité sur la donnée initiale u_0 (même pour le problème de l'existence locale de solutions). D'autre part, elle donne essentiellement des informations "sur la droite" : en général la convergence a lieu dans l'espace $L^\infty(x \geq 0)$ ou $L^\infty(x \geq -t^{1/3})$, et il paraît difficile d'obtenir des informations dans la région $x \leq -t^{1/3}$. Enfin, et c'est le plus rédhibitoire, cette méthode ne s'étend pas aux cas autres que (Kdv) et (mKdV) (ie. $p \neq 2, 3$).

Les résultats présentés dans la suite étudient la dynamique des solutions de (gKdV), et ont été obtenus hors du cadre de la méthode du scattering inverse.

1.1.2 La théorie linéaire

Problème local

Le problème de Cauchy local en temps est bien compris : (gKdV) est bien posée dans H^1 (voir [14]), et même, (cKdV) ($p = 5$) est bien posée dans L^2 localement en temps et globalement en temps pour des données petites.

Théorème 1.1 (Existence locale en temps [14])**.** *Soit $u_0 \in H^1$. Il existe $T = T(\|u_0\|_{H^1})$ et $u \in C^0([0,T), H^1)$ solution de (gKdV), unique dans une classe adaptée. Une telle solution conserve la masse L^2 et l'énergie.*

Dans le cas critique $p = 5$, si $u_0 \in L^2$, il existe T et $u \in C^0([0,T), L^2)$, solution de (cKdV), unique dans une classe adaptée (u conserve la masse L^2).

La preuve de ce résultat s'appuie essentiellement sur l'étude de l'opérateur de KdV linéaire $U(t)$ (c'est pourquoi l'on parle de théorie linéaire) :

$$U(t)\phi \text{ est l'unique solution de} \begin{cases} u_t + u_{xxx} = 0, \\ u(t=0) = \phi, \end{cases} \quad \text{i.e.} \quad \widehat{U(t)\phi} = e^{it\xi^3}\hat{\phi}. \tag{1.4}$$

La formule de Duhamel affirme qu'une solution $u(t)$ de (gKdV) est également solution du problème de point fixe suivant :

$$u(t) = U(t)u_0 + \partial_x \int_0^t U(t-s)|u|^p(s)ds. \tag{1.5}$$

La méthode d'attaque classique est alors de montrer que l'application

$$\begin{aligned} \Phi : \quad B &\to B \\ v(t) &\mapsto u(t) = U(t)u_0 + \partial_x \int_0^t U(t-s)|v|^p(s,x)ds, \end{aligned}$$

est bien définie pour un certain espace métrique complet B bien choisi (en général une boule d'un espace de Banach), et qu'elle est contractante sur B.

Les outils usuels pour traiter le problème de Cauchy local des EDP dispersives sont les estimées de Strichartz : dans le cas de (gKdV), elles s'écrivent

$$\|U(t)\phi\|_{L_t^p L_x^q} \leq \|\phi\|_{L_x^2}. \tag{1.6}$$

où p et q vérifient la condition $\frac{1}{p} + \frac{2}{3q} = \frac{1}{3}$. Ces relations sont obtenues par interpolation de la conservation de la masse $\|U(t)\phi\|_{L^2} = \|\phi\|_{L^2}$ et de l'estimée de dispersion dans L^∞ :

$$\|U(t)\phi\|_{L^\infty} \leq \frac{C}{t^{1/3}}\|\phi\|_{L^1}. \tag{1.7}$$

Dans le cas de l'équation de Schrödinger non-linéaire, par exemple celle avec non-linéarité puissance (dans ce cas $u : \mathbb{R}_t \times \mathbb{R}_x^n \to \mathbb{C}$) :

$$i\partial_t u - \Delta u + |u|^{p-1}u = 0, \tag{NLS}$$

il existe des estimées de Strichartz analogues qui sont étudiées extensivement et sont très efficaces.

Cependant, elles jouent un rôle mineur dans l'étude de (gKdV) : cela est dû au fait que la non-linéarité $|u|^{p-1}u_x$ comporte une dérivé u_x.

Pour pallier ce problème, Kenig, Ponce et Vega ont développé des estimations linéaires où temps et espace sont inversés, du type $L_x^p L_t^q$ (cf. [14, 15]). Introduisons le symbole $D_x = (-\Delta_x)^{1/2}$, ce qui nous permettra de considérer des dérivations fractionnaires. Les deux remarques fondamentales sont que :

$$\forall x \in \mathbb{R}, \quad \int |D_x U(t)\phi|^2 dt = \int |\phi(x)|^2 dx, \quad \text{et} \tag{1.8}$$

$$\sup_{x\in\mathbb{R}} \|U(t)\phi\|_{L_t^4} = \|U(t)\phi\|_{L_x^\infty L_t^4} \leq \|D_x^{1/4}\phi\|_{L^2}. \tag{1.9}$$

Par des développements délicats de la théorie de l'interpolation et un argument TT^*, Kenig, Ponce et Vega obtiennent :

$$\left\| D_x^{\alpha_1} \int_0^t U(t-s)\phi(s,x)dx \right\|_{L_x^{p_1} L_t^{q_1}} \leq C \| D_x^{\alpha_2} \phi \|_{L_x^{p_2'} L_t^{q_2'}}, \tag{1.10}$$

pour tous couples de triplets (α_i, p_i, q_i) $(i = 1, 2)$ où $p_i, q_i \in [1, \infty]$, et $\alpha_i \in \left[-\frac{1}{4}, 1\right]$ vérifient

$$\frac{1}{p_i} + \frac{1}{2q_i} = \frac{1}{4}, \qquad \alpha_i = \frac{2}{q_i} - \frac{1}{p_i}, \qquad \frac{1}{p_i} + \frac{1}{p_i'} = \frac{1}{q_i} + \frac{1}{q_i'} = 1.$$

On a notamment l'estimée linéaire suivante, qui est au centre du résultat dans le cas critique $p = 5$:

$$\left\| \partial_x \int_0^t U(t-s)\phi(s,x)ds \right\|_{L_x^5 L_t^{10}} \leq C \|\phi\|_{L_x^1 L_t^2}. \tag{1.11}$$

Problème global

Le Théorème 1.1 permet alors de définir l'*explosion* pour une solution $u(t)$ de (gKdV) :

- $u(t)$ *explose* si $\limsup_{t \uparrow T} \|u(t)\|_{H^1} = +\infty$ pour un certain $T \in \mathbb{R}^+$ (explosion en temps fini) ou $T = +\infty$ (explosion en temps infini).
- Sinon, $\|u(t)\|_{H^1}$ est uniformément bornée : il existe une constante C telle que pour tout $t \geq 0$, $\|u(t)\|_{H^1} \leq C$, et on dit que u est *globale* (pour les temps positifs).

(on a une caractérisation similaire si on s'intéresse aux temps négatifs). Autrement dit, c'est la quantité $\|u_x(t)\|_{L^2}$ qui détermine s'il y a ou non explosion (car $\|u(t)\|_{L^2}$ est constante) : on dit que $\|u_x(t)\|_{L^2}$ est le taux d'explosion (si $\|u_x(t)\|_{L^2} \to \infty$).

La conservation de la masse L^2 et de l'énergie permet d'obtenir l'existence globale de solutions dans les cas sous-critiques $p < 5$.

Considérons l'inégalité de Gagliardo-Nirenberg (valable pour tout $p \geq 2$) :

$$\forall v \in H^1, \quad \frac{1}{p+1} \int v^{p+1} \leq C(p) \left(\int v^2 \right)^{\frac{p+3}{4}} \left(\int v^2 \right)^{\frac{p-1}{4}}. \tag{1.12}$$

Pour $p < 5$, $\frac{p-1}{4} < 1$: si $\|v\|_{H^1} \to \infty$, alors $E(v) \to \infty$. Par conservation de l'énergie, ceci est impossible pour une solution de (gKdV). On en déduit l'existence globale des solutions H^1 dans le cas $p < 5$, qui est donc appelé sous-critique.

Cela montre également que pour des données initiales suffisamment petites dans H^1, la solution associée est globale (pour tout $p > 1$).

Par contre, dans le cas $p = 5$, l'énergie n'est plus coercive, et cet argument ne s'applique plus. Notons que si $p = 5$, le temps maximal d'existence T n'est pas uniquement une fonction de $\|u_0\|_{L^2}$: si c'était le cas, la conservation de la masse L^2 et un argument de continuité donnerait immédiatement l'existence globale des solutions. Merle [32], Martel et Merle [28, 27] ont démontré l'existence de solutions explosives : nous y reviendrons.

Ainsi $p = 5$ est également un exposant critique du point de vue de l'existence globale. Dans le cas sur-critique $p > 5$, l'existence de solutions explosives est conjecturée.

Scattering linéaire à données petites

On dit qu'il y a scattering (linéaire) si une solution $u(t)$ d'une certaine EDP non-linéaire se comporte comme une solution $v(t)$ de la partie linéaire de cette EDP (dans notre cas $v(t) = U(t)V$ pour une certaine fonction $V(x)$, où $U(t)$ est défini en (1.4)). L'application $u(t) \mapsto v(t)$ (ou $u(t=0) \mapsto v(t=0)$) est appelée opérateur de scattering.

Tout d'abord, notons qu'un corollaire immédiat (de la preuve) du Théorème 1.1 donne, dans le cas critique, l'existence du scattering à petites données.

Corollaire 1.1 ([14]). *Soit $u_0 \in L^2$. Si $\|u_0\|_{L^2}$ est assez petite, la solution associée de (cKdV) $u(t) \in C^0([0,\infty[, L^2)$ est globale et de plus, il existe $V \in L^2$ tel que*

$$\|u(t) - U(t)V\|_{L^2} \to 0 \quad \text{quand} \quad t \to \infty.$$

L'espace L^2 est adéquat pour comprendre le scattering linéaire, car $\|U(t)V\|_{L^2} = \|V\|_{L^2}$ ne tend pas vers 0 quand $t \to \infty$, et ainsi L^2 "conserve l'information" : au contraire la dispersion fait que pour tout $p > 2$, $\|U(t)V\|_{L^p} \to 0$.

Le principal résultat de scattering à petites données est le suivant, dû à Hayashi et Naumkin [13] (voir également des travaux précédents de Stauss [48], Ponce et Vega [40] , Christ et Weinstein [4]). Introduisons les espaces à poids suivants : pour $s \geq 0$ et $m \in \mathbb{N}$,

$$H^{s,m} = \{v \in L^2 | \ (1+|x|^2)^{m/2} v \in H^s\}. \tag{1.13}$$

Théorème 1.2 (Scattering à données petites [13]). *Soit $p > 3$. Il existe $\varepsilon_0 > 0$ assez petit, tel que ce qui suit soit vrai. Soit $u_0 \in H^{1,1} = \{v | v \in H^1 \text{ et } xv \in H^1\}$ tel que $\|u_0\|_{H^{1,1}} \leq \varepsilon_0$. Alors $u \in C_b^0([0,\infty[, H^1)$ existe globablement, et disperse comme une solution linéaire $\|u(t)\|_{L^\infty} \leq Ct^{-1/3}$. Enfin, il existe $V \in L^2$ tel que*

$$\|u(t) - U(t)V\|_{L^2} \leq t^{-(p-3)/3} \quad (\text{quand } t \to \infty).$$

Remarquons que la condition $p > 3$ est certainement optimale : on conjecture qu'il n'y a pas de scattering pour $p \leq 3$, et cela est démontré pour $p \in]1, 2]$, cf. Rammaha [41]. D'autre part, heuristiquement, le scattering signifie que l'accumulation de la force non-linéaire $|u|^{p-1}u_x$ est intégrable en temps. Au vu du taux de dispersion linéaire $\|u(t)\|_{L^\infty} \leq Ct^{-1/3}$, la condition $p > 3$ est naturelle.

Cependant, notons qu'il y a ici quelques pertes : bien que tout se passe essentiellement dans un espace de régularité $H^{1,1}$, le résultat final de scattering est dans L^2. Par ailleurs, tous les solitons sont grands dans $H^{1,1}$ (c'est le minimum à vérifer, car un soliton est l'archétype d'une solution qui ne se comporte pas comme une solution linéaire).

La preuve du Théorème 1.2 repose sur deux arguments : d'une part, des estimées linéaires efficaces, et d'autre part, des opérateurs ayant des propriétés de commutation adéquates.

On introduit ainsi :

$$J^t \phi = U(t)xU(-t)\phi = x\phi - 3t\phi_{xx}, \quad \text{et} \quad I^t \phi = x\phi - 3t \int_{-\infty}^{x} \phi_t(x')dx'.$$

(Pour I^t, ϕ est une fonction de l'espace et du temps). J^t permet de définir la norme M_0^t (qui dépend du temps), qui est au coeur du résultat :

$$M_0^t(\phi) = \|\phi\|_{H^1} + \|D^\alpha J^t \phi\|_{L^2} + \|DJ^t \phi\|_{L^2}. \tag{1.14}$$

($\alpha \in]0,1/2[$ est une constante adéquate, proche de $1/2$). Remarquons que M_0^0 est une norme légèrement moins forte que $H^{1,1}$. Alors on a les estimées linéaires ponctuelles suivantes :

$$|\phi(x)| \leq \frac{C}{(1+t)^{1/3}} M_0^t(\phi) \left(1 + \frac{|x|}{t^{1/3}}\right)^{-1/4},$$
$$|\phi_x(x)| \leq \frac{C}{t^{2/3}} M_0^t(\phi) \left(1 + \frac{|x|}{t^{1/3}}\right)^{1/4}. \tag{1.15}$$

En particulier, $|\phi(x)\phi_x(x)| \leq C(1+t)^{1/3} t^{2/3} M_0^t(\phi)^2$.

Dans un deuxième temps, on estime $D^\alpha I^t u$ et $I^t u_x$ par des méthodes d'énergie (en calculant la dérivée en temps des normes L^2 de ces quantités). Ceci est efficace car I^t satisfait une propriété de type “chain-rule” : pour F une fonction C^1, on a

$$I^t F(\phi)_x = F'(\phi) I^t \phi_x. \tag{1.16}$$

Enfin comme u est solution de (gKdV), $I^t u - J^t u = 3tu^p$, et en utilisant à nouveau les estimations linéaires (1.15), on arrive à estimer $M_0^t(u(t))$ a priori. Par un argument de continuité usuel, on en déduit une borne uniforme :

$$\forall t \geq 0, \quad M_0^t(u(t)) \leq C M_0^0(u_0) \leq C\|u_0\|_{H^{1,1}},$$

pourvu que $\|u_0\|_{H^{1,1}}$ soit assez petit.

Pour conclure par le scattering, il suffit de calculer

$$(U(-t)u(t))_t = U(-t)(|u|^{p-1}u_x).$$

Ainsi $t \mapsto \|(U(-t)u(t))_t\|_{L^2}$ est intégrable, et donc $U(-t)u(t)$ admet une limite V dans L^2 quand $t \to \infty$. Le taux de convergence s'en déduit immédiatement.

Opérateur d'onde linéaire

Issu de l'analyse précédente, on peut dès à présent construire un opérateur d'onde, c'est-à-dire une application, qui à une donnée finale V associe une donnée initiale u_0 telle que la solution associée $u(t)$ de (gKdV) se comporte comme $U(t)V$ quand $t \to \infty$. Ce problème est en quelque sorte réciproque au scattering linéaire.

Théorème 1.3 (Opérateur d'onde linéaire à grandes données [8]). *Soit $p > 3$ et $V \in H^{2,2} = \{v | v \in H^2 \text{ et } x^2 v \in H^2\}$. Il existe $T_0 = T_0(\|V\|_{H^{2,2}}) \geq 1$ et une $u \in C([T_0, \infty[, H^1)$ solution de (gKdV) telle que*

$$\|u^*(t) - U(t)V\|_{H^1} \to 0 \quad \text{quand} \quad t \to \infty.$$

De plus, u est unique dans une classe adaptée et

$$\|u^*(t) - U(t)V\|_{H^1} \leq Ct^{-(p-3)/3}.$$

Supposons que $p = 5$. Alors pour $V \in L^2$, il existe $T_0 = T_0(V) \in \mathbb{R}$ et $u \in C^0([T_0, \infty[, L^2)$ solution de (cKdV) telle que

$$\|u(t) - U(t)V\|_{L^2} \to 0 \quad \text{quand} \quad t \to \infty,$$

et u est unique dans une classe adaptée.

Ce résultat s'appuie sur les estimées utilisées pour démontrer le scattering à petites données. Mais on construit un opérateur d'onde pour de **grandes données** V, ce qui peut paraître surprenant de prime abord.

Dans le cas général $p > 3$, on utilise la méthode des approximations successives. Soit donc S_n une suite de temps telle que $S_n \uparrow \infty$ et u_n la solution de (gKdV) ayant exactement le profil désiré au temps S_n :

$$\begin{cases} u_{nt} + (u_{nxx} + |u_n|^p)_x = 0, \\ u_n(S_n) = U(S_n)V. \end{cases} \tag{1.17}$$

De manière équivalente, on peut considérer le terme d'erreur

$$w_n(t) = u_n(t) - U(t)V, \tag{1.18}$$

qui satisfait :

$$\begin{cases} w_{nt} + (w_{nxx} + |w_n + U(t)V|^p)_x = 0, \\ w_n(S_n) = 0. \end{cases} \tag{1.19}$$

Le coeur du problème est de montrer que pour tout n, u_n est défini sur un intervalle du type $[T_0, S_n]$, où T_0 est fixe (indépendant de n) et que de plus, on a l'estimée uniforme en n :

$$\forall n \in \mathbb{N}, \forall t \in [T_0, S_n], \qquad M_0^t(w_n(t)) \leq \frac{C}{t^\delta}, \tag{1.20}$$

(où $\delta > 0$ est une constante qui apparaît dans [13]).

Admettons un instant ce résultat. En calculant $\frac{d}{dt}\|w_n(t) - w_m(t)\|_{L^2}^2$, on s'aperçoit que la suite $(w_k)_{k\geq n}$ est de Cauchy dans l'espace $C^0([T_0, S_n], L^2)$. On déduit que la suite $w_n(t)$ converge localement uniformément dans l'espace $C^0([T_0, \infty[, L^2)$, vers une limite $w^*(t)$. Par limite faible, $w^*(t) \in L^\infty([T_0, \infty[, H^1)$ et vérifie :

$$M_0^t(w(t)) \leq \frac{C}{t^\delta}.$$

On conclut donc que $u^*(t) = w^*(t) + U(t)V$ satisfait les conditions du théorème.

On est donc ramené à obtenir les estimées uniformes (1.20 sur $w_n(t)$. Considérons un intervalle maximal $[I_n, S_n]$ tel que pour tout temps t dans cet intervalle, $M_0^t(w_n(t)) \leq \varepsilon_0$.

Grâce à cette hypothèse de petitesse sur w_n, et en utilisant les estimations (1.15), on contrôle les dérivées en temps de $\|w_n(t)\|_{H^1}$, $\|D^\alpha I^t w_n\|_{L^2}$ et $\|I^t w_{nx}\|_{L^2}$: cela se fait essentiellement comme dans la preuve du Théorème 1.2 (méthodes d'énergie s'appuyant sur (1.16)). Cependant, les intégrations par parties nécessaires ne fonctionnent pas aussi bien dans le cas présent, et lors de ces calculs apparaît naturellement la condition $V \in H^{2,2}$. Par exemple, si u est solution de (gKdV), alors

$$\frac{d}{dt}\|I^t u_x\|_{L^2}^2 = -p(p-1)\int u^{p-2}u_x(I^t u_x)^2 - 2p\int u^{p-1}u_x I^t u_x.$$

Notons que chaque terme est dérivé au plus une fois. Mais nous devons contrôler l'erreur, et l'on a :

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|I^t w_{nx}\|_{L^2}^2 \\ &= -p\int (U(t)V + w_n)^{p-1}(I^t(U(t)V + w_n)_x)_x I^t w_{nx} \end{aligned}$$

$$
\begin{aligned}
&- p(p-1)\int (U(t)V+w_n)^{p-2}(U(t)V+w_n)_x I^t (U(t)V+w_n)_x I^t w_{nx} \\
&- 2p\int (U(t)V+w_n)^{p-1}(U(t)V+w_n)_x I^t w_{nx} \\
= &-p\int (U(t)V+w_n)^{p-1}(I^t U(t)V_x)_x I^t w_{nx} \\
&+ \frac{p(p-1)}{2}\int (U(t)V+w_n)^{p-2}(U(t)V+w_n)_x (I^t w_{nx})^2 \\
&- p(p-1)\int (U(t)V+w_n)^{p-2}(U(t)V+w_n)_x I^t (U(t)V+w_n)_x I^t w_{nx} \\
&- 2p\int (U(t)V+w_n)^{p-1}(U(t)V+w_n)_x I^t w_{nx}.
\end{aligned}
$$

Dans cette égalité, les trois derniers termes se contrôlent aisément en utilisant les estimées (1.15), $V \in H^{1,1}$, et notre borne a priori $M_0^t(w_n(t)) \leq \varepsilon_0$: par exemple on a

$$
\begin{aligned}
&\left| \int (U(t)V+w_n)^{p-2}(U(t)V+w_n)_x (I^t w_{nx})^2 \right| \\
&\leq \|U(t)V+w_n\|_{L^\infty}^{p-3} \|U(t)V+w_n)(U(t)V+w_n)_x\|_{L^\infty} \|I^t w_{nx}\|_{L^2}^2 \\
&\leq \frac{C}{t^{2/3}(1+t)^{(p-2)/3}} (M_0^t(w_n(t)))^2.
\end{aligned}
$$

Mais la première intégrale

$$
\int (U(t)V+w_n)^{p-1}(I^t U(t)V_x)_x I^t w_{nx}
$$

contient un terme avec deux dérivées, que l'on ne contrôle pas avec la norme M_0^t. Heureusement, il s'agit d'un terme en $U(t)V$, et une solution est de choisir V adéquatement. Plus précisément, nous voulons que $M_0^t(I^t U(t)V_x))$ soit uniformément bornée en t (pour utiliser les estimées ponctuelles (1.15)). Mais

$$
(I^t U(t)V_x) = (J^t U(t)V_x) = U(t)(xV_x).
$$

On voit aisément que pour tout t, pour tout ϕ, $M_0^t(U(t)\phi) \leq C\|\phi\|_{H^{1,1}}$. Il suffit donc de demander $xV_x \in H^{1,1}$.

D'une manière analogue, dans le calcul de la norme $\dot{H}^1$ apparaît la condition $V_{xx} \in L^2$. Ainsi, on peut mener à bien les calculs dès que $V \in H^{2,2}$.

En intégrant ces relations et en tenant compte de ce que $w_n(S_n) = 0$, on en déduit que :

$$
\forall t \in [I_n, S_n], \qquad M_0^t(w_n(t)) \leq \frac{C}{t^\delta},
$$

où C est indépendante de n. Par minimalité de I_n, et par un argument de continuité, on déduit alors que $I_n \leq T_0$ où T_0 est tel que $CT_0^{-\delta} = \varepsilon_0/2$. Ceci conclut la preuve des estimations uniformes.

En ce qui concerne la partie unicité, étant données deux solutions $u_1(t)$ et $u_2(t)$ au problème, on considère la différence $v(t) = u_1(t) - u_2(t)$. Les estimées (1.15) permettent à nouveau de montrer que

$$
\frac{d}{dt}\|v(t)\|_{L^2}^2 \leq \frac{C}{t^{p/3}} \|v(t)\|_{L^2}^2 \left(\sup_t (M_0^t(u_1(t)) + M_0^t(u_2(t))) \right)^{p-1}.
$$

Par le lemme de Gronwall, et comme $v(t) \to 0$ dans L^2 quand $t \to \infty$, on en déduit que $v \equiv 0$.

Dans le cas critique, il est possible d'appliquer la même méthode. Il est également possible de procéder par point fixe. Soit $u(t)$ la solution désirée, on définit le terme d'erreur :

$$u(t) = U(t)V + w(t), \quad \text{i.e.} \quad w(t) = u(t) - U(t)V, \quad \lim_{t\to\infty} \|w\|_{L^2} = 0,$$

et on cherche à exprimer $w(t)$ comme point fixe d'une certaine fonctionnelle. u solution de (gKdV) satisfait la formule de Duhamel (1.5) (que l'on écrit entre les temps τ et 0, et τ et t) :

$$u(\tau) = U(\tau)u(0) + \partial_x \int_0^\tau U(\tau - s)u^5(s)ds = U(\tau - t)u(t) + \partial_x \int_t^\tau U(\tau - s)u^5(s)ds.$$

On en déduit que $w(t)$ satisfait

$$U(\tau)V + w(\tau) = U(\tau - t)(U(t)V + w(t)) + \partial_x \int_t^\tau U(\tau - s)(U(s)V + w(s))^5(s)ds.$$

Les $U(\tau)V$ se simplifient, et après composition par $U(t - \tau)$, on obtient

$$U(t - \tau)w(\tau) = w(t) + \partial_x \int_t^\tau U(t - s)(U(s)V + w(s))^5(s)ds.$$

On fait à présent tendre $\tau \to \infty$, alors notre hypothèse est que $\|U(t - \tau)w(\tau)\|_{L^2} = \|w(\tau)\|_{L^2} \to 0$, et on obtient $w(t)$ comme point fixe :

$$w(t) = -\partial_x \int_t^\infty U(t - s)(U(s)V + w(s))^5(s)ds. \tag{1.21}$$

A partir de là, il suffit de reprendre l'estimée (1.11) de l'espace $L^5_x L^{10}_t$. Si T_0 est assez grand, $\|U(t)V\|_{L^5_x L^{10}_t(t\geq T_0)}$ est aussi petit que l'on veut, et la fonctionnelle associée à (1.21) est contractante dans une boule adéquate de $L^5_x L^{10}_t$. Ainsi le point fixe désiré existe et est unique.

En fait dans les deux cas, le point clé est que l'on travaille pour des temps t grands : ainsi $U(t)V$ a déjà dispersé. C'est cela qui explique que l'on construit un opérateur d'onde pour de grandes données V.

1.1.3 La théorie autour des solitons

On a jusqu'à présent essentiellement étudié des aspects purement linéaires des solutions de (gKdV). Si l'on souhaite étudier les solitons, une telle approche est vouée à l'échec : les solitons sont d'une certaine manière les objets minimaux où la non-linéarité joue un rôle essentiel.

Solitons et dispersion

Le point de départ des résultats qui suivent est une caractérisation variationnelle des solitons (dans le cas critique), due à Weinstein [56].

Théorème 1.4 (Caractérisation variationnelle des solitons [56]). *Soit Q un soliton du cas L^2-critique, c'est-à-dire vérifiant :*

$$Q \in H^1, \quad Q > 0, \quad Q_{xx} + Q = Q^5.$$

Alors on a l'inégalité de Gagliardo-Nirenberg avec constante optimale :

$$\forall v \in H^1, \qquad \frac{1}{6}\int v^6 \leq \left(\frac{\int v^2}{\int Q^2}\right)^2 \int v_x^2. \tag{1.22}$$

En particulier, si $E(v) \leq 0$, alors $\int v^2 \geq \int Q^2$.

Enfin, si $E(v) \leq 0$ et de plus $\int v^2 = \int Q^2$, alors v est un soliton : il existe $\lambda_0 > 0$ et $x_0 \in \mathbb{R}$ tel que

$$v(x) = \lambda_0^{1/2} Q(\lambda_0(x - x_0)).$$

Ce théorème repose essentiellement sur la méthode de régularisation de Schwartz : Q est vu comme un minimiseur de la fonctionnelle $J(v) = \|v\|_{L^2}^4 \|v_x\|_{L^2}^2 \|v\|_{L^6}^{-6}$.

Un corollaire immédiat du Théorème 1.4 est le suivant :

Corollaire 1.2. *Soit $u_0 \in H^1$ tel que $\|u_0\|_{L^2} < \|Q\|_{L^2}$. Alors la solution $u(t)$ associée de (cKdV) est globale en temps.*

Quinze ans après ce résultat, de grandes avancées dans la compréhension du problème critique ont été faites, sous l'impulsion de Yvan Martel et Frank Merle. L'un des premiers résultats est une caractérisation dynamique des solitons : ce sont les seules solutions de (cKdV) qui ne dispersent pas.

Théorème 1.5 (Solutions non dispersives [26]). *Soit $u_0 \in H^1$ tel que*

$$\int u^2 = \int Q^2 + \alpha, \quad 0 \leq \alpha < \alpha_0,$$

et on suppose que la solution $u(t)$ de (gKdV) associée est globale et vérifie pour une certaine constante $C > 0$,

$$\forall t \geq 0, \quad \frac{1}{C} \leq \|u_x(t)\|_{L^2} \leq C,$$

et ne disperse pas dans L^2 :

$$\forall \varepsilon > 0, \exists R \geq 0 \ \ / \ \ \forall t \geq 0, \exists x(t) \in \mathbb{R}, \qquad \int_{|x - x(t)| \geq R} u^2(t,x)dx < \varepsilon.$$

Alors u est un soliton : il existe $\lambda_0 \in \mathbb{R}_+^*$ et $x_0 \in \mathbb{R}$ tels que

$$u(t,x) = \lambda_0^{1/2} Q(\lambda_0(x - x_0) - \lambda_0^3 t).$$

Il s'agit en fait d'un théorème de type Liouville : on met en évidence une structure très rigide pour les solutions de (cKdV) non dispersives. Des énoncés similaires de classification existent pour des équations elliptiques (méthode du plan tournant, voir Gidas, Ni, Nirenberg [10], Gidas et Spruck [11]) et paraboliques (Martel et Zaag [38]). Mais il s'agit ici d'une équation dispersive.

Cas L^2-critique : explosion

Le Théorème de classification précédent 1.5 est en fait un élément crucial des résultats étudiant la dynamique explosive des solutions de (cKdV) ayant des données initiales dans un voisinage des solitons, que nous énonçons à présent.

Dans la suite, $\alpha_0 > 0$ désigne une petite constante qui définit le voisinage de Q pour lequel les théorèmes énoncés seront vrais.

Théorème 1.6 (Explosion [32]). *Soit $u_0 \in H^1$ tel que*

$$\int u^2 = \int Q^2 + \alpha, \quad 0 < \alpha < \alpha_0, \quad et \quad E(u_0) < 0.$$

Alors la solution $u(t)$ de (gKdV) associée explose en temps fini ou infini.

Notons que $\nabla E(Q) = -Q$: l'ensemble des données initiales satisfaisant les hypothèses du théorème est donc non vide, c'est même un ouvert de H^1. On a ainsi exhibé une large classe de solutions explosives.

La condition d'énergie négative doit être mise en perspective avec le cas de l'équation de Schrödinger non-linéaire (NLS). En dimension n, cette équation est L^2-critique pour l'exposant $p = 1 + 4/n$. On a alors formellement la relation suivante, dite du Viriel :

$$\frac{d^2}{dt^2} \int |xu(t,x)|^2 dx = 16E(u_0).$$

Ainsi, une solution d'énergie négative explose nécessairement en temps fini (dans l'espace $\Sigma = H^1 \cap \{u | xu \in L^2\}$). Cette condition qui réapparait pour (gKdV) doit finalement être assez générique.

Théorème 1.7 (Stabilité du profil à l'explosion [28]). *Soit $u_0 \in H^1$ tel que*

$$\int u^2 = \int Q^2 + \alpha, \quad 0 < \alpha < \alpha_0, \quad et \quad E(u_0) < 0.$$

Alors la solution $u(t)$ de (gKdV) associée admet Q pour profil au temps d'explosion T : il existe une $x, \lambda : [0, T[\to \mathbb{R}_+^*$, et $\varepsilon \in \{-1, 1\}$ quand $t \to T$, tels que $\lambda(t) \to \infty$ et

$$\varepsilon \lambda^{1/2}(t) u(t, \lambda(t)x - x(t)) \rightharpoonup Q \quad dans\ H^1 - faible \quad quand \quad t \to T.$$

$\lambda(t) \sim \|u_x(t)\|_{L^2}$ est le taux d'explosion. La présence du $\varepsilon \in \{-1, 1\}$ est uniquement liée au fait que $-Q$ est également solution de (cKdV). Il y a ainsi essentiellement un seul profil à l'explosion, qui est donc stable.

D'autre part, il n'est pas possible d'espérer obtenir une convergence forte vers Q dans H^1, du fait de la conservation de la masse L^2 : il y a nécessairement de la dispersion (sur la gauche).

Théorème 1.8 (Explosion en temps fini [27]). *Soit $u_0 \in H^1$ tel que*

$$\int u^2 = \int Q^2 + \alpha, \quad 0 < \alpha < \alpha_0, \quad et \quad E(u_0) < 0,$$

et admettant la décroissance sur la droite suivante

$$\forall x > 0, \quad \int_{y \geq x} u_0^2(y) dy \leq \frac{C}{x^6}.$$

Alors la solution $u(t)$ de (gKdV) associée explose en temps fini $T < \infty$ et et on a la borne supérieure suivante sur le taux d'explosion : il existe une suite de temps $t_n \uparrow T$ tels que

$$\|u_x(t_n)\|_{L^2} \leq \frac{C}{|T - t_n|}.$$

Un argument simple lié à l'invariance d'échelle donne que le taux d'explosion vérifie $\|u_x(t)\| \geq \frac{C}{\sqrt[3]{T-t}}$. Un corollaire du Théorème 1.7 affirme que l'on a même

$$\|u_x(t)\|_{L^2} \sqrt[3]{T-t} \to \infty \quad \text{quand} \quad t \uparrow T.$$

Il reste tout de même un large espace entre cette borne inférieure et la borne supérieure du Théorème 1.8.

Ces théorèmes utilisent profondément le résultat de classification. Une idée centrale est de construire, à partir de $u(t)$, un objet récurrent $v(t)$, lui aussi solution de (cKdV), mais qui jouit de propriétés supplémentaires. En particulier, une propriété de (presque) monotonie dont nous reparlerons par la suite va se traduire par le fait que $v(t, x)$ est une fonction exponentiellement décroissante (en espace), et ce, uniformément en temps. Ainsi, v ne disperse pas, et par le Théorème 1.5, c'est donc un soliton.

Pour conclure cette étude du comportement explosif de solutions de (cKdV), notons qu'une série de travaux de Merle et Raphael donnent des résultats analogues dans le cas de l'équation (NLS) critique ($p = 1 + 4/n$) : cf. [33, 34, 35, 42, 36, 37].

Cas sous-critique : N-soliton etc.

Dans un esprit très proche, Martel et Merle ont établi, dans le cas de (gKdV) sous-critique, la stabilité asymptotique du soliton [25], un résultat qui fut ensuite étendu par Martel, Merle et Tsai [29] à la stabilité asymptotique d'une somme de N solitons découplés.

Théorème 1.9 (Stabilité de N solitons découplés [29])**.** *Soit $p = 2, 3$ ou 4. Soient $N \in \mathbb{N}$, et $0 < c_1 < \ldots < c_N$. Il existe $\gamma_0 > 0$ et $\alpha_0 > 0$ (petits) et A, L_0 (grands), tels que ce qui suit soit vrai. On suppose qu'il existe $L \geq L_0$, $\alpha < \alpha_0$ et $x_1^0 < \ldots < x_N^0$ tels que :*

$$\left\| u(0) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j^0) \right\|_{H^1} \leq \alpha, \quad \textit{avec} \quad x_j^0 \geq x_{j-1}^0 + L, \quad \textit{pour } j = 2, \ldots, N.$$

Alors il existe $x_1(t), \ldots, x_N(t) \in \mathbb{R}$ tels que

$$\forall t \geq 0, \quad \left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j(t)) \right\|_{H^1} \leq A(\alpha + e^{-\gamma_0 L}).$$

De plus, il existe $c_1^\infty, \ldots, c_N^\infty$ tels que $|c_j^\infty - c_j| \leq A(\alpha + e^{-\gamma_0 L})$ et

$$\forall j = 1, \ldots, N, \qquad \dot{x}_j(t) \to c_j^\infty \quad \textit{quand } t \to \infty,$$

$$\left\| u(t) - \sum_{j=1}^{N} Q_{c_j^\infty}(\cdot - x_j(t)) \right\|_{L^2(x \geq c_1/10)} \to 0 \quad \textit{quand } t \to \infty.$$

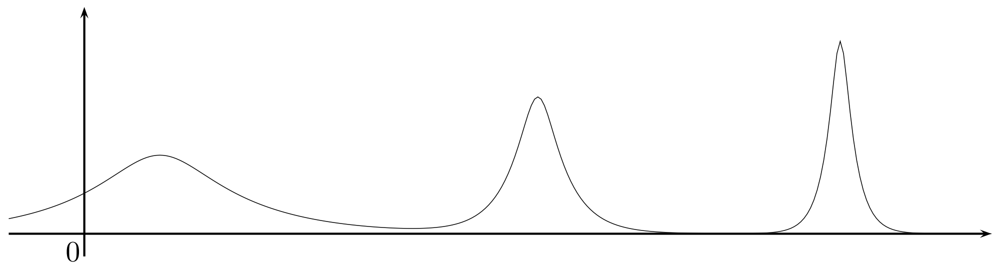

FIG. 1.2 – Un train de solitons découplés.

La première chose à voir est qu'il s'agit d'un résultat au voisinage de solitons découplés, c'est-à-dire que le soliton le plus rapide (et donc de masse L^2 la plus grande) est le plus à droite, puis vient le second plus rapide et ainsi de suite. D'autre part, les solitons sont déjà assez loin les uns des autres. Par ces deux conditions, aucun soliton ne devrait en rattraper un autre, et on espère que les solitons vont finalement interagir faiblement entre eux : le Théorème 1.9 affirme que tel est effectivement le cas.

Essayons d'éclairer un peu la notion de stabilité mise en jeu ici. Il n'est pas possible par exemple, d'espérer montrer que $c_j^\infty = c_j$. En effet, considérons le cas de deux solitons R_{c_1,x_0} et R_{c_2,x_0}. Si c_2 est suffisamment proche de c_1, alors la donnée initiale $u_0 = R_{c_2,x_0}(t=0)$ est dans un voisinage de $R_{c_1,x_0}(t=0)$. Pourtant, les solitons vont à des vitesses différentes et donc

$$\|R_{c_1,x_0}(t) - R_{c_1,x_0}(t)\|_{L^2} \to \|Q_{c_1}\|_{L^2} + \|Q_{c_2}\|_{L^2} \neq 0 \quad \text{quand} \quad t \to \infty.$$

Ainsi, il faut pouvoir tenir compte d'une légère variation de la vitesse (ou de la masse) $(c_j(t))$ des solitons dans la décomposition, ou d'une translation en espace $(x_j(t))$.

Par ailleurs, il n'est pas non plus possible d'espérer une convergence sur tout $\mathbb{R}$, du moins dans L^2. En effet, nous construirons un peu plus tard des solutions se comportant comme la somme de N solitons et d'un terme linéaire $U(t)V$. Si V est choisi adéquatement (et notamment de norme H^1 suffisamment petite), une telle solution $u(t)$ vérifiera la condition proximité de la somme de N solitons découplés. Mais asymptotiquement, la masse L^2 de $U(t)V$ ne disparaît pas (on peut faire le même raisonnement avec un petit soliton $R_{\varepsilon,0}$ se déplaçant très lentement sur la droite).

Au coeur de ce résultat se trouve une propriété de monotonie au niveau de la norme L^2. Soit $\sigma_0 > 0$ assez petit et

$$\psi(x) = 1 - \frac{2}{\pi} \arctan \exp\left(-\frac{\sqrt{\sigma_0}}{2} x \right).$$

Alors on voit aisément que ψ est décroissante, tend vers 1 en $-\infty$ et 0 en $+\infty$. De plus,

$$|\psi'''(x)| \leq -\frac{\sigma_0}{4}\psi'(x).$$

On cherche à séparer les solitons les uns des autres : introduisons donc, pour $j = 1, \dots, N-1$

$$\psi_j(x,t) = \psi(x - m_j(t)), \quad m_j(t) = \frac{c_j + c_{j+1}}{2} t + \frac{x_j + x_{j+1}}{2}.$$

Notons $R_j(t,x) = R_{c_j,x_j}(t,x) = Q_{c_j}(x - c_j t - x_j)$ les solitons.

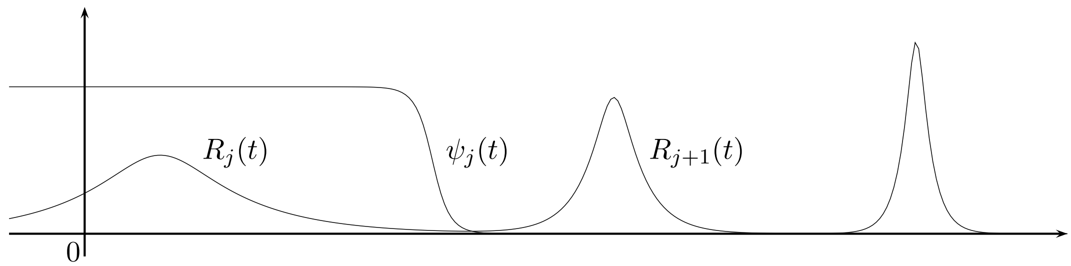

FIG. 1.3 – $\psi_j(t)$ sépare $R_j(t)$ et $R_{j+1}(t)$.

Proposition 1.1 (Presque monotonie)**.** *Soit $\varepsilon_0 > 0$ assez petit et $[a,b]$ un intervalle de temps tel que :*

$$\forall t \in [a,b], \qquad \left\| u(t) - \sum_{j=1}^{N} R_j(t) \right\|_{H^1} \leq \varepsilon_0.$$

Alors on a :

$$\forall j, \ \forall t \in [a,b], \qquad \frac{d}{dt} \int u^2(t,x)\phi_j(t,x) \geq -Ce^{-\gamma t},$$

où $\gamma = \min\{c_1, c_2 - c_1, \ldots, c_N - c_{N-1}\}$.

Grâce à cette propriété de monotonie, et en utilisant finement la conservation de la masse et de l'énergie (qui dans les cas sous-critique varient en fonction de la vitesse du soliton), on réussit à prouver la première partie du Théorème 1.9. Pour la convergence asymptotique, on fait une nouvelle fois appel à la propriété de rigidité du Théorème 1.5.

Énonçons sans plus attendre le pendant du Théorème 1.9, à savoir l'existence d'une unique solution de (gKdV) se comportant comme la somme de N solitons : cette solution s'appelle le N-soliton, et c'est un résultat de Martel [23].

Théorème 1.10 (Existence et unicité du N-soliton [23])**.** *Soit $p \in [2,5]$. Soient $N \in \mathbb{N}$, $0 < c_1 < \ldots < c_N$, et $x_1, \ldots, x_N \in \mathbb{R}$. Il existe $T_0 \in \mathbb{R}$ et une unique solution $u \in C([T_0, +\infty), H^1)$, de (gKdV) telle que :*

$$\left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j - c_j t) \right\|_{H^1} \to 0 \quad \textit{quand} \quad t \to \infty.$$

De plus $u \in C^\infty([T_0, \infty) \times \mathbb{R})$ et la convergence a lieu dans tout les H^s avec un taux exponentiel

$$\exists \gamma > 0, \ \forall s \geq 0, \exists A_s \quad / \quad \left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j - c_j t) \right\|_{H^1} \leq A_s e^{-\gamma t}.$$

Notons que pour $p = 2, 3$, l'existence du N-soliton était un résultat connu de la théorie du scattering inverse. L'unicité est par contre nouvelle dans tous les cas, et c'est un résultat tout à fait étonnant. Remarquons également que le résultat est également vrai dans le cas critique $p = 5$, bien que les solitons soient instables.

1.1.4 Comportement mixte “linéaire” et “non-linéaire”

L’un des problèmes étudiés dans cette thèse est le suivant : étant donné un comportement asymptotique acceptable, peut-on construire une solution de (gKdV) qui admette ce profil ? C’est en quelque sorte le problème réciproque de celui qui, étant donné une donnée initiale, est de comprendre la dynamique, quand $t \to \infty$, de la solution de (gKdV) associée. Le cas purement dispersif (comportement en $U(t)V$) est traité par le Théorème 1.3, et le cas non-dispersif (N-soliton) est l’objet du Théorème 1.10. L’objectif est à présent de construire des solutions ayant un comportement mixte.

Opérateur d’onde non-linéaire

L’unicité du N-soliton (Théorème 1.10) montre que ce comportement est tout à fait exceptionnel : il y a d’une certaine manière une rigidité liée à la structure du N-soliton, ce qui peut se comprendre par le fait que le N-soliton est une solution qui ne disperse pas (en un sens à préciser).

On s’attend donc à ce qu’il y ait d’autres comportements asymptotiques possibles : on a déjà montré que pour $p > 3$, il existe des solutions dispersant complètement, car elles se comportent comme les solutions linéaires. En fait, pour $p = 4$ et $p = 5$, il existe des solutions qui ont un comportement mixte, c’est-à-dire qui se découplent en une somme de soliton et d’une solution linéaire. Nous appelons opérateur d’onde non-linéaire l’application qui, à un tel comportement asymptotique (c’est à dire la donnée des R_j et d’une fonction V), associe une solution de (gKdV), globale, se comportant comme $U(t)V + \sum_j R_j(t)$ quand $t \to \infty$.

Théorème 1.11 (Opérateur d’onde non-linéaire, cas sous-critique [6]). *Soit $p = 4$. Soit $V \in H^{5,1} \cap H^{2,2}$ telle que :*

$$x_+^{4/3} D^5 V \in L^2, \quad x_+^8 V \in H^1.$$

Soient $N \in \mathbb{N}$, $0 < c_1 < \ldots < c_N$ et $x_1, \ldots, x_N \in \mathbb{R}$, on introduit N solitons $R_j(t,x) = Q_{c_j}(x - x_j - c_j t)$. Alors il existe $u^* \in C([T_0, +\infty[, H^4)$, pour un certain $T_0 \in \mathbb{R}$, solution de (gKdV) (avec $p = 4$), tel que :

$$\left\| u^*(t) - U(t)V - \sum_{j=1}^N R_j(t) \right\|_{H^4} + M_0^t\Big(u^*(t) - U(t)V - \sum_{j=1}^N R_j(t)\Big) \le C t^{-(p-3)/3}.$$

Théorème 1.12 (Opérateur d’onde non-linéaire, cas critique [5]). *Soit $p = 5$. Soit $V \in H^1$ tel que*

$$x_+^{2+\delta_0} V \in L^2 \text{ pour un certain } \delta_0 > 0.$$

Soient $N \in \mathbb{N}$, $0 < c_1 < \ldots < c_N$ et $x_1, \ldots, x_N \in \mathbb{R}$, on introduit N solitons $R_j(t,x) = Q_{c_j}(x - x_j - c_j t)$. Alors il existe $u^* \in C([T_0, +\infty[, H^1)$, pour un certain $T_0 \in \mathbb{R}$, solution de (cKdV), tel que :

$$\left\| u^*(t) - U(t)V - \sum_{j=1}^N R_j(t) \right\|_{H^1} \to 0 \quad \text{quand} \quad t \to \infty.$$

La condition de décroissance sur la droite demandée pour V correspond en fait à une interaction faible du terme linéaire $U(t)V$ avec les solitons.

De ce point de vue, le résultat dans le cas critique est probablement presque optimal (à δ_0 près), avec une condition uniquement dans L^2. Dans le cas sous-critique, on est obligé de demander à la fois beaucoup de régularité et de décroissance sur la droite pour V, et de plus, on doit se restreindre à la non-linéarité $x \mapsto x^4$ ($p=4$), ce qui correspond à une meilleure intégrabilité et régularité que celles que donnent $p>3$. Ces défauts sont liés à une moins bonne compréhension du phénomène de scattering dans le cas non-critique.

En contruisant un opérateur d'onde non-linéaire pour de grandes données V, on obtient une classe très générale de comportements. Cependant, il faut remarquer que toutes les solutions ainsi construites sont (globales et) d'énergie positive dans le cas critique. Cela correspond bien à l'idée que les solutions d'énergie négative explosent dans ce cas.

Schéma de preuve

La méthode de construction des solutions pour les Théorèmes 1.11 et 1.12 suit le même schéma que celle du Théorème 1.3 : on définit une suite de solutions u_n qui ont exactement le profil désiré à un temps donné, à savoir

$$\begin{cases} u_{nt} + (u_{nxx} + |u_n|^p)_x = 0, \\ u_n(S_n) = U(S_n)V + \sum_{j=1}^N R_j(S_n), \end{cases} \tag{1.23}$$

où $S_n \to \infty$ quand $n \to \infty$, et on introduit l'erreur

$$w_n(t) = u_n(t) - U(t)V - \sum_{j=1}^N R_j(t), \tag{1.24}$$

qui vérifie le système

$$\begin{cases} w_{nt} + \Big(w_{nxx} + |u_n|^p - \sum_{j=1}^N R_j^p(t) \Big)_x = 0 \\ w_n(S_n) = 0. \end{cases} \tag{1.25}$$

Le coeur du problème est à nouveau de démontrer que w_n est défini sur un intervalle de la forme $[T_0, S_n]$ (avec T_0 fixe) et satisfait des estimées uniformes : par exemple, dans le cas sous-critique, que

$$\forall n, \forall t \in [T_0, S_n], \quad \|w_n(t)\|_{H^4} \le Ct^{-(p-3)/3}, \ \text{ et } M_0^t(w_n(t)) \le Ct^{-\delta}.$$

Comme précédemment, par un argument de continuité usuel, on se ramène a montrer la proposition suivante dans le cas sous-critique (et une proposition similaire dans le cas critique).

Proposition 1.2 (Estimée uniforme sous hypothèse de petitesse)**.** *Il existe $\varepsilon_0 > 0$ tel que ce qui suit soit vrai. Soit $[I_n, S_n]$ un intervalle vérifiant la condition suivante :*

$$\forall t \in [I_n, S_n], \qquad \|w_n(t)\|_{H^4} + M_0^t(w_n(t)) \le \varepsilon_0.$$

Alors en fait on a l'estimée de décroissance :

$$\forall t \in [I_n, S_n], \qquad \|w_n(t)\|_{H^4} \le Ct^{-(p-3)/3}, \ \text{ et } M_0^t(w_n(t)) \le Ct^{-\delta}.$$

La démonstration de ce résultat se découpe en deux grandes parties : estimer les interactions non-linéaires (sur la droite $x \geq ct$), puis estimer les interactions linéaires (sur la gauche $x \leq ct$).

Comparée à la construction de l'opérateur d'onde linéaire (Théorème 1.3) ou du N-soliton (Théorème 1.10), la principale difficulté est bien sûr que lorsqu'on essaye de contrôler les interactions de l'erreur $w_n(t)$ avec les solitons, la partie linéaire $U(t)V$ interfère, et lorsque l'on tente de borner les interactions de $w_n(t)$ avec le terme linéaire $U(t)V$, les solitons interfèrent. Et ces interférences sont a priori très désagréables, car par définition, l'outil linéaire n'est pas approprié pour comprendre le comportement non-linéaire des solitons, et les outils non-linéaires (paraboliques) ne sont pas faits pour traiter des termes linéaires (dispersifs).

Il s'avère cependant que la présence du terme $U(t)V$ perturbe assez faiblement les solitons : cela est dû au fait que l'on a choisi V avec suffisamment de décroissance sur la droite pour que l'interaction avec les solitons soit petite. Plus précisément, introduisons une fonction de coupure entre $x = 0$ et le soliton le plus lent $R_1(t)$:

$$\psi_0(t,x) = \psi(x - \sigma_0 t), \quad \text{où} \quad \sigma_0 = \frac{1}{2}\min\{c_1, c_2 - c_1, \dots, c_N - c_{N_1}\}.$$

On démontre que pour $t \in [I_n, S_n]$,

$$\|w_n(t)\|_{L^2(1-\psi_0(t))} \leq e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + (1 + S_n - t)\|U(S_n)V\|_{L^2(1-\psi_0(S_n))} + \|U(t)V\|_{L^2(1-\psi_0(t))} + \int_t^{S_n} \|U(t)V\|_{L^2(1-\psi_0(t)}dt. \quad (1.26)$$

Supposons par exemple que $\|U(t)V\|_{L^2(1-\psi_0(t)} \leq Ct^{-q}$ (ce qui est le cas dès que $(1+x_+^q)V \in L^2$), (1.26) affirme que

$$\|w_n(t)\|_{L^2(1-\psi_0(t))} \leq Ct^{-q+1}. \quad (1.27)$$

Remarquons que dans le cas du N-soliton (i.e. $V = 0$), on a une décroissance exponentielle. Ici, il ne reste qu'une décroissance polynomiale, qui peut être rendue arbitrairement rapide en demandant suffisamment de décroissance à droite sur V.

La preuve de (1.26) s'appuie sur les outils développés pour les Théorèmes 1.9 et 1.10 : elle est très proche dans les cas critique et sous-critique.

D'une part, grâce au fait que V interagit peu avec les solitons, il est possible de moduler $w_n(t)$ par rapport aux $2N$ directions dégénérées d'une certaine forme quadratique (liée à l'énergie) : la modulation est différente dans les cas critique et sous-critique, mais dans tout les cas, on est en position d'utiliser la presque-positivité de cette forme quadratique. Cette modulation ne tient pas compte de $U(t)V$, et le prix à payer est justement $\|U(t)V\|_{L^2(1-\psi_0(t))}$, à savoir l'interaction de $U(t)V$ et des solitons, d'où en particulier le terme intégral dans (1.26).

D'autre part, du fait que V disperse dans L^∞, la propriété de presque-monotonie est préservée, et c'est le point clé. On peut alors réutiliser un argument de resommation d'Abel, et l'on en déduit, avec la positivité de la forme quadratique précédemment citée, le contrôle (1.26).

Par contre le contrôle sur la gauche (interaction avec $U(t)V$) est plus complexe.

Commençons par considérer le cas critique. On souhaite donc travailler dans l'espace $L^5_x L^{10}_t$. Écrivons l'équation vérifiée par $w_n(t)$, selon la formule de Duhamel :

$$w_n(t) = -\partial_x \int_t^{S_n} U(-s) \left((w_n(s) + U(s)V + \sum_{j=1}^N R_j(s))^5 - \sum_{j=1}^N R_j^5(s) \right) ds.$$

Bien sûr en développant, les termes ne contenant que des $R_j(s)$ s'annulent ou sont exponentiellement petits en espace et en temps, et ceux ne contenant que $w_n(s)$ ou $U(s)V$ sont traités comme dans le cas linéaire. Mais il reste des termes d'interaction, du type :

$$-\partial_x \int_t^{S_n} U(-s)(w_n(s) R_1^4(s)) ds.$$

(où $R_1(t)$ est un soliton, ici le plus lent). On cherche à en estimer la norme $L^5_x L^{10}_t$, et les estimées linéaires (1.10) donnent

$$\left\| \partial_x \int_t^{S_n} U(-s)(w_n(s) R_1^4(s)) ds \right\|_{L^5_x L^{10}_t (t \geq T_0)} \leq C \| w_n(s) R_1^4(s) \|_{L^1_x L^2_t (t \geq T_0)}.$$

Or les solitons ne sont pas seulement grands dans l'espace $L^5_x L^{10}_t$, en fait ils n'appartiennent pas à $L^5_x L^{10}_t (t \geq T)$ pour tout T, car ils ne dispersent pas.

Cependant, on s'aperçoit que l'on a ici affaire à un terme en $w_n(s)$ localisé sur la droite : on peut conclure si l'on arrive à intervertir les sommations en espace et en temps, et ainsi à utiliser l'estimée de décroissance (1.26) obtenue précédemment (sans perte de dérivation toutefois). Cela est possible, et un lemme technique nous dit que quitte à perdre un petit $\delta_0 > 0$, on obtient ce que l'on veut. Plus précisément, si pour $t \in [I_n, S_n]$,

$$\| w_n(t) \|_{L^2(1-\psi_0(t))} \leq C t^{-1+\delta_0},$$

on a :

$$\| w_n(s) R_1(s) \|_{L^1_x L^2_s (s \geq t)} \leq C e^{-\frac{\sigma_0 \sqrt{\sigma_0}}{4} t} + C t^{-\delta_0} \| w_n(s) \|_{L^5_x L^{10}_s (s \geq t)}.$$

On réussit ainsi à boucler les estimées dans $L^5_x L^{10}_t$, puis à revenir dans $C^0_t L^2_x$. La preuve de la Proposition 1.2 est donc terminée dans le cas critique.

Dans le cas sous-critique $p = 4$ cependant, les choses se passent plus mal. Il s'agit d'estimer $M^t_0(w_n(t))$. Le contrôle de la norme H^1 n'est pas difficile en utilisant (1.15).

Mais les difficultés arrivent lorsque l'on calcule la dérivée en temps de $\| I^t w_{nx}(t) \|^2_{L^2}$: des termes problématiques apparaissent, du type

$$\int R_1^2 R_{1x} (I^t w_{nx})^2. \tag{1.28}$$

Il s'agit bien d'un terme localisé (grâce au $R_1(t)$), mais au niveau de régularité H^3 pour w_n (I^t doit être vu comme un opérateur d'ordre 2). On ne peut plus utiliser directement (1.26), et il n'est pas évident d'améliorer cette dernière estimée au niveau H^3, car les arguments "non-linéaires" se placent dans l'espace d'énergie.

L'idée est alors d'utiliser des lois de presque conservation, au niveau H^2, H^3 et également H^4. Pour H^2, on dispose de la relation

$$\frac{1}{2} \frac{d}{dt} \left(\int u_{xx}^2 - \frac{20}{3} \int u_x^2 u^3 \right) = 2 \int u_x^5 + 80 \int u_x^3 u^5,$$

qui est valable si $u(t)$ est une solution de (gKdV) pour $p=4$. Il existe une relation analogue pour $w_n(t)$. Notons que pour cette relation le terme $\int u_x^5$ devra être traité en utilisant l'inégalité de Gagliardo-Nirenberg. On arrive alors à démontrer par un argument de type Gronwall que

$$\|w_n(t)\|_{H^4} \leq \frac{C}{t^{1/3}}.$$

(i.e. un taux $t^{-\frac{p-3}{3}}$ pour $p=4$). Ce taux ne peut être amélioré, et c'est pour cela que l'on doit avoir un contrôle jusqu'au niveau H^4 : par interpolation de cette inégalité avec (1.26), on obtient que $\|w_n\|_{H^3(1-\psi_0(t))}$ a une décroissance polynomiale arbitrairement grande pourvu que V soit choisi adéquatement (cf. (1.27), et l'on peut donc contrôler le terme problématique (1.28) : cela conclut la preuve dans le cas $p=4$.

Il faut noter que cette méthode ne se généralise pas facilement à d'autres p sous-critiques : tout d'abord, l'obtention de relation de presque conservation nécessite une grande régularité sur la non-linéarité (il faut C^4 pour avoir la relation au niveau H^2). D'autre part, p doit être assez grand (intégrabilité assez grande) pour que le lemme de type Gronwall puisse être utilisé : il apparaît que $p=4$ est d'un certaine manière un exposant critique pour cette propriété.

Il reste à conclure les théorèmes : cela se fait dans les deux cas de manière assez proche. On obtient la compacité L^2 de la suite $w_n(T_0)$ par un argument lié à la propriété de monotonie : ainsi $w_n(T_0) \to \varphi$ dans L^2. On définit

$$\begin{cases} u_t^* + (u_{xx}^* + |u^*|^p)_x = 0, \\ u^*(T_0) = \varphi + U(T_0)V + \sum_{j=1}^N R_j(t). \end{cases}$$

Dans le cas sous-critique, il y a en fait convergence dans H^3 par interpolation et la continuité du flot termine la preuve. Dans le cas critique, la continuité du flot L^2 donne la convergence de $u^*(t)$ dans L^2. Pour obtenir la convergence H^1, il faut ajouter un argument utilisant la conservation de l'énergie, associé à un choix astucieux des S_n.

Ouverture

La construction de solutions de (gKdV) avec comportement asymptotique donné doit encore être comprise dans les cas $p \in]3,5[\setminus \{4\}$, et également dans les cas sur-critiques $p>5$. Dans les cas sous-critiques, il s'agira essentiellement d'améliorer la compréhension de l'opérateur de KdV linéaire $U(t)$, et notamment le scattering linéaire à données petites. La question de l'unicité de telles solutions, pour un comportement donné doit également être étudiée. Alors ce problème (rétrograde, avec donnée finale) sera essentiellement résolu car une conjecture plausible est que toute solution de (gKdV) se découple effectivement en une somme de solitons et un terme linéaire.

La question est alors de prouver cette conjecture : étant donnée une condition initiale u_0, quel est le comportement asymptotique de la solution de (gKdV) associée ? Par exemple, on peut se demander s'il y a scattering linéaire dès que l'on sait qu'il n'y a aucun soliton : plus précisément, si l'on suppose que $\|u(t)\|_{L^\infty} \to 0$ quand $t \to \infty$, a-t-on scattering linéaire ?

1.2 Dynamique des wave maps équivariantes en dimension critique

1.2.1 Que sont les Wave maps ?

Wave maps générales

Les wave maps sont un problème modèle pour la compréhension de la dynamique des équations des ondes dans un contexte géométrique. Elles apparaissent naturellement en relativité générale et en théorie des cordes : dans la littérature de Physique, on les étudie sous le nom de σ-modèles (généralisés).

Soit (N, γ) une variété riemmanienne complète de dimension m. On suppose que $N \hookrightarrow \mathbb{R}^\kappa$ de manière isométrique. On dit que $U : \mathbb{R}_t \times \mathbb{R}^n_x \to N$ est une wave map si et seulement si

$$\forall t, x, \quad \Box U(t,x) \perp T_{U(t,x)} N. \tag{1.29}$$

(caractérisation extrinsèque). D'une certaine manière, c'est le problème géométrique le plus simple que l'on puisse imaginer dans le cadre de l'équation des ondes : si $N = \mathbb{R}^m$, une wave map est simplement une solution de l'équation des ondes libres (linéaire) $\Box U = 0$.

Dans un cadre abstrait, on peut aussi voir U comme un point critique de l'action ($\eta = \mathrm{diag}(-1, 1 \ldots, 1)$)

$$S(U) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} \gamma_{ij} \partial_\alpha U^i \partial_\beta U^j \eta^{\alpha\beta} dx^n dt. \tag{1.30}$$

(caractérisation variationnelle). Il faut voir $(\mathbb{R}_t \times \mathbb{R}^n_x, \eta)$ comme l'espace de Minkowsky de dimension $1+n$: d'une manière similaire, on peut définir des wave maps issues de variétés lorenztiennes générales (M, η), comme des points critiques pour l'action

$$\int_M \gamma_{ij} \partial_\alpha U^i \partial_\beta U^j \eta^{\alpha\beta}.$$

Cependant nous n'étudierons que les wave maps issues de l'espace de Minkowsky $\mathbb{R}^{1+n}$.

On déduit de sa caractérisation variationnelle (1.30) que U vérifie l'équation d'Euler-Lagrange associée, à savoir le système suivant (en coordonnées locales) :

$$\forall i = 1, \ldots, m, \qquad \Box U^i + \Gamma^i_{jk}(U) \partial_\alpha U^j \partial^\alpha U^k = 0. \tag{1.31}$$

(les Γ^i_{jk} sont les symboles de Christoffel de la variété (N, γ)). C'est un système d'équations des ondes semi-linéaires (quasi-linéaire dans le cas d'une variété lorentzienne générale).

Notons dès à présent deux propriétés remarquables des wave maps. Tout d'abord, il y a conservation de l'énergie :

$$E(U(t)) = \int_{\mathbb{R}^n} \gamma_{ij} \partial_\alpha U^i \partial_\alpha U^j dx^n = \|DU\|^2_{L^2(\gamma)} = \text{cste}.$$

(où D représente l'ensemble des dérivées, spatiales et temporelles). D'autre part, on a l'invariance d'échelle suivante :

$$U(t,x) \text{ wave map } \iff U_\lambda(t,x) = U(\lambda t, \lambda x) \text{ wave map.}$$

Wave maps à symétrie

On s'intéresse à un cas particulier de wave maps : celles à symétrie dite équivariante. Pour cela, on suppose maintenant que $N = [0, R[\times \mathbb{S}^{k-1}$ avec des coordonnées polaires (ρ, χ), on se donne une fonction g C^1 telle que $g(0) = 0$ et $g'(0) = 1$ et on suppose que la métrique γ s'écrit $ds^2 = d\rho^2 + g^2(\rho)d\chi^2$. On note $x = (r, \omega)$ les coordonnées polaires sur $\mathbb{R}^n$. On dit que U est *équivariante corotationnelle* (ou plus brièvement équivariante) si $n = k$ et s'il existe $u = u(t, r)$ tel que :

$$U(t, r, \omega) = (u(t, r), \omega).$$

Toutes les conditions sur U s'écrivent alors sur $u : \mathbb{R}_t \times \mathbb{R}_r^+ \to \mathbb{R}$, qui vérifie (on note $f = g'g$) :

$$u_{tt} - u_{rr} - \frac{n-1}{r}u_r = -\frac{f(u)}{r^2}. \tag{1.32}$$

L'énergie s'écrit à présent :

$$E(u) = \int \left(u_t^2 + u_r^2 + \frac{g^2(u)}{r^2} \right) r^{n-1}dr = E(u_0, u_1). \tag{1.33}$$

Notons que $E(u) = \|U_t\|_{L^2}^2 + \|\nabla U\|_{L^2}$. Ainsi l'espace d'énergie

$$H = \left\{ (u, v) | E(u, v) = \|(u, v)\|_H^2 = \int \left(v^2 + u_r^2 + \frac{g^2(u)}{r^2} \right) r^{n-1}dr < \infty \right\}$$

correspond à $\dot{H}^1 \times L^2$ pour (U, U_t). Par extension, on notera également, pour une fonction $v : \mathbb{R}_+ \to \mathbb{R}$:

$$E(v) = \int \left(v_r^2 + \frac{g^2(v)}{r^2} \right) r^{n-1}dr.$$

Par ailleurs, on a une invariance d'échelle :

$$u_\lambda(t, r) = u(\lambda t, \lambda r) \quad \text{wave map} \iff u \quad \text{wave map.}$$

Cette équation d'onde propage l'information à vitesse finie : cela s'exprime par exemple par le fait que l'énergie est décroissante sur les cônes. Si on note $E(u, a, b) = \int_a^b (u_t^2 + u_r^2 + g(u)/r^2)rdr$, on a par exemple :

$$\forall R \geq 0, \forall t, \tau, \quad E(u(t), 0, R) \leq E(u(t + \tau), 0, R + |\tau|). \tag{1.34}$$

Wave maps critiques

On restreint l'étude à $n = 2$ et aux wave maps équivariantes. Dans ce cas, $E(u_\lambda) = E(u)$: on dit que la dimension 2 est critique (pour l'énergie). On est amené à l'étude du problème de Cauchy :

$$\begin{cases} u_{tt} - u_{rr} - \dfrac{u_r}{r} = -\dfrac{f(u)}{r^2}, \quad t \in \mathbb{R}, r \in \mathbb{R}^+, \\ (u, u_t)(t = 0) = (u_0, u_1). \end{cases} \tag{1.35}$$

En fait cette restriction n'est pas très forte, car le fait que la dimension soit critique complique plutôt les choses. D'autre part, en dimension petite, et notamment en dimension 2, les estimations de Strichartz donnent moins d'informations (parce que le D'Alembertien est moins dispersif).

Le problème de Cauchy pour les wave maps équivariantes est bien posé dans l'espace d'énergie H (Shatah et Tahvildar-Zadeh [46, Théorème 1.1]).

Théorème 1.13 (Existence locale dans H [46])**.** *Soient $(u_0, u_1) \in H$. Alors il existe $T > 0$ et une unique wave map u solution du problème (1.35) telle que :*

$$(u, u_t) \in L^\infty([0,T), H), \qquad u \in L^{10/3}([0,T), \dot{B}^{1/2}_{10/3,10/3}(\mathbb{R}^+, rdr)).$$

Ce résultat est une conséquence de l'estimée de Strichartz suivante :

$$\|\phi\|_{\dot{B}^s_{q,q}} \leq \|\Box_m \phi\|_{\dot{B}^s_{q',q'}} + \|\phi|_{t=0}\|_{\dot{H}^{s+\frac{1}{2}}} + \|\partial_t \phi|_{t=0}\|_{\dot{H}^{s-\frac{1}{2}}}, \tag{1.36}$$

où $\Box_m$ est le D'Alembertien en dimension m, $q = \frac{2(m+1)}{m-1}$ et $q' = \frac{2(m+1)}{m+3}$, et $s \in \mathbb{R}$. Notons que grâce à la vitesse finie de propagation, cette estimée peut être localisée (sur un cône de lumière). Considérons la fonction $v = u/r$, qui vérifie l'équation des ondes (radiale) non-linéaire suivante :

$$v_{tt} - v_{rr} - \frac{3}{r} v_r = v^3 Z(rv), \tag{1.37}$$

où $Z : \mathbb{R} \to \mathbb{R}$ est une fonction régulière bornée. (1.37) est donc une équation des ondes en dimension 4 avec non-linéarité critique : on applique à v l'estimée de Strichartz précédente (1.36) avec $m = 4$ et $q = 10/3$, ainsi que des estimées d'énergie, et l'on obtient le résultat.

Notons également que la démonstration du Théorème 1.13 prouve que pour toutes données initiales suffisamment petites dans H, la wave map qui en est issue existe globalement en temps.

D'une manière générale, le problème de Cauchy pour les wave maps (critiques ou non) à valeur dans des variétés N, est maintenant bien compris pour les données petites dans $H^{n/2} \times H^{n/2-1}$: c'est l'espace de Sobolev critique pour les wave maps de $\mathbb{R}_t \times \mathbb{R}^n \to N$. Aux travaux fondateurs de Tao [52, 53] (concernant les wave maps à valeur dans la sphère $\mathbb{S}^m$) ont succédé les généralisations de Krieger [20], Klainerman et Rodnianski [17], Tataru [54, 55] (voir également les travaux de Klainerman et Machedon [16], Klainerman et Selberg [18]). Pour une large classe de variétés N, il y a existence globale des wave maps d'énergie petite.

Théorème 1.14 (Existence globale à données petites [54])**.** *Soit $n \geq 2$ et N une variété riemannienne qui s'injecte isométriquement dans $\mathbb{R}^\kappa$. Alors les wave maps $\mathbb{R}_t \times \mathbb{R}^n \to N$ avec données initiales suffisamment petites dans $H^{n/2} \times H^{n/2-1}$ sont globales.*

Revenons à présent aux wave maps équivariantes.

1.2.2 Comportement à l'explosion

La question suivante est donc celle de l'existence globale des données d'énergie grande : elle est ouverte. On conjecture qu'il peut y avoir explosion dans certains cas, notamment pour les wave maps équivariantes. Pour la sphère $N = \mathbb{S}^2$ ($g = \sin$), Bizòn et al. [1] prédisent numériquement la formation de singularités en temps fini. On appelle ce phénomène "explosion".

Concentration d'énergie

La première chose à voir est que l'explosion est liée à un phénomène de concentration de l'énergie. Le Théorème 1.13 donne la condition suivante pour l'explosion :

Condition d'explosion. Il existe $\varepsilon_0 > 0$ tel que u explose au temps T si et seulement si

$$\liminf_{t\uparrow T} E(u(t), 0, T-t) \geq \varepsilon_0. \tag{1.38}$$

(notons que la concentration d'énergie ne peut se faire qu'au point $r = 0$, pour une wave map d'énergie finie, du fait de la condition de décroissance de l'énergie sur les cônes et de la symétrie radiale).

Une idée pour assurer l'existence globale est alors de considérer des métriques qui vont empêcher la concentration d'énergie. Dans [45], Shatah et Tahvildar-Zadeh montrent que si $g' \geq 0$ (cas des variétés géodésiquement convexes), il ne peut y avoir concentration. Cette condition fut étendue au cas $g(\rho) + g'(\rho)\rho > 0$ par Grillakis.

Profil à l'explosion

La deuxième avancée est un résultat de Struwe [50] qui caractérise le profil à l'explosion. Commençons par définir les applications harmoniques : ce sont les solution stationnaires de (1.35), c'est-à-dire les fonctions $Q\ : \mathbb{R}^+ \to \mathbb{R}$ telles que

$$Q_{rr} + \frac{Q_r}{r} = \frac{f(Q)}{r^2}. \tag{1.39}$$

Théorème 1.15 (Stabilité du profil à l'explosion [50])**.** *Soit u une wave map explosant au temps $T = 0$. Alors il existe une suite de temps $t_n \uparrow 0$ et de paramètres d'échelle $\lambda(t_n)$ tels que $\lambda(t)|t| \to \infty$ et :*

$$u\left(t_n + \frac{t}{\lambda(t_n)}, \frac{r}{\lambda(t_n)}\right) \to Q(r) \quad \textit{dans} \quad H^1_{\mathrm{loc}}(\mathbb{R}^{1+2}) \quad \textit{quand} \quad n \to \infty, \tag{1.40}$$

où Q est une application harmonique non-constante.

Ce résultat repose sur une observation astucieuse. Il est bien connu (voir [44, ch. 8]) que la concentration d'énergie se fait plus vite qu'au taux auto-similaire (i.e. sur les cônes de lumière), ou plus précisément,

$$\forall \mu \in]0,1[, \qquad \lim_{t\uparrow 0} E(u(t), \mu t, t) = 0.$$

Une conséquence simple de cette propriété est que $\|u_t(t)\|_{L^2}$ tend vers 0 en moyenne (cf. [44, ch. 8] également) :

$$\lim_{t\uparrow 0} \frac{1}{t} \int_0^t \int u_t^2(\tau, r) r dr d\tau = 0.$$

En reformulant précisément ce fait, on obtient que u (moyennant un changement échelle convenable) tend vers un profil fixe ; au sens des distributions, ce profil satisfait (1.35) et est stationnaire, c'est donc une application harmonique. Enfin, étant donné qu'il y a concentration d'énergie, on déduit que le profil a une énergie non nulle, et est donc non-constant.

Corollaire 1.3. *1) Soit u une wave map explosant au temps $T = 0$. Alors u concentre une énergie au moins égale à $E(Q)$, à un taux super-similaire : il existe $\mu(t) = o(t)$ tel que*

$$\liminf_{t\uparrow 0} E(u(t), 0, \mu(t)) \geq E(Q).$$

En particulier, $E(u) > E(Q)$.
2) On suppose que $g > 0$ *pour* $\rho > 0$*. Alors il n'existe pas d'application harmonique non-constante, et donc toute wave map est globale en temps.*

Il faut penser à $\mu(t_n) = \sqrt{|t_n|/\lambda(t_n)}$, avec les notations du Théorème 1.15. Un résultat très similaire, également dû à Struwe [49, 51], affirme que les wave maps radiales sont globales : cela est dû au fait qu'il n'existe pas d'application harmonique radiale non constante.

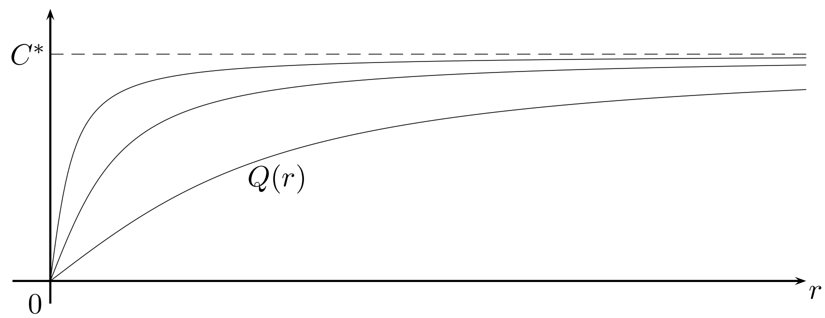

FIG. 1.4 – Q et ses dilatées.

Notons que le point 2) améliore le résultat de Grillakis. Réciproquement, on voit aisément que si g s'annule en un point autre que 0, la variété N admet des fonctions harmoniques (d'énergie finie) non triviales.

Proposition 1.3 ([7])**.** *On suppose que g admet au moins un point d'annulation autre que 0, et que ces points d'annulation sont isolés. Alors il existe des fonctions harmoniques d'énergie finie non triviales : ce sont des fonctions monotones qui joignent deux points d'annulation consécutifs de g.*

1.2.3 Instabilité des fonctions harmoniques

Dans la suite on supposera toujours que g vérifie les hypothèses de la Proposition 1.3. On notera $C^* > 0$ le plus premier point d'annulation de g, et Q une fonction harmonique croissante qui joint 0 à C^*.

Pour étudier la dynamique des wave maps, notre approche sera de considérer des données initiales dans un H-voisinage de Q. Une approche similaire a donné de très fructueux résultats dans l'étude de l'équation de Korteweg-de Vries critique (résultats évoqués dans la première partie de cette introduction [26, 25, 32, 28, 27]), et de Schrödinger non-linéaire critique (résultats de Merle et Raphael [33, 34, 35, 36, 37] et Raphael [42]).

La première motivation de cela est bien sûr le Corollaire 1.3, qui affirme d'une certaine manière que $E(Q)$ est la quantité d'énergie critique pour la formation de singularités.

Décomposition

Cette idée est renforcée par la propriété variationnelle de Q :

Propriété. Soit $v : \mathbb{R}_+ \to \mathbb{R}$ *une fonction d'énergie finie qui joint* 0 *à* C^**, et* $E(v) \leq E(Q)$*. Alors* $E(v) = E(Q)$ *et il existe* $\lambda \in \mathbb{R}_+^*$ *tel que :*

$$v(r) = Q(\lambda r).$$

De cette propriété de rigidité, on déduit l'existence d'une décomposition.

Proposition 1.4 (Décomposition [7])**.** *Il existe $\alpha_0 > 0$ et une fonction continue croissante $\delta : [0, \alpha_0] \to \mathbb{R}_+$ avec $\delta(0) = 0$ telles que ce qui suit soit vrai. Soit $v \in H$ une fonction qui joint 0 à C^*, et $E(v) = E(Q) + \alpha$, $\alpha < \alpha_0$ ($\alpha > 0$). Alors on a la décomposition :*

$$v(r) = Q(\lambda r) + \varepsilon(r), \quad \|\varepsilon\|_H \leq \delta(\alpha), \tag{1.41}$$

où $\lambda \in \mathbb{R}_*^+$.

L'existence d'une décomposition est analogue à celle des cas des équations de Korteweg-de Vries critique et de Schrödinger non-linéaire critique. Elle provient essentiellement du fait que la Hessienne de l'énergie au point Q est une forme bilinéaire qui est presque définie positive. Plus précisément,

$$d^2E(Q).h^2 = \int \left(h_r^2 + \frac{f'(Q)}{r^2} h^2 \right) rdr$$

vérifie, pour un certain $\nu > 0$:

$$\forall h \in H, \quad d^2E(Q).h^2 \geq \nu \int \left(h_r^2 + \frac{h^2}{r^2} \right) rdr - \frac{1}{\nu} \left(\int hrQ_r rdr \right)^2,$$

sous réserve que $rQ_r \in L^2(rdr)$.

La seule direction dégénérée est celle associée à l'invariance d'échelle : $E(Q) = E(Q_\lambda)$ et $\frac{d}{d\lambda} Q_\lambda = rQ_r$). Dans la décomposition, le seul paramètre libre est donc celui lié au scaling $\lambda \mapsto u_\lambda$.

Si u est une wave map dont les données initiales satisfont ces conditions, une telle décomposition est disponible pour tout t. Le paramètre $\lambda(t)$ gouverne la dynamique de la wave map $u(t)$: l'explosion correspond à $\lambda(t) \to \infty$, et on dit que $\lambda(t)$ est le taux d'explosion. $\lambda(t)$ correspond également à la vitesse de concentration de l'énergie, au sens où (en cas d'explosion) $1/\lambda(t) \sim \mu(t)$ avec $\mu(t)$ tel que $E(u(t), 0, \mu(t)) = E(Q)/2$.

Instabilité

Le problème de l'explosion reste ouvert. Cependant, dans cette direction, nous avons démontré l'instabilité géométrique de Q [7] (heuristiquement, $\lambda(t)$ passe de 1 à 2) : c'est un premier pas vers la preuve de l'existence de solutions explosives. Plus précisément :

Théorème 1.16 (Instabilité des solutions harmoniques pour le système des wave maps en dimension critique [7])**.** *Soit $\lambda_0 > 1$. Alors il existe une suite de données initiales (u_n^0, u_n^1) telle que*

$$\|(u_n^0, u_n^1) - (Q, 0)\|_H \to 0 \quad \textit{quand} \quad n \to \infty, \tag{1.42}$$

et en notant u_n les wave maps qui sont issues des données (u_n^0, u_n^1), u_n est définie au moins jusqu'au temps t_n tel que

$$\|(u_n, u_{nt})(t_n) - (Q(\lambda_0 \cdot), 0)\|_H \to 0 \quad \textit{quand} \quad n \to \infty. \tag{1.43}$$

Notons que ce résultat est une forme forte d'instabilité, puisqu'il y a un changement de forme, du profil $\lambda = 1$ au profil $\lambda = \lambda_0 > 1$: ainsi pour toute topologie raisonnable, $(u_n, u_{nt})(t_n)$ reste hors d'un voisinage fixe de $(Q, 0)$. En fait il s'agit presque de l'explosion, car le changement de profil obtenu λ_0 peut être choisi aussi grand que l'on veut. D'autre part, l'équation (1.35) étant réversible, il existe des wave maps dont le profil passe de $\lambda = 1$ à $\lambda = 1/\lambda_0 < 1$.

Les données initiales (u_n^0, u_n^1) sont construites (presque) explicitement : il s'agit de profils régularisés de wave maps auto-similaires. On dit qu'une wave map u est auto-similaire si $u(t,r) = v(r/|t|)$ (explosion à $T = 0$) ou $u(t,r) = v(r/|T-t|)$ (explosion au temps T). En résolvant l'équation (1.35) avec cette ansatz, Q apparaît naturellement, et on obtient une famille à un paramètre b, de wave maps auto-similaires. Celles-ci peuvent s'écrire de la manière suivante :

$$Q_b(t,r) = Q\left(\frac{2(1-bt)r}{1+\sqrt{1-\frac{r^2}{(1/b-t)^2}}}\right). \tag{1.44}$$

(explosion à $T = 1/b$). Remarquons que

$$S(b,r) \overset{\text{def}}{=} Q_b(0,r) = Q\left(\frac{2r}{1+\sqrt{1-b^2r^2}}\right) \to Q(r) \text{ localement quand } b \to 0,$$

et plus précisément,

$$\forall A \geq 0, \quad E(S(b)-Q),0,A) \to 0 \quad \text{quand} \quad b \to 0.$$

Cependant cette convergence n'a pas lieu dans H, car les $S(b)$ sont d'énergie infinie : au voisinage de $r = 1/b$,

$$S_r\left(b, \frac{1}{b} - \varepsilon\right) \sim Q'\left(\frac{1}{b}\right)\sqrt{\frac{b}{\varepsilon}} \quad \text{quand} \quad \varepsilon \downarrow 0.$$

Ainsi, lorsque l'on calcule $\int S_r^2(b,r) r dr$, il y a une divergence logarithmique au point $1/b$, et pour tout $b > 0$, $E(S(b)) = +\infty$.

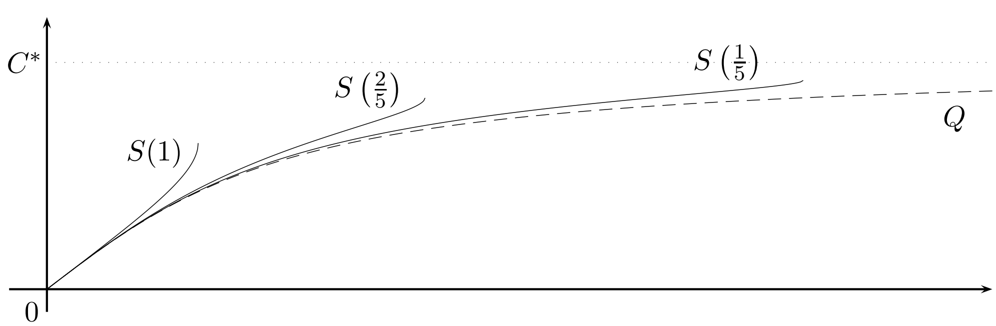

FIG. 1.5 – $Q(r)$ et $S(b,r)$.

Pour approximer Q, il faut donc régulariser $S(b)$ pour $r \geq c_0/b$, où c_0 est une constante que l'on fixe arbitrairement dans l'intervalle $]0,1[$ (proche de 1). Sur l'intervalle $r \in [0, c_0/b]$, on garde $S(b,r)$, sur l'intervalle $r \in [h(b), \infty[$ on choisit $Q(r)$ (ou simplement la constante C^*), et sur l'intervalle $r \in [c_0/b, h(b)]$, on "recolle" à la main, en essayant de gaspiller le moins d'énergie possible (Q minimise l'énergie pour relier 2 points). $h(b)$ est une fonction à choisir convenablement, pour encore une fois ne pas gaspiller trop d'énergie.

On obtient ainsi u_n^1, qui est associée à $S(b_n)$, où b_n est une suite qui tend vers 0 ; u_n^0 est obtenue de manière analogue. Il est possible de faire tout ceci en s'assurant que l'on a bien $\|(u_n^0, u_n^1) - (Q,0)\|_H \to 0$ quand $n \to \infty$.

Ensuite, par vitesse finie de propagation, $u_n(t,r) = Q_{b_n}(t,r)$ dans le cône $\{(t,r)|r \leq c_0/b_n - |t|\}$. Via des arguments liés à la conservation de l'énergie, et au fait que Q minimise l'énergie, on conclut que pour $t_n = c_0(1-c_0)/b_n$, et $\lambda_0 = 1/(1-c_0^2)$,

$$\|(u_n, u_{nt})(t_n) - (Q(\lambda_0 \cdot), 0)\|_H \to 0.$$

Comme c_0 peut être choisi aussi proche de 1 que l'on veut, λ_0 peut être aussi grand que souhaité.

Ouverture

Il est intéressant de souligner que l'on est obligé d'introduire des objets élaborés, les wave maps auto-similaires, pour traiter le problème de l'instabilité (on obtient néanmoins une forme forte d'instabilité). Cette idée avait déjà été exploitée avec succès dans l'étude de l'équation de Schrödinger non-linéaire (cf. [33]). Cependant les résultat obtenus dans ce cas sont extrêmement précis (et même optimaux) et dénotent d'une profonde compréhension de dynamique explosive. De précédents résultats d'explosion avaient été obtenus pour (NLS) en utilisant la décomposition autour de Q, et en localisant une propriété de monotonie (Viriel) : une telle identité existe pour les wave maps mais donne des renseignements considérablement moins précis.

Dans le cas des wave maps, il apparaît qu'il faut probablement utiliser la décomposition associée aux $S(b)$, du type

$$u(t,r) = S(b(t), \lambda(t)r) + \varepsilon(t,r),$$

où plutôt en considérant une régularisation de $S(b)$, obtenue comme dans le Théorème d'instabilité 1.16 : la décomposition associée à Q (1.41) risque d'être trop grossière.

Mais bien sûr, il y a maintenant 2 paramètres $b(t)$ et $\lambda(t)$ à étudier, et cela signifie qu'il faut également comprendre simultanément beaucoup d'aspects de la dynamique.

Organisation du mémoire de Thèse

Le Chapitre 2 est consacré au Théorème 1.3 (construction de l'opérateur d'onde linéaire pour (gKdV). Le Chapitre 3 est consacré au Théorème 1.12 et le Chapitre 4, au Théorème 1.11 (construction de l'opérateur d'onde non-linéaire dans les cas critique et sous-critique respectivement). Le Chapitre 5 est consacré au Théorème 1.16 (instabilité de Q pour le système des wave maps).

Chapitre 2

Large data wave operator for the generalized Korteweg-de Vries equations[1]

2.1 Introduction.

2.1.1 Recall of known results.

We consider the generalized Korteweg-de Vries equation :

$$u_t + (u_{xx} + |u|^p)_x = 0, \quad t, x, \in \mathbb{R}, \tag{2.1}$$

where $p \geq 2$. The case $p = 2$ corresponds to the original equation introduced by Korteweg and de Vries [19] in the context of shallow water waves. For both $p = 2$ and $p = 3$, this equation has many applications to Physics : see for example Miura [39], Lamb [22].

There are two formally conserved quantities for solutions to (2.1) :

$$\int u^2(t) = \int u^2(0) \qquad (L^2 \text{ mass}), \tag{2.2}$$

$$E(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) = E(u(0)) \qquad \text{(energy)}. \tag{2.3}$$

The local Cauchy problem for (2.1) has been intensively studied by many authors. Kenig, Ponce and Vega [14] proved the following existence and uniqueness result in $H^1(\mathbb{R})$: for $u_0 \in H^1(\mathbb{R})$, there exist $T = T(\|u_0\|_{H^1}) > 0$ and a solution $u \in C([0,T], H^1(\mathbb{R}))$ to (2.1) satisfying $u(0) = u_0$, which is unique in some class $Y_T \subset C([0,T], H^1(\mathbb{R}))$. For such solution, one has conservation of mass and energy. Moreover, if T_1 denotes the maximal time of existence for u, then either $T_1 = +\infty$ (global solution) or $T_1 < \infty$ and $\|u(t)\|_{H^1} \to \infty$ as $t \uparrow T_1$ (blow-up solution).

For $p = 2$ and $p = 3$, equation (2.1) is completely integrable, and thus has very special features. The inverse scattering transform method allows to solve the Cauchy problem in an appropriate space (for example if $\|(1 + |x|^2)^5 u_0\|_{C^4} < \infty$) and to find the asymptotic behavior of solutions as $t \to \pm\infty$: see for example Schuur [43], Eckhaus and Schuur [9],

[1] Ce chapitre a fait l'objet d'une publication à *Differential and Integral Equations*, **19** (2006), no. 2, 163–188.

Miura [39]. However, if $p \neq 2$ or 3, the inverse scattering transform method does not longer apply, and the description of solutions as $t \to +\infty$ in the non-integrable case is an open problem. Let us recall some results which are not based on the inverse scattering transform method.

In the case $2 \leq p < 5$, all solutions in H^1 are global and uniformly bounded due to the conservations laws and the Gagliardo-Nirenberg inequality :

$$\forall v \in H^1(\mathbb{R}), \quad \int |v|^{p+1} \leq C(p) \left(\int v^2 \right)^{\frac{p+3}{4}} \left(\int v_x^2 \right)^{\frac{p-1}{4}}.$$

The case $p = 5$ is L^2-critical, in the sense that the mass remains unaffected by scaling. If :

$$u_t + (u_{xx} + u^5)_x = 0, \quad t, x, \in \mathbb{R}. \tag{2.4}$$

Then $u_\lambda(t,x) = \lambda^{1/6} u(\lambda t, \lambda^{1/3} x)$ is also a solution to (2.4), and $\|u_\lambda\|_{L^2} = \|u\|_{L^2}$. In this case, the local existence result of [14] is improved to initial data in L^2 (instead of H^1). However, existence of finite time blow-up solutions was proved by Merle [32] and Martel and Merle [27]. Therefore $p = 5$ also appears as a critical exponent for the long time behavior of solutions to (2.1).

Another problem which was studied by many authors is scattering for small initial data in an appropriate functional space, see for example [48], [40], [4], [13]. Let us recall the result of Hayashi and Naumkin [13]. Introduce the following weighted Sobolev spaces :

$$H^{s,m} = \{ \phi \in \mathcal{S}' |\, \|\phi\|_{H^{s,m}} = \|(1+|x|^2)^{m/2}(1-\partial_x^2)^{s/2}\phi\|_{L^2} < \infty \}. \tag{2.5}$$

Let $p > 3$. Given u_0 small enough in $H^{1,1}$, the arising solution $u(t)$ is global in time, and there is scattering, in the sense that there exists a function $V \in L^2$ so that :

$$\|u(t) - U(t)V\|_{L^2} \to 0.$$

where $U(t)$ denotes the linear operator for the KdV equation, i.e. $v(t) = U(t)V$ satisfies $v_t + v_{xxx} = 0$, $v(0) = V$. This is the description of solutions around with initial data around 0 (in $H^{1,1}$).

Now, there exist also special explicit traveling wave solutions called solitons. If Q denotes the unique solution (up to translation) of :

$$Q > 0, \quad Q \in H^1(\mathbb{R}), \quad Q_{xx} + Q^p = Q, \quad \text{i.e.} \quad Q(x) = \left(\frac{p+1}{2\cosh^2(\frac{p-1}{2}x)} \right)^{\frac{1}{p-1}},$$

then for $c > 0$, the soliton

$$R_{c,x_0} = c^{\frac{1}{p-1}} Q(\sqrt{c}(x - x_0 - ct)) \text{ is a solution to (2.1).}$$

We should notice that for $p > 3$, solitons do not appear in the small data analysis of [13], as :

$$\|R_{c,x_0}\|_{H^{1,1}} \geq c_0, \text{ for some uniform constant } c_0 > 0.$$

Indeed :

$$\int (R_{c,x_0})_x^2 = c^{\frac{p+3}{2(p-1)}} \int Q_x^2, \quad \text{and} \quad \int (xR_{c,x_0})^2 = c^{\frac{7-3p}{2(p-1)}} \int (xQ)^2.$$

For $p > 3$, $\frac{p+3}{2(p-1)} > 0$ but $\frac{7-3p}{2(p-1)} < 0$, so that if $\|R_{c,x_0}\|_{H^1} \to 0$, then $c \to 0$ and thus $\|R_{c,x_0}\|_{H^{0,1}} \to \infty$. (Notice that $H^{1,1}$ is not sharp from this point of view).

Description of solutions around a sum of decoupled solitons is available : Martel and Merle [25], Martel, Merle and Tsai [29] proved stability in H^1 and asymptotic stability (in $L^2(x \geq ct)$ for $c > 0$) of a sum of decoupled solitons, in the sub-critical case $2 \leq p < 5$ (in the critical case $p = 5$, one has blow-up around a soliton [28]).

Our goal is to construct solutions to (2.1) with a given linear asymptotic behavior : that is the construction of a wave operator with respect to the free KdV operator $U(t)$. This problem is reciprocal to (linear) scattering for small initial data.

Let $p > 3$, and $V \in H^{2,2}$ (without smallness assumption). We construct a solution $u(t)$ to (2.1), defined for large enough times, and such that $u(t) - U(t)V \to 0$ in H^1 as $t \to \infty$. Furthermore, $u(t)$ is unique in an adequate space.

In the L^2-critical case ($p = 5$), we obtain an optimal result, in the sense that for $V \in L^2$, we construct $u(t)$ solution to (2.4) such that $u(t) - U(t)V \to 0$ in L^2. Again V need not be small in L^2.

The philosophy underlying these results is the following : the tools needed to prove global existence for *small data* can be applied successfully to construct solutions with a given linear profile, *small or large.*

In the case of non-linear Schrödinger equations, there are many results concerning the construction of wave operators. For a review, see e.g. Ginibre and Velo [12].

2.1.2 Statement of the results.

There are two main results : one in the general case $p > 3$ which uses the framework of Hayashi and Naumkin [13], and one in the critical case $p = 5$, using the framework of Kenig, Ponce and Vega [14].

Let $p > 3$. Fix once for all the three constants :

$$\gamma \in (0, \min\{1/2, (p-3)/3\}), \quad \alpha = \frac{1}{2} - \gamma \in (0, 1/2) \quad \text{and} \quad \delta = \frac{p-3-2\gamma}{3} > 0. \tag{2.6}$$

Following the framework of Hayashi and Naumkin [13], we will use the notation $D = \partial_x = \frac{\partial}{\partial x}$ for the partial differentiation with respect to the space variable x, and :

$$D^\alpha \phi = \mathcal{F}^{-1} \xi^\alpha e^{-(i\pi/2)(1+\alpha)} \hat{\phi}.$$

As in [13], we will use the following two operators :

$$J^t \phi = U(t) x U(-t) \phi = (x - 3t\partial_x^2)\phi, \text{ and } I^t \phi = x\phi + 3t \int_{-\infty}^x \partial_t \phi(t, y) dy.$$

We write J^t and I^t to emphasize that we will always consider norms at a fixed time t although J^t and I^t are space-time operators.

Our working spaces will be defined through the time dependent M_0^t norm :

$$\mathcal{H}_t = \{\phi \in L^2(\mathbb{R}) |\, M_0^t(\phi) = \|\phi\|_{H^1} + \|DJ^t\phi\|_{L^2} + \|D^\alpha J^t \phi\|_{L^2} < \infty\}.$$

J^t only appears in the norm, since it is convenient to do linear estimates (see [13], Lemma 2.3). But we introduced I^t because it is easier to handle when doing energy methods estimates. Notice that M_0^0 is very similar to $\| \cdot \|_{H^{1,1}}$.

Different positive constants might be denoted by the same letter C.
We now state our results.

Theorem 2.1 (Large data wave operator)**.** *Let $p>3$, and $V\in H^{2,2}$. Then :*

1. *There exist $T_0=T_0(\|V\|_{H^{2,2}})\geq 1$ and a unique $u\in C([T_0,\infty),\mathcal{H}_t)$ solution to (2.1) so that :*
$$M_0^t(u(t)-U(t)V)\to 0 \qquad \text{as } t\to\infty.$$
2. *Moreover, there exists a constant C independent of time and V, so that :*
$$\forall t\geq T_0,\quad M_0^t(u(t)-U(t)V)\leq C(1+\|V\|_{H^{2,2}}^p)t^{-\delta}.$$
In particular, $\|u(t)-U(t)V\|_{H^1}\leq Ct^{-\delta}$.

***Remark* 2.1.** *The scattering result of Hayashi and Naumkin [13] is the following : for small initial data in $H^{1,1}$, the associated solution $u(t)$ is global in time, satisfies $\|u(t)\|_{L^\infty}\leq Ct^{-1/3}$ (linear decay rate), and there exists a scattering function V such that $u(t)-U(t)V\to 0$ in L^2 as $t\to\infty$.*

Our point here is the reciprocal problem : we construct a solution u with a given scattering state $U(t)V$. We do not need any smallness assumption ; however some integration by parts do not work as well as in [13], so we need to assume $V\in H^{2,2}$, with basically a convergence result in $H^{1,1}$.

This result can be extended to a more general non-linearity. For example :

Corollary 2.1. *Let us consider the equation :*

$$u_t+(u_{xx}+f(u))_x=0. \tag{2.7}$$

where $f:\mathbb{R}\to\mathbb{R}$ is C^1, with $f(0)=0$ and $f'(x)=O(x^{p-1})$ as $x\to 0$ (and $p>3$). Given $V\in H^{2,2}$, there exist $T_0=T_0(\|V\|_{H^{2,2}})\geq 1$ and a unique $u\in C([T_0,\infty),\mathcal{H}_t)$ solution to (2.7) so that :
$$M_0^t(u(t)-U(t)V)\to 0 \qquad \text{as } t\to\infty.$$
Moreover, there exists a constant C independent of time and V, so that :
$$\forall t\geq T_0,\quad M_0^t(u(t)-U(t)V)\leq C(1+\|V\|_{H^{2,2}}^p)t^{-\delta}.$$

The proof of the corollary follows from that of Theorem 2.1. In particular, although our proof is done for the focusing power case $f(x)=x^p$, it is also true for the defocusing case $f(x)=-|x|^{p-1}x$.

In the critical case $p=5$, we prove an analogous result for $V\in L^2$:

Theorem 2.2 (Wave operator in the critical case)**.** *Let $p=5$. For any $V\in L^2$, there exist $T_0=T_0(V)\in\mathbb{R}$ and $u\in C^0([T_0,\infty),L^2)$ solution to the critical KdV equation (2.4), so that :*
$$\|u(t)-U(t)V\|_{L^2}\to 0.$$
u is unique in the class $\{u|u\in L_t^\infty L_x^2\cap L_x^5L_t^{10}$ and $\partial_x u\in L_x^\infty L_t^2\}$.

***Remark* 2.2.** *The theorem remains true if one weakens the hypothesis to V be such that $\|U(t)V\|_{L_x^5L_t^{10}}+\|\partial_x U(t)V\|_{L_x^\infty L_t^2}<\infty$.*

As previously, our proof extends to the defocusing case $u_t+(u_{xx}-u^5)_x=0$.

The proofs of the two results strongly rely on the scattering analysis of [13] and [14]. However, the arguments developed in each case are of completely different nature, this is why the proofs will be presented in a separate way : Section 2 gives the strategy of the proofs, Section 3 is devoted to the proof of Theorem 2.1, and Section 4, to that of Theorem 2.2.

2.2 Strategy of the proofs.

2.2.1 The general case $p > 3$.

We shall always consider $w(t) = u(t) - U(t)V$. Thus we are looking for w satisfying the equivalent conditions :

$$\begin{cases} w_t + w_{xxx} + (w + U(t)V)^p_x = 0, \\ M_0^t(w(t)) \to 0 \quad \text{as } t \to \infty. \end{cases} \tag{2.8}$$

To construct w, we will use the following approximation scheme. Let $(S_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathbb{R}$ such that $S_n \to \infty$ as $n \to \infty$. Define $w_n(t)$ as a solution to :

$$\begin{cases} w_{nt} + w_{nxxx} + (w_n + U(t)V)^p_x = 0, \\ w_n(S_n) = 0, \end{cases} \tag{2.9}$$

defined on a maximal interval of the form $(I_n, S_n]$: it corresponds to u_n solution to (2.1) with initial condition $u_n(S_n) = U(S_n)V$. Since $V \in H^{2,2}$, $U(S_n)V \in H^2$, so u_n exists and is the unique H^2 solution : the same is true for w_n.

Our method is then to prove that in fact :

1. $I_n \leq T_0$ independent of n : we can define w_n on an interval $[T_0, S_n]$ whose lower bound T_0 is fixed.

2. We have uniform (in n) decay estimates for w_n on the interval $[T_0, S_n]$.

   (Of course, if $3 < p < 5$, there is global existence in H^1, and thus $I_n = -\infty$ is automatic). To prove this, we will make an intensive use of the tools developed by Hayashi and Naumkin : this is the heart of the proof, and it is done in Proposition 2.1.

3. We then prove that the sequence $w_n(t)$ converges to a certain $w(t)$ (in $C^0_t L^2_x$).

4. By weak limit, we improve the regularity of $w(t)$, to conclude that w is a strong solution to (2.8).

This does the existence part of the theorem. For the uniqueness part, we study again the L^2 difference of a solution with the one we constructed, and show, with a Gronwall type argument, that these two solutions coincide where they are defined.

This scheme of proof is very similar to that of [23] and [30]. In [23], Martel also study the problem of constructing solutions with a given asymptotic behavior : there it is proved that given a sum of solitons $R(t)$, there exist a unique $u(t)$ solution to (2.1) with $2 \leq p \leq 5$ such that $\|u(t) - R(t)\|_{H^1} \to 0$ as $t \to \infty$, and furthermore, the convergence takes place in all the Sobolev spaces H^s with an exponential decay. Remark that Martel deeply used ideas of the stability for a sum of decoupled solitons [29].

2.2.2 The L^2-critical case ($p = 5$).

In the critical case, we do not need $H^{1,1}$ regularity. Since Kenig, Ponce and Vega [14] obtained global existence for small data in L^2, we use their setting.

The proof goes in this way : denoting as usual $w(t) = u(t) - U(t)V$, we write formally that an eventual w should solve a fixed point problem with the condition $w(t) \to 0$ as $t \to \infty$.

The fixed point problem is in fact very similar to that of the Cauchy problem, so that we can reuse the linear estimates proved by Kenig, Ponce and Vega [14] for their global existence theorem for small data.

The fact that $w(t) \to 0$ is our smallness condition, which allows us to have a contracting map, and thus a fixed point.

2.3 Proof of Theorem 2.1 : $p > 3$.

We start by recalling the linear estimates of [13] which will be used throughout the proof.

2.3.1 Linear estimates.

Let $p > 3$. Recall our notations : three fixed constants $\gamma \in]0, \min\{(p-3)/3, 1/2\}[$, $\alpha = 1/2 - \gamma$, $\delta = (p - 3 - 2\gamma)/3 > 0$, the operator $J^t\phi = x\phi - 3t\partial_x^2\phi = U(t)xU(-t)\phi$, and our working norm:

$$M_0^t(\phi) = \|\phi\|_{H^1} + \|D^\alpha J^t\phi\|_{L^2} + \|DJ^t\phi\|_{L^2}.$$

First a few remarks on M_0^t. Of course $M_0^0(\phi) \leq C\|\phi\|_{H^{1,1}}$. Second, note that $J^tU(t)V = U(t)xV$ (and $U(t)$ is a H^s isometry), so that if $V \in H^{1,1}$, we have the uniform control in t :

$$M_0^t(U(t)V) \leq C\|V\|_{H^{1,1}}. \tag{2.10}$$

We now recall the linear results obtained in [13] (Lemma 2.2), in a slightly improved form.

Lemma 2.1. *Let $t > 0$ and ϕ be a function so that $M_0^t(\phi)$ is bounded. Then for $r > 4$:*

$$\|\phi\|_{L^r} \leq \frac{C}{(1+t)^{1/3-1/(3r)}} M_0^t(\phi).$$

And one also has the pointwise inequalities :

$$|\phi(x)| \leq \frac{CM_0^t(\phi)}{(1+t)^{1/3}} \left(1 + \left|\frac{x}{t^{1/3}}\right|\right)^{-\frac{1}{4}}, \quad |\phi_x(x)| \leq \frac{CM_0^t(\phi)}{t^{2/3}} \left(1 + \left|\frac{x}{t^{1/3}}\right|\right)^{\frac{1}{4}}.$$

As a simple consequence, for $V \in H^{1,1}$, we have similar decay estimates on $U(t)V$.

Proof. See [13], Lemma 2.2 and its proof (especially inequalities 2.16, 2.17. and 2.18). For completeness, the proof is given in Appendix A. □

We will also need the polarized version of Lemma 2.3 of [13] :

Lemma 2.2. *Let $h, k : \mathbb{R} \to \mathbb{R}$. Then the following inequalities are valid if their right-hand side is bounded :*

$$\begin{aligned} \|D^\alpha k^p\|_{L^2} &\leq C\|k^{p-1}\|_{L^2}(\|kk_x\|_{L^\infty}^{1/2} + \|k\|_{L^\infty}^{3\gamma}\|kk_x\|_{L^\infty}^{(1-3\gamma)/2}), \\ \|D^\alpha |k|^{p-1}h_x\|_{L^2} &\leq C(\|D^\alpha h\|_{L^2} + \|h_x\|_{L^2})(\|k\|_{L^\infty}^{p-3}\|kk_x\|_{L^\infty} \\ &\quad + \|k\|_{L^\infty}^{p-3-2\gamma}\|k\|_{L^2}^{2\gamma}\|kk_x\|_{L^\infty} + \|k\|_{L^\infty}^{p-3+2\gamma}\|kk_x\|_{L^\infty}^{1-\gamma}). \end{aligned}$$

Proof. See [13], Lemma 2.3 and its proof (case $\sigma = 0$). □

2.3.2 Uniform estimates on w_n.

Recall that w_n is the solution to the problem :

$$\begin{cases} w_{nt} + w_{nxxx} + (w_n + U(t)V)^p_x = 0, \\ w_n(S_n) = 0, \end{cases} \tag{2.9}$$

where $S_n \to \infty$ is an increasing sequence of times. Using estimates developed in [13], we have the following proposition for w_n :

Proposition 2.1 (Uniform estimates on w_n). *Let $p > 3$. There exists $T_0 \geq 1$ independent of n, so that for all $n \in \mathbb{N}$ such that $S_n \geq T_0$, $w_n \in C([T_0, S_n], H^1(\mathbb{R}))$, and :*

$$\forall t \in [T_0, S_n], \quad M_0^t(w_n(t)) \leq C(1 + \|V\|^p_{H^{2,2}})t^{-\delta} \tag{2.11}$$

where $\delta = (p - 3 + 2\gamma)/3 > 0$ and C are independent of n and V.

Of course Proposition 2.1 is the heart of the proof of Theorem 2.1.

Proof. Define $I_n^* \geq 1$ minimal so that :

$$\forall t \in [I_n^*, S_n], \quad M_0^t(w_n(t)) \leq 1. \tag{2.12}$$

Lemma 2.3. *Suppose we can prove :*

$$\forall t \in [I_n^*, S_n], \quad M_0^t(w_n(t)) \leq C(1 + \|V\|^p_{H^{2,2}})t^{-\delta}.$$

Then $I_n^* \leq T_0$ independent of n and Proposition 2.1 holds true :

$$\forall t \in [T_0, S_n], \quad M_0^t(w_n(t)) \leq C(1 + \|V\|^p_{H^{2,2}})t^{-\delta}.$$

Proof of Lemma 2.3. This follows from a continuity argument. First, due to Theorem 2.3, proved in Appendix B, $t \mapsto M_0^t(u_n(t))$ is upper semi-continuous : since $M_0^t(w_n(S_n)) = 0$, $I_n^* < S_n$.

Let $T_0 \geq 1$ be such that $C(1 + \|V\|^p_{H^{2,2}})T_0^{-\delta} \leq 1$. If $I_n^* > T_0$, $C(1 + \|V\|^p_{H^{2,2}})I_n^{*-\delta} < 1$, our hypothesis gives that :

$$\forall t \in [I_n^*, S_n], \quad M_0^t(w_n(t)) \leq C(1 + \|V\|^p_{H^{2,2}})t^{-\delta} \leq C(1 + \|V\|^p_{H^{2,2}})I_n^{*-\delta} < 1.$$

By our upper semi-continuity argument (again Theorem 2.3), there would be a $t' < I_n^*$ so that for $t \in [t', S_n]$, $M_0^t(w_n(t)) \leq 1$. This contradicts the minimality of I_n^*, and so $I_n^* \leq T_0$.

The decay estimate follows immediately. □

So it is enough to prove :

$$\forall t \in [I_n^*, S_n], \quad M_0^t(w_n(t)) \leq C(1 + \|V\|^p_{H^{2,2}})t^{-\delta},$$

assuming (2.12). Recall :

$$M_0^t(\phi) = \|\phi\|_{L^2} + \|D\phi\|_{L^2} + \|D^\alpha J^t\phi\|_{L^2} + \|DJ^t\phi\|_{L^2}.$$

We will now estimate each one of the 4 norms involved. Let us denote $K = 1 + \|V\|_{H^{2,2}}$. We will successively prove that for $t \in [I_n^*, S_n]$, we have :

(i) $\|w_n\|_{L^2} \leq CK^p t^{-(p-3)/3}$.

(ii) $\|w_n\|_{H^1} \leq CK^p t^{-(p-3)/3}$.

(iii) $\|D^\alpha I^t w_n\|_{L^2} \leq CE^p t^{-\delta}$.

(iv) $\|I^t w_{nx}\|_{L^2} \leq CK^p t^{-(p-3)/3}$.

(v) $M_0^t(w(t)) \leq CK^p t^{-\delta}$.

Remark that we first do estimates on I^t (step (iii) and (iv)), since it easier to handle when doing energy methods estimates. In step (v) we shall go back to estimates on J^t.

Before we do this, we have to notice that Lemma 2.1 applies for $t \in [I_n^*, S_n]$ to give :

$$|w_n(t,x)| \leq Ct^{-1/3}\left(1+\frac{|x|}{t^{1/3}}\right)^{-\frac{1}{4}}, |w_{nx}(t,x)| \leq Ct^{-2/3}\left(1+\frac{|x|}{t^{1/3}}\right)^{\frac{1}{4}}.$$

Then we can add up $U(t)V$ in these estimates to get :

$$|(U(t)V + w_n(t))(x)| \leq CKt^{-1/3}\left(1+\frac{|x|}{t^{1/3}}\right)^{-\frac{1}{4}}, \tag{2.13}$$

$$|(U(t)V + w_n(t))_x(x)| \leq CKt^{-2/3}\left(1+\frac{|x|}{t^{1/3}}\right)^{\frac{1}{4}}, \tag{2.14}$$

$$\|U(t)V + w_n(t)\|_{L^r} \leq CKt^{-1/3-1/(3r)}, \quad r > 4. \tag{2.15}$$

Proof of (i).

Let us multiply (2.9) by w_n and integrate in x :

$$\frac{1}{2}\frac{d}{dt}\int w_n^2 = -\int (U(t)V + w_n)^p_x w_n = -p\int (U(t)V+w_n)_x (U(t)V+w_n)^{p-1} w.$$

So (after simplification by $\|w_n\|_{L^2}$) :

$$\frac{d}{dt}\|w_n\|_{L^2} \leq \|U(t)V + w_n\|_{L^2}\|(U(t)V+w_n)(U(t)V+w_n)_x\|_{L^\infty}\|U(t)V+w_n\|^{p-3}_{L^\infty}.$$

For $t \in [I_n^*, S_n]$, we get :

$$\frac{d}{dt}\|w_n\|_{L^2} \leq C(K + \|w_n(t)\|_{L^2}) \cdot \frac{K^2}{t} \cdot \frac{K^{p-3}}{t^{(p-3)/3}} \leq CK^p t^{-\frac{p}{3}}.$$

Thus, by integration in time on $[t, S_n]$, using $w_n(S_n) = 0$, this gives :

$$\|w_n(t)\|_{L^2} \leq C(p)K^p\left(t^{-(p-3)/3} - S_n^{-(p-3)/3}\right) \leq CK^p t^{-(p-3)/3}. \tag{2.16}$$

Proof of (ii).

Let us differentiate (2.9) with respect to x :

$$w_{nxt} + w_{nxxxx} + (U(t)V + w_n)^p_{xx} = 0.$$

Now, multiply by w_{nx} and integrate in x. After an integration by part, we get :

$$\frac{1}{2}\frac{d}{dt}\int {w_n}_x^2 = \int (U(t)V+w_n)^p_x w_{nxx} = p\int w_{nxx}(U(t)V+w_n)_x(U(t)V+w_n)^{p-1}.$$

The point is to rule out w_{nxx}, term that we cannot control : for this we will split the second term and do integrations by parts. First :

$$2\int w_{nxx}w_{nx}(U(t)V+w_n)^{p-1} = -p\int w_{nx}^2(U(t)V+w_n)_x(U(t)V+w_n)^{p-2},$$

which we can easily control. Indeed :

$$\begin{aligned} &p\left|\int w_{nxx}w_{nx}(U(t)V+w_n)^{p-1}\right| \\ &\leq p\|w_{nx}\|_{L^2}^2\|(U(t)V+w_n)_x(U(t)V+w_n)\|_{L^\infty}\|U(t)V+w_n\|_{L^\infty}^{p-3} \\ &\leq CK^{p-1}\|w_{nx}(t)\|_{L^2}^2 t^{\frac{p}{3}} \leq CK^p\|w_{nx}\|_{L^2}t^{\frac{p}{3}}. \end{aligned}$$

(as $\|w_n\|_{H^1} \leq M_0^t(w(t)) \leq 1$). For the second term :

$$\begin{aligned} \int w_{nxx}(U(t)V)_x(U(t)V+w_n)^{p-1} = &-\int w_{nx}(U(t)V)_{xx}(U(t)V+w_n)^{p-1} \\ &-\int w_{nx}(U(t)V)_x(U(t)V+w_n)^{p-2}(U(t)V+w_n)_x. \end{aligned}$$

The first integral would be troublesome (with the double derivative on $U(t)V$), this is why we made the hypothesis $V \in H^{2,2}$ (in [13], this phenomenon is avoided because the integration by parts works better). The second integral is fine. So we can do the estimate (note that we use the L^∞ decay estimate (2.16) on $(U(t)V)_{xx} = (U(t)V_x)_x$, as $V_x \in H^{1,1}$), for all $t \in [I_n^*, S_n]$:

$$\begin{aligned} &p\left|\int w_{nxx}(U(t)V)_x(U(t)V+w_n)^{p-1}\right| \\ &\leq C\|w_{nx}\|_{L^2}\|(U(t)V+w_n)(U(t)V_x)_x\|_{L^\infty}\|U(t)V+w_n\|_{L^2}\|U(t)V+w_n\|_{L^\infty}^{p-3} \\ &\quad + C\|w_{nx}\|_{L^2}\|(U(t)V)_x\|_{L^2}\|(U(t)V+w_n)(U(t)V+w_n)_x\|_{L^\infty}\|U(t)V+w_n\|_{L^\infty}^{p-3} \\ &\leq CK^p\|w_{nx}\|_{L^2}t^{-p/3}. \end{aligned}$$

(we used estimate (2.16), and $\|w_n\|_{H^1} \leq 1$). So finally, after simplifying by $\|w_{nx}\|_{L^2}$:

$$\frac{d}{dt}\|w_{nx}(t)\|_{L^2} \leq CK^p t^{-p/3}.$$

After integration between t and S_n ($w_n(S_n) = 0$) :

$$\|w_{nx}(t)\|_{L^2} \leq CK^p\left(t^{-(p-3)/3} - S_n^{-(p-3)/3}\right).$$

And so, we neglect the S_n term, and when adding up with (2.16), we obtain :

$$\|w_n(t)\|_{H^1} \leq C(p)K^p t^{-(p-3)/3}. \tag{2.17}$$

Proof of (iii).

We now turn to the estimates on $I^t w_n$. Let us denote $L = \partial_t + \partial_{xxx}$ the linear KdV operator, and recall some commutation relations of the different operators involved . Recall the definition of the dilation operator $I^t\phi = x\phi + 3t\int_{-\infty}^x \phi_t dx$ and of $J^t\phi = x\phi - 3t\partial_{xx}\phi$. Then :

$$I^t\phi - J^t\phi = 3t\int_{-\infty}^x L\phi dx.$$

We have the following commutation relations :

$$[L, J^t] = 0, \quad [L, I^t]\phi = 3\int_{-\infty}^x L\phi dx, \quad [J^t, \partial_x] = [I^t, \partial_x] = -Id.$$

Again notice that $I^tU(t)V - J^tU(t)V = 3t\int_{-\infty}^x LU(t)V dx = 0$ so that $\|D^\alpha I^tU(t)V\|_{L^2} + \|DI^tU(t)V\|_{L^2} \le C\|V\|_{H^{1,1}}$.

Let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function and ϕ be such that $f'(\phi)I^t\phi_x$ has a sense. Then :

$$I^t(f(\phi)_x) = xf(\phi)_x + 3tf(\phi)_t = xf'(\phi)\phi_x + 3tf'(\phi)\phi_t = f'(\phi)I^t\phi_x.$$

We will use this formula for $f(x) = x^p$ and $\phi = U(t)V + w_n(t)$.

We compute :

$$\begin{aligned} LI^tw_n = I^tLw_n + 3\int_{-\infty}^x Lw_n &= -I^t(U(t)V + w_n)^p_x - 3(U(t)V + w_n)^p \\ &= -p(U(t)V + w_n)^{p-1}I^t(U(t)V + w_n)_x - 3(U(t)V + w_n)^p \\ &= -p(U(t)V + w_n)^{p-1}(I^t(U(t)V + w_n))_x - (3-p)(U(t)V + w_n)^p. \end{aligned}$$

Apply the operator D^α, multiply by $D^\alpha I^tw_n$ and integrate in space (recall $[D^\alpha, L] = 0$ and $(L\phi, \phi) = \frac{1}{2}\frac{d}{dt}\int\phi^2$) :

$$\begin{aligned} \frac{d}{dt}\|D^\alpha I^tw_n\|^2_{L^2} \le \|D^\alpha I^tw_n\|_{L^2}\|D^\alpha(U(t)V + w_n)^{p-1}(I^t(U(t)V + w_n))_x\|_{L^2} \\ + C\|D^\alpha I^tw_n\|_{L^2}\|D^\alpha(U(t)V + w_n)^p\|_{L^2}. \quad (2.18) \end{aligned}$$

We now apply Lemma 2.2 with $k = U(t)V + w_n$ and $h = I^t(U(t)V + w_n)$. So that (along with the linear estimates (2.13), (2.14), (2.15)) :

$$\begin{gathered} \|kk_x\|_{L^\infty} \le CK^2t^{-1}, \ \|k\|_{L^2} \le CK, \quad \|k\|_{L^\infty} \le CKt^{-\frac{1}{3}}, \\ \|k^{p-1}\|_{L^2} = \|k\|^{p-1}_{L^{2(p-1)}} \le CK^{p-1}t^{(p-1)(-\frac{1}{3}+\frac{1}{6(p-1)})} \le CK^{p-1}t^{-(2p-3)/6}, \end{gathered}$$

and :

$$\|D^\alpha h\|_{L^2} \le CK, \quad \|h_x\|_{L^2} \le CK.$$

We can compute :

$$\begin{aligned} &\|D^\alpha(U(t)V + w_n)^{p-1}(I^t(U(t)V + w_n))_x\|_{L^2} \\ &\le CK^p(t^{-(p-3)/3+1} + t^{-(p-3-2\gamma)/3+1} + t^{-(p-3+2\gamma)/3-1+\gamma}) \\ &\le CK^pt^{-(p-2\gamma)/3}. \end{aligned}$$

(as $t \ge I_n^* \ge 1$; the point being $(p-2\gamma)/3 > 1$). And for the second term :

$$\begin{aligned} \|D^\alpha(U(t)V + w_n)^p\|_{L^2} &\le CK^p(t^{-(2p-3)/6-1/2} + t^{-(2p-3)/6-\gamma-(1-3\gamma)/2}) \\ &\le CK^pt^{-p/3+\gamma/2}. \end{aligned}$$

Finally, after simplification by $\|D^\alpha I^t w_n\|_{L^2}$, (2.18) gives :

$$\frac{d}{dt}\|D^\alpha I^t w_n\|_{L^2} \le CK^p t^{-(p-2\gamma)/3}.$$

And as before, we integrate on $[t, S_n]$:

$$\|D^\alpha I^t w_n(t)\|_{L^2} \le CK^p t^{-(p-3-2\gamma)/3}. \tag{2.19}$$

Proof of (iv).
Again, we compute :

$$\begin{aligned} LI^t(w_{nx}) &= I^t L w_{nx} + 3Lw_n = -I^t(U(t)V + w_n)^p_{xx} - 3(U(t)V + w_n)^p_x \\ &= -(I^t(U(t)V + w_n)^p_x)_x - 2(U(t)V + w_n)^p_x \\ &= -p(U(t)V + w_n)^{p-1}(I^t(U(t)V + w_n)_x)_x \\ &\quad - p(p-1)(U(t)V + w_n)^{p-2}(U(t)V + w_n)_x I^t(U(t)V + w_n)_x \\ &\quad - 2p(U(t)V + w_n)^{p-1}(U(t)V + w_n)_x. \end{aligned} \tag{2.20}$$

We want to multiply by $I^t w_{nx}$ and integrate in space. There is essentially one troublesome term, the double derivative one. We split it :

$$(U(t)V + w_n)^{p-1}(I^t(U(t)V + w_n)_x)_x = (U(t)V + w_n)^{p-1}((I^t U(t)V_x)_x + (I^t w_{nx})_x). \tag{2.21}$$

The second term will have only first order terms after integration by parts, which is fine :

$$\begin{aligned} &\left|\int (U(t)V + w_n)^{p-1}(I^t w_{nx})_x I^t w_{nx}\right| \\ &= \frac{(p-1)}{2}\left|\int (U(t)V + w_n)^{p-2}(U(t)V + w_n)_x (I^t w_{nx})^2\right| \\ &\le C\|(U(t)V + w_n)(U(t)V + w_n)_x\|_{L^\infty}\|U(t)V + w_n\|_{L^\infty}^{p-3}\|I^t w_{nx}\|_{L^2}^2 \\ &\le CK^{p-1}t^{-p/3}\|I^t w_{nx}\|_{L^2}^2. \end{aligned}$$

However the first term on the right hand side of (2.21) requires extra regularity on V. Indeed

$$(I^t U(t)V_x)_x = (J^t U(t)V_x)_x = (U(t)xV_x)_x,$$

and ($xV_x \in H^{1,1}$), $M_0^t(U(t)(xV_x)) \le \|V\|_{H^{2,2}}$ so that by Lemma 2.1 :

$$|(I^t U(t)V_x)_x(x)| \le \frac{C\|V\|_{H^{2,2}}}{t^{2/3}}\left(1 + \left|\frac{x}{t^{1/3}}\right|\right)^{\frac{1}{4}}.$$

We can now estimate (essentially in the same way that for the H^1 estimate) :

$$\begin{aligned} &\left|\int (U(t)V + w_n)^{p-1}(I^t U(t)V_x)_x I^t w_{nx}\right| \\ &\le \|U(t)V + w_n\|_{L^\infty}^{p-3}\|U(t)V + w_n\|_{L^2} \\ &\qquad \|(U(t)V + w_n)(I^t U(t)V_x)_x\|_{L^\infty}\|I^t w_{nx}\|_{L^2} \\ &\le CK^{p-1}t^{p/3}\|I^t w_{nx}\|_{L^2}. \end{aligned}$$

The remaining two terms in (2.20) are simpler and can be treated directly (after multiplication by $I^t w_{nx}$ and integration in x) :

$$\begin{aligned}
&\left|\int (U(t)V+w_n)^{p-2}(U(t)V+w)_x I^t(U(t)V+w_n)_x I^t w_{nx}\right| \\
&\leq \|U(t)V+w_n\|_{L^\infty}^{p-3}\|(U(t)V+w_n)(U(t)V+w_n)_x\|_{L^\infty} \\
&\qquad\qquad \|I^t(U(t)V+w_n)_x\|_{L^2}\|Iw_{nx}\|_{L^2} \\
&\leq CK^p t^{p/3}\|I^t w_{nx}\|_{L^2}, \\
&\left|\int (U(t)V+w_n)^{p-2}(U(t)V+w_n)_x I^t w_{nx}\right| \\
&\leq \|U(t)V+w_n\|_{L^\infty}^{p-3}\|(U(t)V+w_n)(U(t)V+w_n)_x\|_{L^\infty} \\
&\qquad\qquad \|U(t)V+w_n\|_{L^2}\|I^t w_{nx}\|_{L^2} \\
&\leq CK^p t^{p/3}\|I^t w_{nx}\|_{L^2}.
\end{aligned}$$

Hence, (after simplifying by $\|I^t w_{nx}\|_{L^2}$) we get :

$$\frac{d}{dt}\|I^t w_{nx}\|_{L^2} \leq CK^{p-1}(K+\|I^t w_{nx}\|_{L^2})t^{p/3} \leq CK^p t^{-p/3}.$$

And Gronwall's Lemma (between t and S_n) gives :

$$\|I^t w_{nx}(t)\|_{L^2} \leq CK^p t^{-(p-3)/3}. \tag{2.22}$$

Proof of (v).

We now have to go back to

$$J^t w_n(t) = I^t w_n(t) - 3t\int_{-\infty}^{x} Lw_n(t)dx' = I^t w_n(t) + 3t(U(t)V+w_n(t))^p.$$

Using the commutation relations, we get :

$$\begin{aligned}
\|D^\alpha J^t w_n\|_{L^2} + \|DJ^t w_n\|_{L^2} &\leq \|D^\alpha I^t w_n\|_{L^2} + 3t\|D^\alpha(U(t)V+w_n(t))^p\|_{L^2} \\
&+ \|I^t w_{nx}\|_{L^2} + \|w_n\|_{L^2} + 3pt\|(U(t)V+w_n(t))^{p-1}(U(t)V+w_n(t))_x\|_{L^2}.
\end{aligned}$$

Now, using again Lemma 2.2 with $k = U(t)V + w_n(t)$:

$$\begin{aligned}
\|D^\alpha(U(t)V+w_n(t))^p\| &\leq C\|k^{p-1}\|_{L^2}(\|kk_x\|_{L^\infty}^{1/2} + \|k\|_{L^\infty}^{3\gamma}\|kk_x\|_{L^\infty}^{(1-3\gamma)/2}) \\
&\leq CK^p t^{-p/3+\gamma/2}.
\end{aligned}$$

And as usual :

$$\begin{aligned}
&\|(U(t)V+w_n)^{p-1}(U(t)V+w_n)_x\|_{L^2} \\
&\leq \|U(t)V+w_n\|_{L^\infty}^{p-3}\|(U(t)V+w_n)(U(t)V+w_n)_x\|_{L^\infty}\|U(t)V+w_n\|_{L^2} \\
&\leq CK^p t^{p/3}.
\end{aligned}$$

So using these last two inequalities, along with (2.16), (2.19) and (2.22), we get :

$$\|D^\alpha J^t w_n\|_{L^2} + \|DJ^t w_n\|_{L^2} \leq CK^p t^{(p-3+2\gamma)/3}. \tag{2.23}$$

Recall $\delta = (p-3-2\gamma)/3 > 0$. So adding up (2.17) and (2.23) gives :

$$\forall t \in [I_n^*, S_n], \quad M_0^t(w_n(t)) \leq CK^p t^{-\delta}. \qquad \square$$

2.3.3 Construction and uniqueness of u.

Proof of Theorem 2.1. Existence of u.

Proposition 2.1 provides us with a sequence $w_n(t)$ solution to (2.9) satisfying uniform estimates in n :

$$\forall t \in [T_0, S_n], \quad M_0^t(w_n(t)) \leq CK^p t^{-\delta}.$$

In particular, for $t \in [T_0, S_n]$, estimates (2.13) and (2.14) are valid.

Let us prove that for all $k \in \mathbb{N}$, $(w_n)_{n \geq k}$ is a convergent sequence in $C^0([T_0, S_k], L^2(\mathbb{R}))$. For this we show that $(w_n)_{n \geq k}$ is in fact a Cauchy sequence in $C^0([T_0, S_k], L^2(\mathbb{R}))$: let us consider $v_{n,m}(t) = w_n(t) - w_m(t)$ in L^2. Without loss of generality, we can suppose that $m > n \geq k$. First : $\|v_{n,m}(S_n)\|_{L^2} \leq CK^p S_n^{-(p-3)/3}$ (see (2.16)).

$v_{n,m}$ satisfies (we denote $v = v_{n,m}$ for simplicity in the computations) :

$$v_t + v_{xxx} + (U(t)V + w_n)^p_x - (U(t)V + w_m)^p_x = 0. \tag{2.24}$$

We multiply by v and integrate in x :

$$\frac{1}{2}\frac{d}{dt}\int v^2 = \int v((U(t)V + w_n)^p_x - (U(t)V + w_m)^p_x).$$

Now, for any functions ϕ and ψ :

$$\phi^p_x - \psi^p_x = p\phi^{p-1}\phi_x - p\psi^{p-1}\psi_x = p\phi^{p-1}(\phi - \psi)_x + p(\phi^{p-1} - \psi^{p-1})\psi_x,$$

and :

$$|\phi^{p-1} - \psi^{p-1}| \leq C|\psi - \phi|(|\phi|^{p-2} + |\psi|^{p-2}).$$

So that with $\phi = U(t)V + w_n$, $\psi = U(t)V + w_m$ (after integration by parts on the first term) :

$$\frac{1}{2}\frac{d}{dt}\int v^2 = -\frac{p-1}{2}\int v^2 (U(t)V + w_n)^{p-2}(U(t)V + w_n)_x$$
$$+ \int v(U(t)V + w_n)^{p-1} - (U(t)V + w_m)^{p-1})(U(t)V + w_m)_x.$$

Treating each term separately (for $t \in [T_1, S_n]$) :

$$\left|\int v^2(U(t)V + w_n)^{p-2}(U(t)V + w_n)_x\right| \leq CK^{p-1}t^{-p/3}\|v\|^2_{L^2},$$
$$\left|\int v(U(t)V + w_n)^{p-1} - (U(t)V + w_m)^{p-1})(U(t)V + w_m)_x\right|$$
$$\leq C\int v^2(|U(t)V + w_n|^{p-2} + |U(t)V + w_m|^{p-2}|(U(t)V + w_m)_x|$$
$$\leq CK^{p-1}t^{-p/3}\|v\|^2_{L^2}.$$

So that :

$$\frac{d}{dt}\|v\|^2_{L^2} \leq CK^{p-1}t^{-p/3}\|v\|^2_{L^2}.$$

Now, using Gronwall's Lemma on $[t, S_n]$:

$$\|v_{n,m}(t)\|_{L^2} \leq C\|v_{n,m}(S_n)\|_{L^2} \leq CS_n^{-(p-3)/3}.$$

This proves that $(w_n)_{n\geq k}$ is a Cauchy sequence in the space $C([T_0,S_k],L^2)$ and so converges to a certain $w(t,x)$. Since this can be done for arbitrarily large n (and $S_n\to\infty$), $w\in C([T_0,\infty),L^2)$ is the only possible weak limit of $(w_n)_{n\in\mathbb{N}}$.

Given a fixed $t\geq T_0$, $M_0^t(w_n(t))\leq CK^pt^{-\delta}$, so by weak limit :

$$M_0^t(w(t))\leq CK^pt^{-\delta}<\infty.$$

Now w_n satisfies (2.9), hence :

$$w_n(t)=w_n(T_0)+\left(\int_{T_0}^t U(t-s)(w_n(s))^p ds\right)_x.$$

$w_n(T_0)\to w(T_0)$ in L^2 and $w_n(t)\to w(t)$ in L^2. Furthermore, $w_n\to w$ in $C_b([T_0,t],L^2)$ and w_n is a bounded sequence in $L^\infty([T_0,t],H^1)$ (for all $t\geq T_0$), so that by interpolation $w_n^p\to w_n^p$ in $C([T_0,t],L^2)$. Thus :

$$\left(\int_{T_0}^t U(t-s)(w_n(s))^p ds\right)_x \to \left(\int_{T_0}^t U(t-s)(w(s))^p ds\right)_x \quad \text{in } H^{-1}.$$

Hence, w satisfies the integral formulation of (2.8). As a consequence, when defining $u(t)=w(t)+U(t)V$, $u\in C_b([T_0,\infty),H^1)$ satisfies the conditions of Theorem 2.1.

Uniqueness of u.

Let $\tilde u$ be another solution. We switch to $\tilde w(t,x)=\tilde u(t,x)-(U(t)V)(x)$. Then of course :

$$\begin{cases} \tilde w_t+\tilde w_{xxx}+(U(t)V+\tilde w)^p_x=0,\\ M_0^t(\tilde w(t))\to 0 \quad \text{as } t\to\infty.\end{cases}$$

Introduce T_1 such that for $t\geq T_1$, $M_0^t(\tilde w(t))\leq 1$. Let us introduce again $v(t,x)=w(t,x)-\tilde w(t,x)$. Then :

$$v_t+v_{xxx}+(U(t)V+w)^p_x-(U(t)V+\tilde w)^p_x=0,$$

which is basically identical to (2.24). So multiplying by v and integrating in x, one can do the same computations as previously, so that for $t\geq T_1$:

$$\left|\frac{d}{dt}\|v(t)\|^2_{L^2}\right|\leq C(V)t^{-p/3}\|v\|^2_{L^2}.$$

By Gronwall's Lemma between t and τ :

$$\|v(t)\|_{L^2}\leq C(V)\|v(\tau)\|_{L^2}.$$

By hypothesis, $M_0^\tau(v(\tau))\to 0$ so that $\|v(\tau)\|_{L^2}\to 0$, and letting $\tau\to\infty$ gives :

$$\forall t\geq T_1,\quad v(t,x)=0.$$

So $w(t,x)=\tilde w(t,x)$, $u(t,x)=\tilde u(t,x)$ for $t\geq T_1$ and by H^1 uniqueness, for all t such that u and $\tilde u$ are defined, $u(t,x)=\tilde u(t,x)$. □

2.4 Proof of Theorem 2.2 : the critical case $p = 5$.

2.4.1 Linear estimates.

Let us now set $p = 5$, and recall the linear estimates in [14].

Lemma 2.4. *Let ϕ be a function of space and ψ be a function of time and space. Then the following inequalities hold, assuming their right-hand side term is bounded :*

1. $\|\partial_x U(t)\phi\|_{L^\infty_x L^2_t([T,\infty))} \leq \|\phi\|_{L^2_x}$.
2. $\sup_{t\geq T} \left\| \partial_x \int_t^\infty U(t-s)\psi(s)ds \right\|_{L^2_x} \leq \|\psi\|_{L^1_x L^2_t([T,\infty))}$.
3. $\left\| \partial^2_{xx} \int_t^\infty U(t-s)\psi(s)ds \right\|_{L^\infty_x L^2_t([T,\infty))} \leq \|\psi\|_{L^1_x L^2_t([T,\infty))}$.
4. $\|U(t)\phi\|_{L^5_x L^{10}_t([T,\infty))} \leq C\|\phi\|_{L^2_x}$.
5. $\left\| \int_t^\infty U(t-s)\psi(s)ds \right\|_{L^5_x L^{10}_t([T,\infty))} \leq \|\psi\|_{L^{5/4}_x L^{10/9}_t([T,\infty))}$.

Proof. See [14]. In the proof, $T = -\infty$; for general T, the estimates 1. and 4. are clear, and for estimates 2., 3. and 5., replace ψ by $\psi \mathbb{1}_{t\geq T}$. □

2.4.2 Construction of u.

Proof of Theorem 2.2. Let us do formal computations first : suppose u is the desired solution, and write :

$$u(t) = U(t)V + w(t), \quad \text{i.e.} \quad w(t) = u(t) - U(t)V, \quad \lim_{t\to\infty} \|w\|_{L^2} = 0.$$

u satisfies :

$$u(\tau) = U(\tau)u(0) + \partial_x \int_0^\tau U(\tau - s)u^5(s)ds = U(\tau - t)u(t) + \partial_x \int_t^\tau U(\tau - s)u^5(s)ds.$$

Thus, w satisfies :

$$U(\tau)V + w(\tau) = U(\tau - t)(U(t)V + w(t)) + \partial_x \int_t^\tau U(\tau - s)(U(s)V + w(s))^5(s)ds.$$

$U(\tau)V$ cancels and we obtain after composing by $U(t-\tau)$:

$$U(t-\tau)w(\tau) = w(t) + \partial_x \int_t^\tau U(t-s)(U(s)V + w(s))^5(s)ds.$$

Now, let $\tau \to \infty$, $\|U(t-\tau)w(\tau)\|_{L^2} = \|w(\tau)\|_{L^2} \to 0$, so we obtain w as a fixed point :

$$w(t) = -\partial_x \int_t^\infty U(t-s)(U(s)V + w(s))^5(s)ds.$$

This explains the following scheme : we will show that such fixed point exists by a contracting map argument. Let us introduce the function Φ.

$$\Phi : w \mapsto -\partial_x \int_t^\infty U(t-s)(U(s)V + w(s))^5 ds. \tag{2.25}$$

Our goal is to find a fixed point for Φ.

Let us introduce the following norm :

$$\|\phi\|_{X_T} = \|\phi\|_{L^5_x L^{10}_t([T,\infty))} + \|\partial_x \phi(t)\|_{L^\infty_x L^2_t([T,\infty))}.$$

From estimate 5. of Lemma 2.4, we get :

$$\begin{aligned}\|\Phi(w)(t)\|_{L^5_x L^{10}_t([t,\infty))} &\le \|(U(t)V + w(t))^4 (U(t)V + w(t))_x\|_{L^{5/4}_x L^{10/9}_t([t,\infty))} \\ &\le \|(U(t)V + w(t))^4\|_{L^{5/4}_x L^{10/4}_t([t,\infty))} \|(U(t)V + w(t))_x\|_{L^\infty_x L^2_t([t,\infty))} \\ &\le C\|U(t)V + w(t)\|^4_{L^5_x L^{10}_t([t,\infty))} \|\partial_x(U(t)V + w(t))\|_{L^\infty_x L^2_t([t,\infty))}.\end{aligned}$$

By estimate 3. :

$$\begin{aligned}\|\partial_x \Phi(w)(t)\|_{L^\infty_x L^2_t([t,\infty))} &\le \|(U(t)V + w(t))^5\|_{L^1_x L^2_t([t,\infty))} \\ &\le \|U(t)V + w(t)\|^5_{L^5_x L^{10}_t([t,\infty))}.\end{aligned}$$

Adding up the last two estimates (and using $|a+b|^k \le 2^{k-1}(|a|^k + |b|^k)$, $|a^4 b| \le |a|^5 + |b|^5$):

$$\begin{aligned}\|\Phi(w)\|_{X_T} &\le C\|U(t)V + w(t)\|^4_{L^5_x L^{10}_t([T,\infty))} \\ &\quad \times \Big(\|U(t)V + w(t)\|_{L^5_x L^{10}_t([T,\infty))} + \|\partial_x(U(t)V + w(t))\|_{L^\infty_x L^2_t([T,\infty))}\Big) \\ &\le 2^3(\|U(t)V\|^4_{L^5_x L^{10}_t([T,\infty))} + \|w\|^4_{X_T})(\|U(t)V\|_{X_T} + \|w\|_{X_T}) \\ &\le C(\|U(t)V\|^4_{L^5_x L^{10}_t([T,\infty))}\|U(t)V\|_{X_T} + \|U(t)V\|_{X_T}\|w\|^4_{X_T} + \|w\|^5_{X_T}).\end{aligned}$$

Let us prove the following simple lemma :

Lemma 2.5. *Let V be such that $U(t)V \in X_0$ (in particular, this holds if $V \in L^2$). Then $\|U(t)V\|_{L^5_x L^{10}_t([T,\infty))} \to 0$ as $T \to \infty$.*

Proof. Consider

$$f(\tau, x) = \|U(t)V\|^5_{L^{10}_t[\tau,\infty)}, \text{ and } h(x) = \|U(t)V\|^5_{L^{10}_t[0,\infty)} = f(0,x).$$

f and h are finite x-a.e. as $\int_x f < \infty$ and $\int_x h < \infty$. Then $h \in L^1_x$ and for all $\tau \ge 0$, $0 \le f(\tau,x) \le h(x)$. Now, since f is an exhausting integral, we have :

$$x-\text{a.e.}, \lim_{\tau\to\infty} f(\tau,x) = 0.$$

Lebesgue's dominated convergence theorem applies : $\lim_{\tau\to\infty} \|f(\tau,x)\|_{L^1_x} = 0$, which is exactly $\lim_{\tau\to\infty} \|U(t)V\|_{L^5_x L^{10}_t[\tau,\infty)} = 0$. □

As $V \in L^2$, by the previous lemma, $\|U(t)V\|_{L^5_x L^{10}_t([T,\infty))} \to 0$ as $T \to \infty$. Moreover, $\|U(t)V\|_{X_T} \leq C\|V\|_{L^2}$. So our estimate writes :

$$\|\Phi(w)\|_{X_T} \leq C(\|U(t)V\|^4_{L^5_x L^{10}_t([T,\infty))}\|V\|_{L^2} + \|V\|_{L^2}\|w\|^4_{X_T} + \|w\|^5_{X_T}). \tag{2.26}$$

This shows that for T_0 large enough, there exists $\delta > 0$ so that Φ maps $B_{X_{T_0}}(0,\delta)$ to itself :

$$\Phi : B_{X_{T_0}}(0,\delta) \to B_{X_{T_0}}(0,\delta).$$

The same computations show that for T_0 large enough, $\delta > 0$ small enough, $\Phi : B_{X_{T_0}}(0,\delta) \to B_{X_{T_0}}(0,\delta)$ is a contraction. Thus, Φ has a unique fixed point, which we denote v.

$v = \Phi(v)$ writes :

$$v(t) = -\partial_x \int_t^\infty U(t-s)(U(s)V + v(s))^5 ds,$$

and by construction $u(t) = U(t)V + v(t)$ is a solution to (2.4). Now by (2.26), if δ has been chosen small enough, we have :

$$\|v\|_{X_T} \leq C\|U(t)V\|^4_{L^5_x L^{10}_t([T,\infty))}\|V\|_{L^2} \to 0 \quad \text{as} \quad T \to \infty.$$

And by estimate 2. of Lemma 2.4 :

$$\begin{aligned}\|v\|_{L^\infty_t([T,\infty),L^2_x)} &\leq \|U(t)V + v\|^5_{L^5_x L^{10}_t([T,\infty))} \\ &\leq C(\|U(t)V\|^5_{L^5_x L^{10}_t([T,\infty))} + \|v\|^5_{X_T}) \to 0 \text{ as } T \to \infty.\end{aligned}$$

To conclude, v satisfies the decay estimate $\|v(t)\|_{L^2} \to 0$ and moreover :

$$\|v\|_{L^\infty_t([T,\infty))L^2_x} + \|v\|_{L^5_x L^{10}_t([T,\infty))} + \|\partial_x v\|_{L^\infty_x L^2_t([T,\infty))} \to 0 \quad \text{as} \quad T \to \infty. \qquad \square$$

Appendix A.

For the sake of completeness, we present the proof of lemma 2.1 of [13].

Proof of Lemma 2.1. Let us note $v(t,x) = U(-t)\phi$. We have the identity :

$$\begin{aligned}\phi(x) = U(t)v &= \frac{1}{\pi}\Re \int_0^\infty e^{ipx+ip^3t/3}\hat{v}(t,p)dp \\ &= \frac{1}{\pi\sqrt[3]{t}}\Re \int_0^\infty e^{iq\eta+iq^3/3}\left(\hat{v}(t,\chi) + \left(\hat{v}\left(t,\frac{q}{\sqrt[3]{t}}\right) - \hat{v}(t,\chi)\right)\right)dq \\ &= \frac{1}{\sqrt[3]{t}}\Re \mathrm{Ai}\left(\frac{x}{\sqrt[3]{t}}\right)\hat{v}(t,\chi) + \mathcal{R}(t,x),\end{aligned}$$

where we made the change of variable $q = p\sqrt[3]{t}$ and $\eta = x/\sqrt[3]{t}$, and we introduce $\chi = \sqrt{-x/t}$ if $x \leq 0$ and $\chi = 0$ if $x \geq 0$, and :

$$\mathcal{R}(t,x) = \frac{1}{\pi\sqrt[3]{t}}\Re \int_0^\infty e^{iq\eta+iq^3/3}\left(\hat{v}\left(t,\frac{q}{\sqrt[3]{t}}\right) - \hat{v}(t,\chi)\right)dq.$$

We now estimate $\mathcal{R}(t,x)$, and consider two cases : $x \geq 0$ and $x \leq 0$. For this, we will do appropriate integration by parts to use the decay related to the oscillatory integral.

Let us introduce a final notation $\mu = \sqrt{|\eta|}$. We have the identity :

$$e^{iq\eta+iq^3/3} = \frac{1}{1+iq(q^2+\mu^2)}\frac{\partial}{\partial q}(qe^{iq\eta+iq^3/3}). \tag{2.27}$$

Consider the case $x \geq 0$, that is $\chi = 0$. We can do an integration by parts using (2.27) in the remainder term $\mathcal{R}$:

$$\mathcal{R}(t,x) = \frac{1}{\pi\sqrt[3]{t}}\Re\int_0^\infty \left(\frac{iq(q^2+\mu^2)(\hat{v}(t,q/\sqrt[3]{t})-\hat{v}(t,0))}{1+iq(q^2+\mu^2)} - \frac{q}{\sqrt[3]{t}}\hat{v}_p\left(t,\frac{q}{\sqrt[3]{t}}\right)\right)\frac{e^{iq\eta+iq^3/3}dq}{1+iq(q^2+\mu^2)}.$$

Using the identity

$$\|D^\alpha J^t\phi\|_{L^2} = \|U(t)D^\alpha(xU(-t)\phi)\|_{L^2} = \|D^\alpha(xv)\|_{L^2} = \||p|^\alpha\hat{v}_p\|_{L^2},$$

and the Cauchy-Schwarz inequality, we get :

$$\begin{aligned}|\hat{v}(t,p)-\hat{v}(t,0)| &\leq \int_0^p |\hat{v}_p(t,\rho)|d\rho \leq \left(\int_0^p |\rho|^{-2\alpha}d\rho\int_0^p |\hat{v}_p^2(t,\rho)\rho^{2\alpha}|d\rho\right)^{\frac{1}{2}}\\ &\leq C|p|^\gamma\||p|^\alpha\hat{v}_p(t,p)\|_{L^2} \quad\leq\quad C|p|^\gamma\|D^\alpha J^t\phi\|_{L^2}.\end{aligned} \tag{2.28}$$

On the other hand, using the change of variable $z = q(1+\mu^2)$, a computation shows that, for any $a, b > 0$ such that $3b - a > 1$, we have :

$$\int_0^\infty \frac{q^a dq}{(1+q(q^2+\mu^2))^b} \leq \int_0^{\mu+1}\frac{2q^a dq}{(1+q(1+\mu^2))^b} + \int_{\mu+1}^\infty \frac{q^a dq}{1+q^{3b}} \leq \frac{C}{(1+\mu)^{3b-a-1}}. \tag{2.29}$$

(we used the inequality $1+q(q^2+\mu^2) \geq \frac{1}{2}(1+q(1+\mu^2))$ for the first term and $q^2 \leq q^2+\mu^2$ for the second). Thus we can estimate :

$$\begin{aligned}|\mathcal{R}(t,x)| &\leq \frac{C}{\sqrt[3]{t}}\int_0^\infty \frac{(|\hat{v}(t,q/\sqrt[3]{t})-\hat{v}(t,0)| + q/\sqrt[3]{t}|\hat{v}_p(t,q/\sqrt[3]{t})|)dq}{|1+iq(q^2+\mu^2)|}\\ &\leq \frac{C}{t^{(1+\gamma)/3}}\|D^\alpha J^t\phi\|_{L^2}\int_0^\infty \frac{q^\gamma dq}{1+q(q^2+\mu^2)}\\ &\quad + \frac{C}{\sqrt[3]{t}^2}\left(\int_0^\infty q^{2\alpha}\hat{v}_p^2\left(t,\frac{q}{\sqrt[3]{t}}\right)dq\int_0^\infty\frac{q^{1+2\gamma}dq}{(1+q(q^2+\mu^2))^2}\right)^{1/2}\\ &\leq \frac{C\|D^\alpha J^t\phi\|_{L^2}}{t^{(1+\gamma)/3}(1+\mu)^{2-\gamma}}.\end{aligned} \tag{2.30}$$

Let us now consider the case $x \leq 0$, that is, $\eta = -\mu^2 \leq 0$ (recall $\mu = \sqrt{|x|}/\sqrt[6]{t}$, $\chi = \mu/\sqrt[3]{t}$). We will now use the identity :

$$e^{iq\eta+iq^3/3} = \frac{1}{1+i(q^2-\mu^2)(q+\mu)}\frac{\partial}{\partial q}((q-\mu)e^{iq\eta+iq^3/3}). \tag{2.31}$$

We integrate by parts the remainder term $\mathcal{R}$:

$$\mathcal{R}(t,x) = \frac{1}{\pi\sqrt[3]{t}}\Re\int_0^\infty\left(\frac{i(q-\mu)^2(3q+\mu)}{1+i(q^2-\mu^2)(q+\mu)}\left(\hat{v}\left(t,\frac{q}{\sqrt[3]{t}}\right)-\hat{v}(t,\chi)\right)\right.$$

$$-\frac{q-\mu}{\sqrt[3]{t}}\hat{v}_p\left(t,\frac{q}{\sqrt[3]{t}}\right)\Bigg)\frac{e^{iq\eta+iq^3/3}dq}{1+i(q+\mu)(q-\mu)^2}.$$

A computation similar to (2.29) gives (provided that $3c-a-b-1>0$) :

$$\begin{aligned}\int_0^\infty \frac{|q-\mu|^a q^b dq}{(1+(q-\mu)^2(q+\mu))^c} &\le \int_0^{2(1+\mu)} + \int_{2(1+\mu)}^\infty \\ &\le C\int_{-1-\mu}^{1+\mu}\frac{|z|^a(1+\mu)^b dz}{1+(z^2(1+\mu))^c} + \int_{1+\mu}^\infty \frac{|z|^{a+b}dz}{1+z^{3c}} \\ &\le C(1+\mu)^{a+b+1-3c}.\end{aligned}$$

(with the change of variable $z=q-\mu$ and $z'=z\sqrt{1+\mu}$). Thus, by the Cauchy-Schwarz inequality, we obtain :

$$\begin{aligned}|\mathcal{R}(t,x)| &\le \frac{C}{\sqrt[3]{t}}\int_0^\infty \left(\left|\hat{v}\left(t,\frac{q}{\sqrt[3]{t}}\right)-\hat{v}(t,\chi)\right| + \frac{|q-\mu|}{\sqrt[3]{t}}\left|\hat{v}_p\left(t,\frac{q}{\sqrt[3]{t}}\right)\right|\right)\frac{dq}{|1+i(q-\mu)^2(q+\mu)|} \\ &\le \frac{C}{t^{(1+\gamma)/3}}\|D^\alpha J^t\phi\|_{L^2}\Bigg(\int_0^\infty \frac{|q-\mu|^\gamma dq}{1+(q-\mu)^2(q+\mu)} \\ &\qquad + \left(\int_0^\infty \frac{(q-\mu)^2 dq}{(1+(q-\mu)^2(q+\mu))^2 q^{2\alpha}}\right)^{1/2}\Bigg) \\ &\le \frac{C\|D^\alpha J^t\phi\|_{L^2}}{t^{(1+\gamma)/3}(1+\mu)^{2-\gamma}}.\end{aligned}\tag{2.32}$$

(recall $\alpha = 1/2-\gamma)$). It remains to bound $\|\hat{v}(t,\cdot)\|_{L^\infty}$:

$$\begin{aligned}|\hat{v}(t,p)| &\le \int_{\mathbb{R}}|\hat{v}_p(t,\rho)|d\rho \le \int_{|\rho|\ge 1}|\hat{v}_p(t,\rho)|d\rho + \int_{|\rho|\le 1}|\hat{v}_p(t,\rho)|d\rho \\ &\le \left(\int_{|\rho|\ge 1}|\hat{v}_p^2\rho^2|d\rho\int_{|\rho|\ge 1}\frac{d\rho}{\rho^2}\right)^{\frac12} + \left(\int_{|\rho|\le 1}|\hat{v}_p^2\rho^{2\alpha}|d\rho\int_{|\rho|\le 1}\frac{d\rho}{\rho^{2\alpha}}\right)^{\frac12} \\ &\le C(\|DJ^t\phi\|_{L^2}+\|D^\alpha J^t\phi\|_{L^2}).\end{aligned}\tag{2.33}$$

Thus, adding up (2.30), (2.32), along with the estimate $|\mathrm{Ai}(\eta)| \le C(1+|\eta|)^{-1/4}$ and (2.33), we obtain (recall $\eta = x/\sqrt[3]{t}$) :

$$|\phi(x)| \le \frac{C}{t^{1/3}}(1+|\eta|)^{-1/4}M_0^t(\phi), \tag{2.34}$$

which is the first pointwise estimate. The L^r-estimate follows :

$$\begin{aligned}\|\phi\|_{L^r} &\le \frac{C}{t^{1/3}}M_0^t(\phi)\left(\int\left(1+\frac{|x|}{\sqrt[3]{t}}\right)^{r/4}dx\right)^{1/r} \\ &\le \frac{C}{t^{1/3-1/(3r)}}M_0^t(\phi)\left(\int(1+|\eta|)^{r/4}d\eta\right)^{1/r} \le \frac{C}{t^{1/3-1/(3r)}}M_0^t(\phi).\end{aligned}$$

Now we switch to the derivative ϕ_x :

$$\phi_x(x) = \frac{i}{\pi}\Re\int_0^\infty e^{iq\eta+iq^3t/3}\hat{v}\left(t,\frac{q}{\sqrt[3]{t}}\right)qdq.$$

Using identity (2.27), we obtain analogously to (2.30), in the domain $x \geq 0$ ($\eta = \mu^2 \geq 0$) :

$$\begin{aligned}
|\phi_x(x)| &\leq \frac{C}{t^{2/3}} \int_0^\infty \frac{|\hat{v}(t, q/\sqrt[3]{t})| q dq}{|1 + iq(q^2 + \mu^2)|} + \frac{C}{t} \int_0^\infty \frac{|\hat{v}_p(t, q/\sqrt[3]{t})| q^2 dq}{|1 + iq(q^2 + \mu^2)|} \\
&\leq \frac{C}{t^{2/3}} (\|\hat{v}(t)\|_{L^\infty} + \||p|^\alpha \hat{v}_p(t,p)\|_{L^2}) \\
&\leq \frac{C}{t^{2/3}} (\|DJ^t\phi\|_{L^2} + \|D^\alpha J^t \phi\|_{L^2}). \qquad (2.35)
\end{aligned}$$

(we used the Cauchy-Schwarz inequality as in (2.30), and in the last inequality, we used again (2.33)). In the domain $x \leq 0$ ($\eta = -\mu^2 \leq 0$), we use identity (2.31) to get analogously to (2.32) :

$$\begin{aligned}
|\phi_x(x)| &\leq \frac{C}{\sqrt[3]{t}^2} \int_0^\infty \left(\left|\hat{v}\left(t, \frac{q}{\sqrt[3]{t}}\right) + \frac{|q-\mu|}{\sqrt[3]{t}} \left|\hat{v}\left(t, \frac{q}{\sqrt[3]{t}}\right)\right|\right) \right| \frac{qdq}{|1 + i(q-\mu)^2(q+\mu)|} \\
&\leq \frac{C}{\sqrt[3]{t}^2} \|\hat{v}\|_{L^\infty} \int_0^\infty \frac{qdq}{|1 + i(q-\mu)^2(q+\mu)|} \\
&\quad + \frac{C}{\sqrt[3]{t}^2} \||p|^\alpha \hat{v}_p(t,p)\|_{L^2} \left(\int_0^\infty \frac{(q-\mu)^2 q^{2\alpha} dq}{(1 + (q-\mu)^2(q+\mu))^2} \right)^{\frac{1}{2}}.
\end{aligned}$$

The integral of the second term can be estimated by our regular computations, but we have to be more careful with the first term :

$$\begin{aligned}
\int_0^\infty \frac{qdq}{1 + (q-\mu)^2(q+\mu)} &\leq C \int_0^{2(\mu+1)} \frac{(1+\mu)dq}{1 + (q-\mu)^2\mu} + C \int_{2(\mu+1)}^\infty \frac{(q-\mu)dq}{1 + (q-\mu)^3} \\
&\leq C\sqrt{1+\mu} \int \frac{dz}{1+z^2} + \frac{C}{1+\mu} \leq C\sqrt{1+\mu}.
\end{aligned}$$

So that we obtain, for $x \leq 0$, the estimate :

$$|\phi_x(x)| \leq \frac{C}{\sqrt[3]{t^2}} (\|DJ^t\phi\|_{L^2} + \|D^\alpha J^t\phi\|_{L^2}) \sqrt{1+\mu}. \qquad (2.36)$$

Finally, combining (2.35) and (2.36) gives the second pointwise estimate (recall $\eta = x/\sqrt[3]{t}$) :

$$|\phi_x(x)| \leq \frac{CM_0^t(\phi)}{t^{2/3}} (1+\eta)^{1/4}. \qquad (2.37)$$

□

Appendix B.

To conclude, we present a proof of a local existence theorem in M_0^t, which is needed in the discussion of Theorem 2.1. The proof is done for forward times, but of course, it is also true backwards.

Theorem 2.3. *Let $t_0 \in \mathbb{R}$ and $u_0 \in H^1(\mathbb{R})$ such that $M_0^{t_0}(u_0) < \infty$. Then there exist $T = T(M_0^{t_0}(u_0))$ and a unique solution $u : [t_0, t_0 + T) \times \mathbb{R} \to \mathbb{R}$ to :*

$$\begin{cases} u_t + u_{xxx} + (u^p)_x = 0, \\ u|_{t=t_0} = u_0. \end{cases}$$

Furthermore :

$$\forall t \in [t_0, T + t_0[, \quad M_0^t(u(t)) < \infty, \text{ and } \quad \limsup_{t \downarrow t_0} M_0^t(u(t)) \leq M_0^{t_0}(u_0).$$

Proof. Uniqueness is straightforward as there is already uniqueness in H^1.

For the existence part, we proceed by regularization. Let $u_0^n \in H^3$ with compact support such that :

$$M_0^{t_0}(u_0^n - u_0) \to 0.$$

(u_0^n exists by standard density arguments). In particular, we can suppose that :

$$\forall n, \quad M_0^{t_0}(u_0^n) \leq 2M_0^{t_0}(u_0) = K.$$

For every u_0^n, the local existence theorem in H^3 ensures the existence of $u^n(t)$ on an interval $[t_0, t_0 + T^n]$. Recall that existence in H^3 has the same time span as in H^1. Since the initial data sequence $(u_0^n)_n$ is uniformly bounded in H^1, $T_n = T_n(\|u_0^n\|_{H^1}) \geq T^* > 0$.

The point in working in H^3 is that $I^t(u^n(t)) = J^t(u^n(t)) - t(u^n)^p$ is well defined in H^1. Indeed, one computes :

$$\left| \frac{d}{dt} \int x(u^n)^2 \right| = \left| -3 \int (u_x^n)^2 - \frac{2p}{p+1} \int (u^n)^{p+1} \right| \leq C \|u^n(t)\|_{H^1},$$

so that $\forall t \in [t_0, t_0 + T^*)$, $\|\sqrt{x} u^n(t)\|_{L^2} < \infty$ (recall $u_0(t_0)$ has compact support, so that quantity is initially well defined at time t_0). In the same way, the derivatives in time of $\|xu^n\|_{L^2} + \|\sqrt{x} u_x^n(t)\|_{L^2}$ are controlled by $\|u^n(t)\|_{H^2}$, and that of $\|xu_x^n\|_{L^2}$ by $\|u^n(t)\|_{H^3}$. So finally, the H^3 bound on $u^n(t)$ ensures that for $t \in [t_0, t_0 + T^*]$:

$$\|J^t u^n(t)\|_{H^1} \leq \|xu^n(t)\|_{H^1} + t\|u_{xx}^n(t)\|_{H^1} < \infty.$$

And for $I^t u_n(t)$:

$$\|I^t(u^n(t))\|_{H^1} \leq \|J^t(u^n(t))\|_{H^1} + t\|(u^n(t))^p\|_{H^1} < \infty.$$

So now it is possible to do the a priori computations of Naumkin and Hayashi [13] for $u^n(t)$. Let $I = [t_0, t_0 + T_n^*)$ such that for all $t \in I$, $M_0^t(u^n(t)) \leq 2K$ (and I maximal for this property) : by H^3 continuity, $T_n^* > 0$. Their computations give (see equations 3.3, 3.8 and 3.9 of [13]).

$$\frac{d}{dt} \|u^n\|_{H^1}^2 \leq C \frac{M_0^t(u^n(t))^{p-2}}{t^{2/3}(1+t)^{(p-2)/3}} \|u^n\|_{H^1}^2 \leq \frac{CK^p}{t^{2/3}(1+t)^{(p-2)/3}}, \tag{2.38}$$

and similar estimates for $\|D^\alpha I^t u^n\|_{L^2}$ and $\|DI^t u^n\|_{L^2}$.

Let T be such that :

$$\int_{t_0}^{t_0+T} \frac{CK^p}{t^{2/3}(1+t)^{(p-2)/3}} dt \leq \frac{K}{3}.$$

(There are 3 estimates). By a standard continuity argument, $T^* \geq T_n^* \geq T$ (independent of n), and :

$$\forall n, \ \forall t \in [t_0, t_0 + T], \quad M_0^t(u_n(t)) \leq 2K.$$

In the same way, (2.38) gives that $t \mapsto M_0^t(u_n(t))$ is equicontinuous (in n).

Now, standard computations show that $u^n(t)$ is a Cauchy sequence in $C([t_0, t_0+T], L^2)$, so it converges to $u(t)$. Indeed, with $v = u^n - u^m$:

$$v_t + v_{xxx} + (u^n)^p_x - (u^m)^p_x = 0.$$

Multiply by v, and integrate in space :

$$\frac{d}{dt}\|v\|^2_{L^2}(t) \leq \frac{(2K)^p}{t^{2/3}(1+t)^{(p-2)/3}}\|v\|^2_{L^2}(t).$$

And by Gronwall's Lemma (taking the supremum) :

$$\|v\|_{C^0([t_0,t_0+T],L^2)} \leq CK^p\|v(0)\|_{L^2}.$$

But $v(0) \to 0$ as $m, n \to \infty$ (in H^1, so in L^2).

$u(t)$ can also be seen as weak limit in H^1, and in M_0^t, of $u^n(t)$ (for t fixed) : therefore we get the bounds in H^1 and M_0^t for $u(t)$. Finally, we take the H^{-1}-limit in the integral formulation :

$$u^n(t) = S(t-t_0)u_0^n + \int_{t_0}^t S(t-\tau)(u^n)^p_x(\tau)d\tau.$$

There is no problem for $S(t-t_0)u_0^n$, and for the second term :

$$\int_{t_0}^t S(t-\tau)\left((u^n)^p_x(\tau) - (u^m)^p_x\right)d\tau = \left(\int_{t_0}^t S(t-\tau)\left((u^n)^p(\tau) - (u^m)^p\right)d\tau\right)_x.$$

$(u^n)^p(\tau) - (u^m)^p \to 0$ in $C^0([t_0, t_0+T], L^2)$: so we can take the limit in L^2 in the integral, and then in the H^{-1} sense for the whole term.

$u(t)$ is a solution to (2.1) in the integral sense, $u(t_0) = u_0$, and $u \in L^\infty([t_0, t_0+T], H^1)$: so u is the unique $C^0([t_0, t_0+T], H^1)$ solution. By weak limit :

$$\forall t \in [t_0, t_0+T[, \quad M_0^t(u(t)) \leq 2K, \quad \text{and} \quad \limsup_{t\downarrow t_0} M_0^t(u(t)) \leq M_0^{t_0}(u_0).$$ □

Chapitre 3

Construction of solutions to the L^2-critical KdV equation with a given asymptotic behaviour[1]

3.1 Introduction.

3.1.1 General setting

We consider the critical Korteweg-de Vries equation :

$$u_t + (u_{xx} + u^5)_x = 0, \quad t, x \in \mathbb{R}. \tag{3.1}$$

It is a special case of the generalized KdV equation :

$$u_t + (u_{xx} + u^p)_x = 0, \quad t, x \in \mathbb{R}, \tag{3.2}$$

where $p \geq 2$. The case $p = 2$ corresponds to the original equation introduced by Korteweg and de Vries [19] in the context of shallow water waves. For both $p = 2$ and $p = 3$, this equation has many applications to Physics : see for example Miura [39], Lamb [22].

There are two formally conserved quantities for solutions to (3.2) :

$$\int u^2(t) = \int u^2(0) \qquad (L^2 \text{ mass}), \tag{3.3}$$

$$E(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) = E(u(0)) \qquad (\text{energy}). \tag{3.4}$$

The local Cauchy problem for (3.2) has been intensively studied by many authors. Kenig, Ponce and Vega [14] proved the following existence and uniqueness result in $H^1(\mathbb{R})$: for $u_0 \in H^1(\mathbb{R})$, there exist $T = T(\|u_0\|_{H^1}) > 0$ and a solution $u \in C([0,T], H^1(\mathbb{R}))$ to (3.2) satisfying $u(0) = u_0$, which is unique in the class $Y_T \subset C([0,T], H^1(\mathbb{R}))$. Moreover, if T_1 denotes the maximal time of existence for u, then either $T_1 = +\infty$ (global solution) or $T_1 < \infty$ and $\|u(t)\|_{H^1} \to \infty$ as $t \uparrow T_1$ (blow-up solution). For such a solution, one has conservation of mass and energy. In the critical case, problem 3.1), this result is improved to local well-posedness in L^2 (see [14] and [15]).

[1] Ce chapitre a fait l'objet d'une prépublication soumise.

The next problem is to know whether these solutions to (3.2) are global in time, or blow-up. In the case $2 \leq p < 5$ (sub-critical), all solutions in H^1 are global and uniformly bounded thanks to the conservation laws and the Gagliardo-Nirenberg inequality :

$$\forall v \in H^1(\mathbb{R}), \qquad \int |v|^{p+1} \leq \kappa(p) \left(\int v^2\right)^{\frac{p+3}{4}} \left(\int v_x^2\right)^{\frac{p-1}{4}}. \tag{3.5}$$

The case $p = 5$ is L^2-critical, in the sense that mass remains unaffected by scaling. Indeed, $u_\lambda(t,x) = \lambda^{1/6} u(\lambda t, \lambda^{1/3} x)$ is also a solution to (3.1), and $\|u_\lambda\|_{L^2} = \|u\|_{L^2}$. Moreover, existence of finite time blow-up solutions was proved by Merle [32] and Martel and Merle [27]. Therefore $p = 5$ also appears as a critical exponent for the long time behaviour of solution to (3.2).

For $p > 5$ (super-critical case), numerics predict blow-up.

A fundamental property of (3.2) is the existence of a family of explicit traveling wave solutions. If Q denotes the only solution (up to translation) of :

$$Q > 0, \quad Q \in H^1(\mathbb{R}), \quad Q_{xx} + Q^p = Q, \quad \text{i.e.} \quad Q(x) = \left(\frac{p+1}{2\cosh^2\left(\frac{p-1}{2}x\right)} \right)^{1/(p-1)},$$

then for $c > 0$ the soliton

$$R_{c,x_0} = c^{1/(p-1)} Q(\sqrt{c}(x - x_0 - ct)) \text{ is a solution to (3.2).}$$

Solitons are stable in H^1 in the sub-critical case $p \in [2,5)$ (see [25], and unstable in the $p > 5$ super-critical (see [47]) and $p = 5$ critical case (see [26]).

For $p = 2$ and $p = 3$, equation (3.2) is completely integrable, and thus has very special features. The inverse scattering transform method allows to solve the Cauchy problem in an appropriate space (for example if $u_0 \in H^4$ and $xu_0 \in L^1$) and the qualitative behaviour of solutions is well understood. For example, given u_0 smooth and with rapid decay, there exist N solitons R_{c_j,x_j} such that

$$\left\| u(t) - \sum_{j=1}^{N} R_{c_j,x_j}(t) \right\|_{L^\infty(x \geq -t^{1/3})} \leq \frac{C}{t^{1/3}} \quad (\text{as } t \to \infty).$$

See for example Schuur [43], Eckhaus and Schuur [9], Miura [39].

However, if $p \neq 2$ or 3, the inverse scattering transform method does not longer apply, and the description of solutions in the general, non-integrable case is a widely open problem, especially in the critical case. It can be decomposed in two types of problems.

Problem 1 : Asymptotic behavior. In the sub-critical case, given an initial data u_0, can we describe the behavior of the solution $u(t)$ to (3.2) ? In the critical and super-critical cases, does $u(t)$ blow-up ? Can we determine the blow up rate and profile ?

Problem 2 : Construction of a non-linear wave operator. Given some reasonable behavior at $t \to \infty$, can we find a solution $u(t)$ to (3.2) defined for large enough t, with this behaviour ? Is there uniqueness for $u(t)$?

3.1.2 Recent results on Problems 1. and 2. for the critical KdV equation

From now on, we will focus only on equation (3.1), that is, the L^2-critical case.

Let us now develop some results which will be the base to our result. The first result deals with scattering for small initial data. One wants to prove that given an initial data u_0, small in an adequate functional space, the arising solution to the non-linear equation (3.1) behaves likes $U(t)v_0$, the solution to the linear equation $u_t + u_{xxx} = 0$, with initial data v_0 ($U(t)$ is the linear KdV group). The map $u_0 \mapsto v_0$ is called the scattering operator. The following result is an easy corollary of Kenig, Ponce and Vega [14].

Scattering operator. Given u_0 small enough in L^2, the solution $u(t)$ to (3.1) is global in time, and there is scattering, in the sense that there exists a function $V \in L^2$ so that

$$\|u(t) - U(t)V\|_{L^2} \to 0 \quad \text{as} \quad t \to \infty.$$

This is the description of solutions with initial data around 0 (in L^2), a result which can be understood as stability around 0.

The second point answers the question of behavior of solutions with initial data close to a soliton. As we are in the critical case, one does not have stability : contrary to the sub-critical case (see [26]), one has instability and blow-up. Let us cite a result of Merle [32] and Martel and Merle [28].

Blow-up solutions to (3.1). There exists $\alpha_0 > 0$ such that the following is true. Suppose

$$E(u) < 0 \quad \text{and} \quad \int u(t)^2 \leq \int Q^2 + \alpha_0.$$

Then $u(t)$ blows-up in finite or infinite time $T \in (0, \infty]$. Furthermore, there exist $\lambda(t) > C(T-t)^{-1/3}$, $\varepsilon \in \{-1, 1\}$ and $x(t) \in \mathbb{R}$ such that

$$\varepsilon \lambda^{1/2}(t) u(t, \lambda(t)x + x(t)) \rightharpoonup Q \quad \text{in} \quad H^1\text{-weak} \quad \text{as} \quad t \uparrow T.$$

These results are related to Problem 1. Let us now turn to results concerning Problem 2. A surprising result of Martel [23] is the existence and uniqueness of N-solitons in the critical case :

Existence and uniqueness of the N-soliton. Let $p \in [2,5]$. Let $N \in \mathbb{N}$, $0 < c_1 < \ldots < c_N$, and $x_1, \ldots, x_N \in \mathbb{R}$. There exist $T_0 \in \mathbb{R}$ and a unique function $u \in C([T_0, +\infty), H^1)$ solution to (3.2), and such that

$$\left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j - c_j t) \right\|_{H^1} \to 0 \quad \text{as} \quad t \to \infty.$$

Furthermore, $u \in C^\infty([T_0, \infty) \times \mathbb{R})$ and convergence takes place in H^s for all $s \geq 0$, with an exponential decay :

$$\forall s \geq 0, \exists A_s \ \ / \quad \left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j - c_j t) \right\|_{H^1} \leq A_s e^{-\gamma t},$$

where $\gamma = \sigma_0 \sqrt{\sigma_0}/32$ and $\sigma_0 = \min(c_1, c_2 - c_1, \ldots, c_N - c_{N-1})$.

This result appears as a development of monotonicity properties and a dynamical argument, ideas which where used by Martel and Merle [25] and Martel, Merle and Tsai [29]. It is a surprise that the argument applies also in the critical case $p = 5$, although it fails in the proof of stability (failure which isn't due to a lack in the proof, but to true instability : see [26], [28]). The second surprise is the uniqueness of a solution behaving as a sum of N solitons.

The last result solves the case of a linear behavior, that is the existence of a wave operator (see [8]).

Large data wave operator. Let $V \in L^2$. There exist a time $T_0 \in \mathbb{R}$ and a function $u \in C([T_0, \infty), L^2)$ solution to (3.1) such that

$$\|u(t) - U(t)V\|_{L^2} \to 0 \quad as \quad t \to \infty.$$

Furthermore u is unique in an adapted class.

In the same way that the result of Martel [23] was based on considerations of Martel, Merle and Tsai [29], this result relies on the analysis of Kenig, Ponce and Vega [14].

3.1.3 Statement of the main result

Our goal in this article is to construct solutions which behave like a sum of a linear term $U(t)V$, and of N solitons, for the L^2-critical Korteweg-de Vries equation (3.1). Our main result is the following.

Theorem 3.1 (Nonlinear wave operator for (3.1))**.** *Let $V \in H^1$ have sufficient decay on the right, i.e. such that $(1 + x_+)^{2+\delta_0} V(x) \in L^2$ for some $\delta_0 > 0$ (we denote $x_+ = \max\{0, x\}$).*

Let $N \in \mathbb{N}$, $0 < c_1 < \ldots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. Let $R_j(t, x) = Q_{c_j}(x - x_j - c_j t)$, for $j = 1, \ldots, N$, be N solitons.

Then there exists $u^* \in C([T_0, +\infty), H^1)$, for some $T_0 \in \mathbb{R}$, solution to (3.1) and such that $u^*(t)$ is uniformly bounded in H^1 and

$$\left\| u^*(t) - U(t)V - \sum_{j=1}^{N} R_j(t) \right\|_{H^1} \to 0 \quad as \quad t \to \infty. \tag{3.6}$$

Theorem 3.1 allows to work with large data (V large in L^2), which is both surprising and satisfactory. The decay on the right we assume for V is to ensure low interaction with the solitons. This result should be viewed as a step in the solving process of Problem 2.

***Remark* 3.1.** *This result essentially units the linear approach contained in [15] and [8], and the solitons related approach, developed in [30] and [23]. The difficulty is to mix both methods together, so that they do not break down.*

An important change in the method of proof when considering [23] is the following. Solitons have an exponential decay, and so integrability (in time) is always automatic. Here the linear term $U(t)V$ will interact with the solitons to produce a polynomial decay in time, which will require to be taken care of.

***Remark* 3.2.** *This result is analogous to that obtained in [6], where a non-linear wave operator is constructed in the sub-critical case $p = 4$. However, in the sub-critical case,* *much more decay and smoothness are required on* V*. This is due to the fact that the linear scattering analysis of [15] is no longer available if* $p \neq 5$.

In the sub-critical case,we have to relie on the scattering theory of Hayashi and Naumkin [13]. There it is proved scattering for small data $u_0 \in H^{1,1} = \{u \in H^1 | xu \in H^1\}$ *: for* $p > 3$*, given such a* u_0 *the solution* $u(t)$ *to (3.2) is global, satisfies the linear decay rate* $\|u(t)\|_{L^\infty} \leq Ct^{-1/3}$*, and there exists* $V \in L^2$ *such that* $\|u(t) - U(t)V\|_{L^2} \to 0$ *as* $t \to \infty$*. Their method is a very beautiful clock-work, but breaks down at some point when constructing the non-linear wave operator. To recover from this, the setting must be strengthen, and hence, the conditions on* V *must be reinforced.*

Here, the methods of [23] and [15] can be smoothly adapted to take care of the interaction between non-linear terms (the solitons) and the linear term ($U(t)V$)*, to provide an almost sharp result.*

Indeed, our smoothness assumption H^1 *is a natural setting to work with the solitons, and especially to have bounded energy. On the other side, notice that the decay assumption only concerns the* L^2 *level for* V*, and only decay on the right. The assumption on* $U(t)V$ *should be understood in this way : to handle the interaction of the solitons, we need one degree of decay on* V *so that its interference is low enough. To prevent the solitons from interfering too much when handling the linear term* $U(t)V$*, we need a second order of decay on* V.

An optimal result for our framework would then be $(1 + x_+^2)V(x) \in L^2(dx)$*. In view of this, our assumption appears to be almost optimal.*

***Remark* 3.3.** *The uniqueness of solutions to (3.1) with a given asymptotic behaviour of the form* $U(t)V + R(t)$ *is not clear. Recall that for* $V = 0$*, that is, the* N*-soliton, one has uniqueness in* H^1 *(see [23]) : it is linked to the fact the constructed solution is smooth and converges exponentially fast in* H^s *for all* $s \geq 0$ *(*$s = 4$ *would be enough). If* V *belong to* H^1 *but not more, this is not possible. However, one might be able to prove uniqueness for smoother* V.

***Remark* 3.4.** *There are some related results for the (critical) non-linear Schrödinger equation (NLS) : due to the pseudo-conformal transform, the construction of a non-linear wave operator is equivalent to the construction of solutions which blow up with a given behavior. For (NLS), the results are expressed in the latter form, see Bourgain and Wang [3], Krieger and Schlag [21], Merle [30]. In this case also do conditions on the linear term regarding smoothness and low interaction with the solitons appear.*

3.2 Strategy of the proof.

Following a usual convention, different positive constants might be denoted by the same letter C.

Let V as in the hypothesis of Theorem 3.1, $0 < c_1 < \ldots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. Denote the soliton with speed c_j and shift x_j

$$R_j(t,x) = Q_{c_j}(x - x_j - c_j t).$$

Define also $R(t) = \sum_{j=1}^N R_j(t)$.

Let S_n be an increasing sequence of time, so that $S_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} E(U(S_n) + R(S_n)) = \liminf_{t\to\infty} E(U(S_n) + R(S_n)). \tag{3.7}$$

(such a sequence obviously exists ; the condition on the energy appears when concluding the proof of Theorem 3.1). For $n > 0$, we define $u_n(t)$, the solution of

$$\begin{cases} u_{nt} + (u_{nxx} + u^5)_x = 0, \\ u_n(S_n) = U(S_n) + R(S_n). \end{cases} \tag{3.8}$$

Equivalently, we introduce $w_n(t)$ the error term

$$w_n(t) = u_n(t) - U(t)V - R(t),$$

so that $w_n(t)$ satisfies the equation

$$\begin{cases} w_{nt} + w_{nxxx} + \left(u^5 - \sum_{j=1}^N R_j^5(t)\right)_x = 0, \\ w_n(S_n) = 0. \end{cases} \tag{3.9}$$

As $u(S_n) \in H^1$, $u_n(t)$ is well defined, at least on a small interval of time around S_n.

The heart of the proof of Theorem 3.1 is the following result :

Proposition 3.1 (Uniform estimates)**.** *There exist T_0, K_0 and a continuous function η : $[1,\infty) \to \mathbb{R}_*^+$, depending on V, with*

$$\eta(t) \downarrow 0 \quad as \quad t \to \infty,$$

such that the following is true. For all n such that $S_n \geq T_0$, the solution $u_n(t)$ to (3.8) and the solution $w_n(t)$ to (3.9) belong to $C([T_0, S_n], H^1)$. Furthermore, we have the uniform decay estimate and control (in n) :

$$\forall t \in [T_0, S_n], \quad \|w_n(t)\|_{L^2} \leq \eta(t), \quad and \quad \|w_n(t)\|_{H^1} \leq K_0. \tag{3.10}$$

The proof of this proposition requires several steps.

The first remark allows us to further assume smallness on $w_n(t)$, in order to get the decay (3.10).

Proposition 1' (Reduction of proof)**.** *There exist $\varepsilon_0 > 0$, $T_0 \geq 1$ and a decreasing continuous function $\eta : [1,\infty) \to \mathbb{R}_*^+$, depending on V, with*

$$\eta(t) \downarrow 0 \quad as \quad t \to \infty,$$

such that the following is true. Introduce the norm

$$\|f(t,x)\|_{\mathcal{N}([A,B])} = \|f\|_{L_x^5 L_t^{10}(t\in[A,B])} + \sup_{t\in[A,B]} \|f(t)\|_{L_x^2}.$$

Let $n \in \mathbb{N}$ so that $S_n \geq T_0$. Let $I_n \in [T_0, S_n]$ such that $\|w_n\|_{\mathcal{N}([I_n,S_n])} \leq \varepsilon_0$. Then in fact,

$$\forall t \in [I_n, S_n], \quad \|w_n\|_{\mathcal{N}([t,S_n])} \leq \eta(t), \quad and \quad \|w_n(t)\|_{H^1} \leq K_0.$$

We introduce the $L_x^5 L_t^{10}$ space as it is necessary in the control of the linear term $U(t)V$: see [15] for further details.

Proof of Proposition 3.1 assuming Proposition 1'. This is a continuity argument. Let

$$T_0 = \inf\{\tau : \tau \geq 1 \text{ and } \eta(\tau) \leq \varepsilon_0\},$$

and define

$$I_n^* = \inf\Big\{\tau : \tau \in [1, S_n], \text{ and } \|w_n\|_{\mathcal{N}([\tau,S_n])} \leq \varepsilon_0\Big\}.$$

We now use the continuity the norm $L_x^5 L_t^{10} \cap C^0 H^1$ under the flow of (3.1), (see [15]). As $w_n(S_n) = 0$, we obtain that the set on which we do the infimum is non-empty, so that $I_n^* < S_n$.

Then of course, this allows us to apply Proposition 1' with $I_n = I_n^*$ so that

$$\forall t \in [I_n^*, S_n], \quad \|w_n\|_{\mathcal{N}([t,S_n])} \leq \eta(t), \quad \text{and} \quad \|w_n(t)\|_{H^1} \leq K_0. \tag{3.11}$$

By minimality of I_n^*, if $I_n^* > 1$, we also get that

$$\limsup_{t \downarrow I_n^*} \|w_n\|_{\mathcal{N}([t,S_n])} \geq \varepsilon_0.$$

In particular, this gives

$$\varepsilon_0 \leq \limsup_{t \downarrow I_n^*} \|w_n\|_{\mathcal{N}([t,S_n])} \leq \limsup_{t \downarrow I_n^*} \eta(t) \leq \eta(I_n^*).$$

So that $\eta(I_n^*) \geq \varepsilon_0$.

In any case, we get that $I_n^* \leq T_0$ (as η is decreasing) : (3.11) allows us to conclude. □

Proof of Proposition 1'.
Step 1 : Monotonicity and non-linear tools. We obtain L^2 estimates on the right. Let us introduce the cut-off speed

$$\sigma_0 \in (0, 1/2 \min\{c_1, c_2 - c_1, \dots, c_N - c_{N-1}\}), \tag{3.12}$$

to be determined in the proof of the following Proposition 3.2 below, and the cut-off function

$$\psi(x) = \frac{2}{\pi} \arctan\left(\exp\left(-\frac{\sqrt{\sigma_0}}{2} x\right)\right), \qquad \psi_0(t, x) = \psi(x - \sigma_0 t - 2|x_1|). \tag{3.13}$$

$\psi_0(t)$ allows us to separate the solitons interaction from the $U(t)V$ interaction.

Proposition 3.2 (Interaction with the solitons)**.** *There exist $\sigma_1 > 0$, ε_1, T_1, C_1 and K_0 such that the following is true. If $\sigma_0 \leq \sigma_1$, $\varepsilon_0 \leq \varepsilon_1$ and $T_0 \geq T_1$, then, for all $n \in \mathbb{N}$ and all $t \in [I_n, S_n]$,*

$$\begin{aligned} \|w(t)\|_{L^2(1-\psi_0(t))} &\leq C_1 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{8} t} + C_1 \|U(t)V\|_{L^2(1-\psi_0(t))} \\ &\quad + C_1(S_n - t + 1)\|U(t)V\|_{L^2(1-\psi_0(S_n))} + C_1 \int_t^{S_n} \|U(t)V\|_{L^2(1-\psi_0(t))} dt, \end{aligned}$$

and

$$\|w_n(t)\|_{H^1} \leq K_0.$$

The control of the H^1-norm simply relies on uniform bounds of the energy, and on the smallness assumption on $\|w(t)\|_{L^2}$. The deep result is the first estimate.

Essentially we obtain a polynomial decay on $\|w_n(t)\|_{L^2(1-\psi_0(t))}$ (instead of an exponential decay in the case of solely soliton). However the good point is that we can choose this polynomial decay to be as fast as we want by lowering the interaction of $U(t)V$ with the solitons, that is, by requiring sufficient decay on the right for V : see Lemma 3.2.

Step 2 : Linear theory. Essentially we have to take care of the interaction of $U(t)V$ and w_n. For this, we use the linear estimates and the setting of [14] and [15].

Proposition 3.3 (Interaction with the linear term). *There exists $\varepsilon_2 > 0$, T_2, C_2 such that the following is true. Suppose that for some C and $\delta_0 > 0$, we have for all n such that $S_n \geq T_2$:*

$$\forall t \in [I_n, S_n], \quad \|w_n(t) + U(t)V\|_{L^2(1-\psi_0(t)} \leq \frac{C}{t^{1+\delta_0}}.$$

Then there exists C_2 such that if we denote :

$$\eta(t) = C_2\|U(\tau)V\|^5_{L^5_x L^{10}_\tau(\tau\geq t)} + C_2 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|U(\tau)V\|_{L^5_x L^{10}_\tau(\tau\geq t)} + \frac{C_2}{t^{\delta_0}},$$

we have :

$$\|w_n(t)\|_{\mathcal{N}([I_n,S_n])} \leq \eta(t).$$

Of course, $\eta(t)$ decreases to 0 as $t \to \infty$, and so satisfies the conditions of Proposition 1'.

Finally, Proposition 3.2, Lemma 3.2 and estimates (3.42) and (3.43) ensure that the assumptions of Proposition 3.3 are fulfilled if V is chosen as in Theorem 3.1, that is $V \in H^1$ and $x_+^{2+\delta_0}V \in L^2$. Fix $\sigma_0 < \sigma_1$, $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$ and $T_0 = \min\{T_1, T_2\}$: this completes the proof of Proposition 1', and so, of Proposition 3.1 .

Proof of Theorem 3.1. From Proposition 3.1, we are able to prove some compactness property in L^2 on the sequence $u_n(T_0)$. The limit of a subsequence yields an initial data φ_0, from which $u^*(t)$ is the arising solution to (3.1). Then Proposition 3.1 allows to conclude that

$$\|u^*(t) - U(t)V - R(t)\|_{L^2} \to 0.$$

To obtain the H^1 convergence, we need another argument. We compare $E(U(S_n) + R(S_n))$ and $E(u^*(t))$, taking advantage of (3.7). By developping

$$E(u^*(t)) = E(w^*(t) + U(t)V + R(t)),$$

and studying carefully all the obtained terms, we prove that the error term $\|w_x^*(t)\|_{L^2} \to 0$ as $t \to \infty$: this completes the proof of Theorem 3.1.

The proof of Theorem 3.1 assuming Proposition 3.1 is done in Section 3. The rest of the proof completes the proof of Proposition 1' and thus that of Proposition 3.1. In Section 4., we give some preliminary estimates to be used both in Section 5. and Section 6. Section 5. is devoted the proof of Proposition 3.2. Finally, Proposition 3.3 is proved in Section 6.

3.3 Proof of Theorem 3.1 assuming Proposition 3.1

In this section, we assume Proposition 3.1 holds, and from this we conclude the proof of Theorem 3.1.

3.3.1 A compactness result linked with the monotonicity Lemma 3.5

From Proposition 3.1, we dispose of a sequence $u_n(t)$ defined on $[T_0, S_n]$ (we dropped the terms with $S_n < T_0$), solutions to (3.2), such that

$$u_n(S_n) = U(S_n)V + \sum_{j=1}^{N} R_j(S_n) = U(S_n) + R(S_n),$$

and that the following uniform estimates holds ($w_n(t) = u_n(t) - U(t)V - R(t)$) :

$$\forall n \in \mathbb{N},\ \forall t \in [T_0, S_n], \quad \|w_n(t)\|_{L^2} \leq \eta(t) \quad \text{and} \quad \|w(t)\|_{H^1} \leq K_0.$$

Claim. $u_n(T_0)$ is a compact sequence in the sense that

$$\lim_{A\to\infty} \sup_{n\in\mathbb{N}} \int_{|x|\geq A} u_n^2(T_0, x)dx = 0.$$

Proof of the Claim. Indeed, let $\varepsilon > 0$, and $T(\varepsilon)$ such that $\eta(T(\varepsilon)) \leq \sqrt{\varepsilon}$. Then

$$\int (u_n(T(\varepsilon)) - U(T(\varepsilon))V - R(T(\varepsilon))^2 \leq \varepsilon.$$

Let $A(\varepsilon)$ be such that $\int_{|x|\geq A(\varepsilon)} (U(T(\varepsilon))V + R(T(\varepsilon)))^2(x)dx \leq \varepsilon$; we get

$$\int_{|x|\geq A(\varepsilon)} u_n^2(T(\varepsilon), x)dx \leq 2\varepsilon.$$

Let $g \in C^3$ a function such that $g(x) = 0$ if $x \leq 0$, $g(x) = 1$ if $x \geq 2$, and furthermore $0 \leq g'(x) \leq 1$, $0 \leq g'''(x) \leq 1$.

Recall that if $f \in C^3$ does only depend on x, we have

$$\frac{d}{dt}\int u_n^2 f = -3\int u_{nx}^2 f_x + \int u_n^2 f_{xxx} + \frac{2p}{p+1}\int u_n^{p+1} f_x.$$

(See Lemma 3.5 and its proof). For $C(\varepsilon)$ to be determined later, we then have

$$\frac{d}{dt}\int u_n^2(t,x) g\left(\frac{x - A(\varepsilon)}{C(\varepsilon)}\right) = -\frac{3}{C(\varepsilon)}\int u_{nx}^2 g'\left(\frac{x - A(\varepsilon)}{C(\varepsilon)}\right) + \frac{1}{C(\varepsilon)^3}\int u_n^2 g'''\left(\frac{x - A(\varepsilon)}{C(\varepsilon)}\right) + \frac{2p}{(p+1)C(\varepsilon)}\int u_n^{p+1} g'\left(\frac{x - A(\varepsilon)}{C(\varepsilon)}\right).$$

As $t \geq T_0$, u_n satisfies $\|u_n(t)\|_{H^1} \leq K_0 + \|V\|_{H^1} + \sum_{j=1}^{N} \|Q_{c_j}\|_{H^1} \leq C^0$. So that :

$$\left|\frac{d}{dt}\int u_n^2(t,x) g\left(\frac{x - A(\varepsilon)}{C(\varepsilon)}\right)\right| \leq \frac{1}{C(\varepsilon)}\left(3\int u_{nx}^2(t) + \int u_n^2(t) + \frac{2p}{p+1}\|u_n\|_{L^\infty}^{p-1}\int u_n^2(t)\right) \leq \frac{1}{C(\varepsilon)}\left(3{C^0}^2 + \frac{2p}{p+1}2^{(p-1)/2}{C^0}^{p+1}\right).$$

Now choose $C(\varepsilon) = \max\left\{1, \frac{T(\varepsilon)-T_0}{\varepsilon}\left(3{C^0}^2 + \frac{2p}{p+1}2^{(p-1)/2}{C^0}^{p+1}\right)\right\}$, from which we derive

$$\left|\frac{d}{dt}\int u_n^2(t,x)g\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right)\right| \le \frac{\varepsilon}{T(\varepsilon)-T_0}.$$

And after integration in time between T_0 and $T(\varepsilon)$,

$$\int_{x\ge 2C(\varepsilon)+A(\varepsilon)} u_n^2(T_0,x) \le \int u_n^2(T_0,x)g\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right) \le 3\varepsilon.$$

Now considering $\frac{d}{dt}\int u_n^2(t,x)g\left(\frac{-A(\varepsilon)-x}{C(\varepsilon)}\right)$, we get in a similar way

$$\int_{x\le -2C(\varepsilon)-A(\varepsilon)} u_n^2(T_0,x) \le 3\varepsilon.$$

So that if we denote $A_\varepsilon = 2C(\varepsilon/6) + A(\varepsilon/6)$, we obtain

$$\forall n\in\mathbb{N}, \qquad \int_{|x|\ge A_\varepsilon} u_n^2(T_0,x) \le \varepsilon,$$

as claimed. $\square$

3.3.2 Construction of u^* and L^2 convergence to the profile

Now $u_n(T_0)$ is a bounded sequence in $H^1(\mathbb{R})$, and so converges weakly up to a subsequence, to some φ_0 in $H^1(\mathbb{R})$ (we suppose for convenience that the whole sequence converges weakly). The previous compactness result ensures that the convergence is strong in $L^2(\mathbb{R})$. Indeed, let $\varepsilon > 0$, and A such that $\int_{|x|\ge A}\varphi_0^2(x)dx \le \varepsilon$ and

$$\forall n\in\mathbb{N}, \qquad \int_{|x|\ge A} u_n^2(T_0,x) \le \varepsilon.$$

As the embedding $H^1([-A,A]) \hookrightarrow L^2([-A,A])$ is compact, as $n\to\infty$, $\int_{|x|\le A}|u_n(T_0,x) - \varphi_0(x)|^2dx \to 0$. We thus derive that

$$\limsup_{n\to\infty} \|u_n(T_0) - \varphi_0\|^2_{L^2(\mathbb{R})} \le 4\varepsilon.$$

As this is true for all $\varepsilon > 0$, $u_n(T_0) \to \varphi_0$ in $L^2(\mathbb{R})$.

Denote $u^*(t)$ the solution to

$$\begin{cases} u^*_t + (u^*_{xx} + u^{*p})_x = 0, \\ u^*(T_0) = \varphi_0. \end{cases}$$

The Cauchy problem being well-posed in $L^2(\mathbb{R})$, u^* is well defined, at least for t in a neighborhood $\mathcal{V}$ of T_0. Now the flow is continuous in L^2 (in fact it is Lipschitz), so that for all $t\in\mathcal{V}$, $u_n(t) \to u^*(t)$ in L^2. As $(u_n(t))_n$ is a bounded sequence in H^1, this proves that the whole sequence converges weakly to $u^*(t)$ in H^1 :

$$\forall t\in\mathcal{V}, \quad \lim_{n\to\infty} u_n(t) = u^*(t) \quad \text{in } L^2(\mathbb{R}) - \text{strong and } H^1(\mathbb{R}) - \text{weak.} \tag{3.14}$$

Thus, we can take the limit in the estimates (3.10) (with t fixed), to get

$$\forall t \in \mathcal{V}, \quad \|u^*(t) - U(t)V - R(t)\|_{L^2} \leq \eta(t), \quad \text{and} \quad \|u^*(t) - U(t)V - R(t)\|_{H^1} \leq K_0.$$

This shows that $u^*(t)$ is H^1 uniformly bounded on $\mathcal{V}$, so that by the Cauchy problem theory and a standard continuity argument, u^* is defined for all $t \geq T_0$. Hence, $w^*(t)$ is uniformly bounded in H^1, and satisfies the expected L^2 decay estimate :

$$\|w^*(t)\|_{H^1} \leq K_0 \quad \text{and} \quad \|w^*(t)\|_{L^2} \leq \eta(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{3.15}$$

3.3.3 H^1 convergence of $u^*(t)$ to its profile

The H^1 convergence comes essentially from an analysis of the energy $E(u^*(t))$.

From (3.14), $u_n(t) \rightharpoonup u^*(t)$ H^1-weak and $u_n(t) \to u^*(t)$ in L^6 as $n \to \infty$, and we deduce that

$$\begin{aligned} E(u^*(T_0)) &\leq \liminf_{n\to\infty} E(u_n(T_0)) \leq \liminf_{n\to\infty} E(u_n(S_n)) \\ &\leq \liminf_{n\to\infty} E(U(S_n)V + R(S_n)). \end{aligned}$$

Now, conservation of energy gives for $E(u^*(t)) = E(u^*(T_0))$, for $t \geq T_0$. By (3.7), and in view of the previous computation, we have

$$\begin{aligned} &\liminf_{t\to\infty} \big(E(U(t)V + R(t)) - E(u^*(t))\big) \\ &\qquad = \liminf_{t\to\infty} E(U(t)V + R(t)) - E(u^*(T_0)) \\ &\qquad = \lim_{n\to\infty} E(U(S_n)V + R(S_n)) - E(u^*(T_0)) \\ &\qquad \geq 0. \end{aligned} \tag{3.16}$$

Thus, let us estimate $E(U(t)V + R(t)) - E(u^*(t))$:

$$\begin{aligned} &E(U(t)V + R(t)) - E(u^*(t)) \\ &= E(U(t)V + R(t)) - E(w^*(t) + U(t)V + R(t)) \\ &= E(U(t)V + R(t)) - E(w^*(t)) - E(U(t)V + R(t)) - \int w_x^*(t)U(t)V_x \\ &\quad - \int w_x^*(t)R_x(t)) + \frac{1}{6}\sum_{k=1}^{5} C_6^k \int w^*(t)^k (U(t)V + R(t))^{6-k} \\ &= -\frac{1}{2}\int |w_x^*(t)|^2 - \int w_x^*(t)U(t)V_x - \int w_x^*(t)R_x(t) \\ &\quad + \frac{1}{6}\sum_{k=1}^{6} C_6^k \int w^*(t)^k (U(t)V + R(t))^{6-k}. \end{aligned} \tag{3.17}$$

Recall (3.15) : by interpolation L^2-H^1, we get that for all $p \geq 2$, $\|w^*(t)\|_{L^p} \to 0$ as $t \to \infty$.

Let us first control the second line in (3.17) : for $k = 2, \ldots, 6$,

$$\left| \int w^*(t)^k (U(t)V + R(t))^{6-k} \right| \leq \|w^*(t)\|_{L^k}^k \|U(t)V + R(t)\|_{L^\infty}^{6-k} = o_{t\to\infty}(1).$$

For $k=1$, we have also

$$\left|\int w^*(t)(U(t)V+R(t))^5\right| \le \|w^*(t)\|_{L^2}\|U(t)V+R(t)\|_{L^2}\|U(t)V+R(t)\|_{L^\infty}^4 = o_{t\to\infty}(1).$$

Now,

$$\left|\int w_x^*(t)R_x(t)\right| = \left|\int w^*(t)R_{xx}(t)\right| \le \|w^*(t)\|_{L^2}\|R(t)\|_{H^2} = o_{t\to\infty}(1).$$

The last term $\int w_x^*(t)U(t)V_x$ requires a little more attention. Consider the function $t\mapsto U(-t)(w^*(t))$: then $\|U(-t)(w^*(t))\|_{L^2} = \|w^*(t)\|_{L^2} \to 0$ as $t\to\infty$ and furthermore $\|U(-t)(w^*(t))\|_{H^1} = \|w^*(t)\|_{H^1}$ is uniformly bounded in t. Hence, the only possible weak limit of $U(-t)(w^*(t))$ in H^1 (as $t\to\infty$) is 0. This proves that :

$$\begin{cases} U(-t)(w^*(t)) \to 0 & \text{in } L^2-\text{strong} \quad \text{as} \quad t\to\infty, \\ U(-t)(w_x^*(t)) \rightharpoonup 0 & \text{in } L^2-\text{weak} \quad \text{as} \quad t\to\infty. \end{cases}$$

Hence,

$$\int w_x^*(t)U(t)V_x = \int U(-t)(w_x^*(t))V_x = o_{t\to\infty}(1).$$

We can conclude from (3.17) that

$$E(U(t)V+R(t)) - E(u^*(t)) = -\frac{1}{2}\int |w_x^*(t)|^2 + o_{t\to\infty}(1),$$

and in view of (3.16),

$$\begin{aligned} 0 &\le \liminf_{t\to\infty}\big(E(U(t)V+R(t)) - E(u^*(t))\big) \\ &\le \liminf_{t\to\infty}\left(-\frac{1}{2}\int |w_x^*(t)|^2 + o_{t\to\infty}(1)\right) \\ &\le -\frac{1}{2}\limsup_{t\to\infty}\int |w_x^*(t)|^2. \end{aligned}$$

This proves that $\|w_x^*(t)\|_{L^2}\to 0$, and along with (3.15), we get that

$$\|u^*(t) - U(t)V - R(t)\|_{H^1} \to 0 \quad \text{as} \quad t\to\infty.$$

This concludes the proof of Theorem 3.1.

The following is devoted the proof of Proposition 3.1, or more precisely of Proposition 1'. We will now only work on the interval $[I_n, S_n]$.

3.4 Preliminaries

3.4.1 Cut-off functions and localized quantities

We already introduced $\sigma_0 \in (0, 1/2\min\{c_1, c_2-c_1, \dots, c_N-c_{N-1}\})$, and the cut-off function :

$$\psi(x) = \frac{2}{\pi}\arctan\left(e^{-\frac{\sqrt{\sigma_0}}{2}x}\right). \tag{3.13}$$

We can check that $\lim_{+\infty}\psi = 0$, $\lim_{-\infty}\psi = 1$, and ψ is decreasing. Furthermore, by direct computations,

$$\psi'(x) = -\frac{\sqrt{\sigma_0}}{2\pi\cosh\left(\frac{\sqrt{\sigma_0}}{2}x\right)}, \quad \psi''' = \frac{\sigma_0}{4}\psi'(x)\left(1 - \frac{2}{\cosh\left(\frac{\sqrt{\sigma_0}}{2}x\right)}\right),$$

and so,

$$|\psi'''(x)| \leq -\frac{\sigma_0}{4}\psi'(x). \tag{3.18}$$

We introduce, for $j = 1, \dots, N-1$,

$$m_j(t) = \frac{c_j + c_{j+1}}{2}t + \frac{x_j + x_{j+1}}{2}, \quad m_0(t) = \sigma_0 t - 2|x_1|, \quad m_{-1}(t) = \frac{\sigma_0}{2}t - 2|x_1|.$$

So that we can define, for $j = -1, \dots, N-1$:

$$\psi_j(t,x) = \psi(x - m_j(t)), \quad \psi_N(t,x) = 1.$$

Then we set, for $j = 1, \dots, N-1$,

$$\phi_0(t) = \psi_0(t), \quad \phi_j(t) = \psi_j(t) - \psi_{j-1}(t), \quad \phi_N(t) = 1 - \psi_{N-1}(t).$$

By construction, $\sum_{k=1}^{j}\phi_k = \psi_j$. Finally, we define some local quantities related to mass and energy, for $j = 0, \dots, N$,

$$M_j(t) = \int u_t^2(t)\phi_j(t), \quad E_j(t) = \int \left(\frac{1}{2}u_x^2(t) - \frac{1}{p+1}u^{p+1}(t)\right)\phi_j(t),$$
$$F_j(t) = E_j(t) + \frac{1}{100}M_j(t).$$

For $j \geq 1$, the ϕ_j separates the solitons R_j from one another. $\psi_0(t)$ separates the solitons from the linear term $U(t)V$. The aim of $\psi_{-1}(t)$ is different : it provides an interval on which $U(t)V$ is small in H^1 an so in L^∞ (see Lemma 3.2 hereafter). This will be crucially used in the almost monoticity Lemma 3.5 (it is in fact the only place where $\psi_{-1}(t)$ plays a role).

Observe that $\|U(t)V\|_{L^\infty} \leq C\|V\|_{L^1}t^{-1/3}$, so that pointwise smallness on $U(t)V$ is automatic if $V \in L^1$. However, this hypothesis is not part of Theorem 3.1.

3.4.2 Preliminary bounds on $w_n(t)$

Notice that from the uniform bound on the energy, we get a uniform control on $w_n(t)$ for $t \in [I_n, S_n]$. This is the purpose of the following lemma. This preliminiary result will be very important in the proof of the almost monotonicity Lemma 3.5.

Lemma 3.1 (Bound on the H^1 norm of $w_n(t)$)**.** *There exists a constant K_0 independent of $\varepsilon_0 \in]0, \kappa(6)^{-1/4}]$ ($\kappa(p)$ denotes the sharp constant in the Gagliardo-Nirenberg inequality (3.5)), such that*

$$\forall n \in \mathbb{N},\ \forall t \in [I_n, S_n], \quad \|w_n(t)\|_{H^1} \leq K_0. \tag{3.19}$$

In particular, the $\|w(t)\|_{L^\infty}$ can be made arbitrarily small :

$$\forall n \in \mathbb{N}, \forall t \in [I_n, S_n], \quad \|w(t)\|_{L^\infty} \leq \sqrt{\varepsilon_0 K_0}. \tag{3.20}$$

Remark that this lemma gives the second estimate of Proposition 1'.

Proof. We combine smallness of $w_n(t)$ in L^2 along with uniform bounds (in n) on $E(u_n)$. The energy is preserved so that $E(u_n(S_n)) = E(u_n(t))$. Then we have

$$\begin{aligned} E(u_n(S_n)) &= E(U(S_n)V + R(S_n)) \\ &\leq C\int |U(S_n)V_x|^2 + C\sum_{j=1}^N \int {Q_{c_j}}_x^2 + C\int |U(S_n)V|^6 + C\sum_{j=1}^N \int Q_{c_j}^6 \leq C. \end{aligned}$$

So that the energy $E(u_n(t))$ is uniformly bounded (in n). Now we have the following.

Claim. Let $f, \varepsilon \in H^1$, with $\|\varepsilon\|_{L^2} \leq \kappa(6)^{-1/4}$. Then there is a function $G : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\varepsilon\|_{H^1} \leq 3E(f+\varepsilon) + G(\|f\|_{H^1}).$$

To conclude, it suffice to apply the claim for $\varepsilon = w_n(t)$ and $f = U(t)V + R(t)$ (whose H^1-norm is uniformly bounded in t).

Let us prove the claim. Indeed, we compute :

$$\begin{aligned} E(f+\varepsilon) &= \frac{1}{2}\int (f+\varepsilon)_x^2 - \frac{1}{6}\int (f+\varepsilon)^6 \\ &= \frac{1}{2}\int f_x^2 + \int f_x\varepsilon_x + \frac{1}{2}\int \varepsilon_x^2 - \frac{1}{6}\sum_{k=0}^{5} C_6^k \int \varepsilon^k f^{6-k} - \frac{1}{6}\int \varepsilon^6. \end{aligned}$$

Now, we have $\int f_x^2 \leq \|f\|_{H^1}^2$, $|\int f_x\varepsilon_x| \leq \|f\|_{H^1}\|\varepsilon_x\|_{L^2}$, $\int f^6 \leq \|f\|_{H^1}^6$, and $|\int \varepsilon f^5| \leq \|f\|_{H^1}^5\|\varepsilon\|_{L^2}$. For $k = 2, \dots, 5$, we have the Gagliardo-Nirenberg inequality (whose sharp constant is $\kappa(k)$) :

$$\int \varepsilon^k f^{6-k} \leq \kappa(k)\|f\|_{L^\infty}^{6-k}\|\varepsilon\|_{L^2}^{k/2+1}\|\varepsilon_x\|_{L^2}^{k/2-1}.$$

For $k = 6$, the Gagliardo-Nirenberg inequality also applies, but gives an exponent 2 for $\|\varepsilon\|_{L^2}$:

$$\frac{1}{6}\int \varepsilon^6 \leq \frac{\kappa(k)}{6}\|\varepsilon\|_{L^2}^4\|\varepsilon_x\|_{L^2}^2 \leq \frac{1}{6}\int \varepsilon_x^2.$$

So that we get from the energy equality :

$$\begin{aligned} \frac{1}{2}\int \varepsilon_x^2 \leq E(f+\varepsilon) &+ \|f\|_{H^1}^2 + \|f\|_{H^1}\|\varepsilon_x\|_{L^2} + \|f\|_{H^1}^6 + \|f\|_{H^1}^5\|\varepsilon\|_{L^2} \\ &+ \sum_{k=2}^{5} \|f\|_{L^\infty}^{6-k}\|\varepsilon\|_{L^2}^{k/2+1}\|\varepsilon_x\|_{L^2}^{k/2-1} + \frac{1}{6}\int \varepsilon_x^2. \end{aligned}$$

This can be rewritten as ($\|\varepsilon\|_{L^2} \leq 1$)

$$\frac{1}{3}\|\varepsilon_x\|_{L^2}^2 \leq E(f+\varepsilon) + 2^5(\|f\|_{H^1} + \|f\|_{L^\infty}^6)(1 + \|\varepsilon_x\|_{L^2}^{3/2}).$$

If $a^2 \leq K_1 + K_2 a^{3/2}$, then obviously $a^2 \leq K_1 + K_2^4$, so that we get

$$\|\varepsilon_x\|_{L^2}^2 \leq 3E(f+\varepsilon) + 3\cdot 2^6(\|f\|_{H^1} + \|f\|_{L^\infty}^{24}).$$

□

3.4.3 Estimates of $U(t)V$ on the right

We now obtain bounds for $U(t)V$ on the right, which is will be crucial for the monotonicity Lemma 3.5, and also in Section 5 (analysis of the interaction of the linear term $U(t)V$).

Lemma 3.2 ($U(t)V$ estimates on the right)**.** *Let $f \in L^2$, then*

$$\|U(t)f\|_{L^2(1-\psi_{-1}(t))} \leq \|f\|_{L^2(1-\psi_{-1}(t/2))} \to 0 \quad \text{as} \quad t \to \infty. \tag{3.21}$$

In particular, if $f \in H^1$, then

$$\sup_{x \geq m_{-1}(t)} |U(t)f(x)|^2 \leq 4\|f\|_{L^2}\|f_x\|_{L^2(1-\psi_{-1}(t/2))} \to 0 \quad \text{as} \quad t \to \infty. \tag{3.22}$$

Suppose that $(1+x_+^q)f(x) \in L^2(dx)$, for some $q > 0$. Then there exists a constant $C = C(\sigma_0, x_1)$ independent of f such that

$$\forall t \geq 1, \quad \|U(t)f\|_{L^2(1-\psi_0(t))} \leq \frac{C}{t^q}\|(1+x_+^q)f(x)\|_{L^2(dx)}. \tag{3.23}$$

If $(1+x_+^{1/2})f(x) \in L^2(dx)$, we have furthermore that

$$\int_{t\geq 0} \|U(t)f\|^2_{L^2(1-\psi_0(t))}dt \leq C\|(1+x_+^{1/2})f(x)\|^2_{L^2(dx)} < \infty. \tag{3.24}$$

We will apply this result to V and V_x.

Proof. The key remark is that $U(t)$ "pushes" the L^2-mass on the left. Let $\varphi = \psi_{-1}$ or $\varphi = \psi_0$. Let us consider, for t fixed, the function

$$\tau \mapsto \int |U(2\tau - t)f|^2(x)\varphi(\tau, x)dx.$$

We compute (we dropped the x for clarity) :

$$\begin{aligned}
&\frac{d}{d\tau}\int |U(2\tau-t)f|^2\varphi(\tau) \\
&= 2\int (U(2\tau-t)f)_\tau U(2\tau-t)f\varphi(\tau) + \int |U(2\tau-t)f|^2\varphi_\tau(\tau) \\
&= -4\int U(2\tau-t)f_{xxx}U(2\tau-t)f\varphi(\tau) + \int |U(2\tau-t)f|^2\varphi_\tau(\tau) \\
&= 4\int U(2\tau-t)f_{xx}U(2\tau-t)f_x\varphi(\tau) + 4\int U(2\tau-t)f_{xx}U(2\tau-t)f\varphi_x(\tau) \\
&\quad + \int |U(2\tau-t)f|^2\varphi_\tau(\tau) \\
&= -6\int |U(2\tau-t)f_x|^2\varphi_x(\tau) - 4\int U(2\tau-t)f_xU(2\tau-t)f\varphi_{xx}(\tau) \\
&\qquad + \int |U(2\tau-t)f|^2\varphi_\tau(\tau) \\
&= -6\int |U(2\tau-t)f_x|^2\varphi_x(\tau) + \int |U(2\tau-t)f|^2(2\varphi_{xxx}(\tau) + \varphi_\tau(\tau)).
\end{aligned}$$

As $\psi_{xxx} \le \frac{\sigma_0}{4}|\psi_x|$, and $\psi_x < 0$, we have that, for $\varphi = \psi_{-1}$ or ψ_0,

$$\forall \tau,\ \forall x, \quad \varphi_x(\tau, x) < 0 \quad \text{and} \quad 2\varphi_{xxx}(\tau, x) + \varphi_\tau(\tau, x) \ge 0.$$

So that $\tau \mapsto \int U(2\tau - t)f(x)^2\varphi_0(\tau, x)dx$ is an increasing function of τ. In particular, when comparing for $\tau = t$ and $\tau = t/2$ ($t \ge 0$), we have :

$$\forall t \ge 0, \quad \int |U(t)f|^2(x)\varphi(t, x) \ge \int f^2(x)\varphi(t/2, x).$$

As the flow $U(t)$ preserves the L^2-mass, we get in each case $\varphi = \psi_{-1}$ or ψ_0 :

$$\forall t \ge 0, \quad \int |U(t)f|^2(x)(1 - \psi_{-1}(t, x))dx \le \int f^2(x)(1 - \psi_{-1}(t/2, x))dx, \tag{3.25}$$

$$\int |U(t)f|^2(x)(1 - \psi_0(t, x))dx \le \int f^2(x)(1 - \psi_0(t/2, x))dx. \tag{3.26}$$

(3.25) immediatly gives (3.21). Let $x \ge m_{-1}(t)$. Then for $y \ge x$, $1 - \psi_{-1}(t, y) \ge 1 - \frac{2}{\pi}\arctan(1) = \frac{1}{2}$. Thus,

$$\begin{aligned}
|U(t)f(x)|^2 &= -2\int_y^\infty U(t)f(y)U(t)f_x(y)dy \\
&\le 2\left(\int_y^\infty |U(t)f(y)|^2dy\right)^{1/2}\left(\int_y^\infty |U(t)f_x(y)|^2dy\right)^{1/2} \\
&\le 8\left(\int_y^\infty |U(t)f(y)|^2(1 - \psi_{-1}(t, y))dy\right)^{1/2} \\
&\qquad\qquad \times \left(\int_y^\infty |U(t)f_x(y)|^2(1 - \psi_{-1}(t, y))dy\right)^{1/2} \\
&\le 8\|U(t)f\|_{L^2(1-\psi_{-1}(t))}\|U(t)f_x\|_{L^2(1-\psi_{-1}(t))} \\
&\le 8\|f\|_{L^2(1-\psi_{-1}(t/2))}\|f_x\|_{L^2(1-\psi_{-1}(t/2))}.
\end{aligned}$$

This is (3.22).

We will now use (3.26). Suppose that for some $q > 0$, $(1 + x_+^q)f(x) \in L^2(dx)$. Then for $t \ge 1$,

$$\begin{aligned}
\int f^2(1 - \psi_0(t/2)) &= \int_{x \le \sigma_0 t/4} f^2(1 - \psi_0(t/2)) + \int_{x \ge \sigma_0 t/4} f^2(1 - \psi_0(t/2)) \\
&\le \sup_{x \le \sigma_0 t/4}(1 - \psi_0(t/2, x))\int f^2 + \left(\frac{\sigma_0 t}{4}\right)^{-2q}\int_{x \ge \sigma_0 t/4} x^{2q}f^2 \\
&\le C(x_0)e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|f\|_{L^2}^2 + C(\sigma_0)t^{-2q}\|x_+^q f\|_{L^2}^2.
\end{aligned}$$

And we get

$$\forall t \ge 1, \quad \|U(t)f\|_{L^2(1-\psi_0(t))} \le \frac{C}{t^q}\|(1 + x_+)^q f\|_{L^2},$$

which is (3.23).

Suppose now that $(1 + x_+^{1/2})f(x) \in L^2(dx)$. Then

$$\int_{t=0}^\infty \int_x |U(t)f(x)|^2(1 - \psi_0(t, x))dxdt \le \int_{t=0}^\infty \int_x f^2(x)(1 - \psi_0(t/2, x))dxdt$$

$$\leq \int_x f^2(x) \int_{t=0}^{\infty} (1-\psi_0(t/2,x))dtdx$$
$$\leq C \int f^2(x) \frac{1+x_+}{\sigma_0} dx$$
$$\leq C\|(1+x_+^{1/2})f(x)\|^2_{L^2(dx)},$$

and this proves (3.24). □

3.5 Control of the interaction of $w_n(t)$ with the solitons

This section is devoted to the proof of Proposition 3.2. We develop arguments very similar to those of [29] and [28].

3.5.1 Modulation close to the asymptotic profile

Lemma 3.3. *There exist T_1 large enough and $\varepsilon_1 > 0$ small enough such that if $T_1 \geq T_1$ and $\varepsilon_0 \leq \varepsilon_1$, the following is true.*

There exist $2N$ C^1 functions $y_j, \gamma_j : [I_n, S_n] \to \mathbb{R}$ such that if we denote :

$$\tilde{R}_j(t,x) = Q_{\gamma_j(t)}(x - y_j(t)), \qquad \tilde{R}(t,x) = \sum_{j=1}^{N} \tilde{R}_j(t,x),$$
$$\tilde{w}_n(t) = u_n(t,x) - U(t)V - \tilde{R}(t,x),$$

we have for all $j = 1, \ldots, N$:

$$\int \tilde{w}_n(t,x)\tilde{R}_{j_x}(t,x)dx = 0 \quad \text{and} \quad \int \tilde{w}_n(t,x)\tilde{R}_j^3(t,x)dx = 0.$$

Moreover, there exists C_{11} such that :

$$\|\tilde{w}_n(t)\|_{L^2} + \sum_{j=1}^{N} |\gamma_j(t) - c_j| + \sum_{j=1}^{N} |y_j(t) - x_j - c_j t| \leq C_{11}\varepsilon_0, \tag{3.27}$$

$$|y_j'(t) - c_j| + |\gamma_j'(t)| \leq C_{11} e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C_{11}\|U(t)V\|_{L^2(1-\psi_0(t))}$$
$$+ C_{11}\left(\int \tilde{w}_n^2(t) e^{-\sqrt{\sigma_0}|x-c_j t|}\right)^{1/2}. \tag{3.28}$$

Proof. The existence of the modulation is essentially an application of the implicit function theorem. Consider the C^∞ functional

$$F : [I_n, S_n] \times L^2 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n,$$
$$(t, u, (y_j)_j, (\gamma_j)_j) \mapsto (F_{1j}(t, u, (y_j)_j, (\gamma_j)_j), F_{2j}(t, u, (y_j)_j, (\gamma_j)_j)),$$

with

$$F_{1j}(t, u, (y_j)_j, (\gamma_j)_j) = \int (u - U(t)V - \tilde{R}(t))\tilde{R}_{j_x}(t)dx,$$
$$F_{2j}(t, u, (y_j)_j, (\gamma_j)_j) = \int (u - U(t)V - \tilde{R}(t))\tilde{R}_j^3(t)dx,$$

locally on a neighborhood of the curve $y_j(t) = x_j + c_j t$, $\gamma_j(t) = c_j$, $u = U(t)V + R(t)$. To express y_j, γ_j in function of u, t, we apply the implicit function theorem stated in the Appendix : let us prove that $\partial_{y_j,\gamma_j} F$ is invertible at points $(t, U(t)V + R(t), x_j + c_j t)_j, (c_j)_j)$, compute $\partial_u F$, and do some uniform (in t) estimates.

For all t, α being y_j or γ_k we compute

$$\partial_\alpha F_{1j}(t) = -\int (\partial_\alpha \tilde{R}_j)(t) \tilde{R}_{j_x}(t) + \int (u - U(t)V - \tilde{R}_j(t))(\partial_\alpha \tilde{R}_{j_x})(t),$$
$$\partial_\alpha F_{2j}(t) = \int (\partial_\alpha \tilde{R}_j)(t) \tilde{R}_j^3(t) + 3\int (u - U(t)V - \tilde{R}_j(t))(\partial_\alpha \tilde{R}_j)(t) \tilde{R}_j^2(t),$$

and

$$(\partial_{y_j} \tilde{R}_j)(t,x) = -\tilde{R}_{j_x}(t,x),$$
$$(\partial_{\gamma_j} \tilde{R}_j)(t,x) = \frac{1}{4c_j}\tilde{R}_j(t,x) + \frac{1}{2c_j}(x - x_j - c_j t)\tilde{R}_{j_x}(t,x).$$

Let u, y_j and γ_j be such that

$$\|u - U(t)V - R(t)\|_{L^2} + \sum_{j=1}^N |y_j| + |\gamma_j| \le \varepsilon_0.$$

We get that

$$\left| \partial_{y_j} F_{1j} - \int Q_{c_j}{}_x^2 \right| \le C\varepsilon_0,$$

(recall $\|Q_{c_j}\|_{L^2} = \|Q\|_{L^2}$) and for $k \neq j$, using the exponential decay :

$$|\partial_{y_k} F_{1j}(t)| \le C e^{-\frac{\sigma\sqrt{\sigma_0}}{4}t} + C\varepsilon_0,$$
$$|\partial_{y_k} F_{2j}(t)| + |\partial_{\gamma_k} F_{1j}(t)| + |\partial_{\gamma_k} F_{2j}(t)| \le C e^{-\frac{\sigma\sqrt{\sigma_0}}{4}t} + C\varepsilon_0.$$

Now Q is an even function, so that

$$|\partial_{\gamma_j} F_{1j}(t)| \le C\delta, \qquad |\partial_{y_j} F_{2j}(t)| \le C\varepsilon_0.$$

Finally, for $\partial_{\gamma_j} F_{2j}$, we have

$$\left| \partial_{\gamma_j} F_{2j} - \frac{1}{4c_j}\int Q_{c_j}{}^4 \right| \le C\varepsilon_0.$$

Hence, for T_1 large enough and $\varepsilon_1 > 0$ small enough, the conditions of the implicit function theorem are fufilled, and we obtain the existence and regularity of $y_j(t)$, $\gamma_j(t)$, along with the first estimate (3.27).

For the second estimate (3.28), we compute the equation satisfied by $\tilde{w}(t)$ and do the scalar product with every $\tilde{R}_{j_x}$ and every $\tilde{R}_j^3$: the relations obtained in this way will yield the result. The equation satisfied by $\tilde{R}_k$ (using $-c_k R_{kx} + R_{kxxx} + R_k^5)_x = 0$) is now

$$\tilde{R}_{kt} + \tilde{R}_{kxxx}$$
$$= (-y_k'(t) + c_k)\tilde{R}_{kx} + \frac{\gamma_k'(t)}{\gamma_k(t)}\left(\frac{\tilde{R}_k(t)}{4} + (x - y_k(t))\frac{\tilde{R}_{kx}(t)}{2}\right) - c_k\tilde{R}_{kx} + \tilde{R}_{kxxx}$$

$$= (-y_k'(t) + c_k)\tilde{R}_{kx} + \frac{\gamma_k'(t)}{\gamma_k(t)}\left(\frac{\tilde{R}_k(t)}{4} + (x - y_k(t))\frac{\tilde{R}_{kx}(t)}{2}\right) - (\tilde{R}_k^5)_x.$$

So that with $\tilde{w}_n = u_n(t) - U(t)V - \tilde{R}(t)$, we get

$$\tilde{w}_{nt} + \tilde{w}_{nxxx} = \sum_{k=1}^{N}(y_k'(t) - c_k)\tilde{R}_{kx} - \sum_{k=1}^{N}\frac{\gamma_k'}{\gamma_k}\left(\frac{\tilde{R}_k}{4} + (x - y_k(t))\frac{\tilde{R}_{kx}}{2}\right) - \left((\tilde{w}_n + U(t)V + \tilde{R})^5 - \sum_{k=1}^{N}\tilde{R}_k^5\right)_x. \quad (3.29)$$

Now, if we express $\tilde{R}_j$ in terms of R_j, we get

$$\tilde{R}_{jxt} = -y_j'(t)\tilde{R}_{jxx} + \frac{\gamma_j'(t)}{\gamma_j(t)}\left(\frac{\tilde{R}_{jx}(t)}{4} + (x - y_j(t))\frac{\tilde{R}_{jxx}(t)}{2} + \frac{\tilde{R}_{jx}(t)}{2}\right).$$

And keeping in mind that $\frac{d}{dt}\int \tilde{w}_n\tilde{R}_{jx} = \int \tilde{w}_n\tilde{R}_{jx} = 0$, we get

$$\int \tilde{w}_{nt}\tilde{R}_{jx} = -\int \tilde{w}_n\tilde{R}_{jxt} = \int \tilde{w}_n\left(y_j'(t) - \frac{\gamma_j'(t)}{\gamma_j(t)}\frac{x - y_j(t)}{2}\right)\tilde{R}_{jxx}.$$

We multiply (3.29) by $\tilde{R}_{jx}$, integrate in x, and do integration by parts :

$$\begin{aligned}
&(y_j'(t) - c_j)\int \tilde{R}_{jx}^2 \\
&\quad = -y_j'(t)\int \tilde{w}(t)\tilde{R}_{jxx} + \frac{\gamma_j'(t)}{2\gamma_j(t)}\int \tilde{w}_n(t)(x - y_k(t))\tilde{R}_{jxx} - \int \tilde{w}_n(t)\tilde{R}_{jxxxx} \\
&\quad - \sum_{k,k\neq j}(c_k - y_k'(t))\int \tilde{R}_{jx}\tilde{R}_{kx} + \sum_{k=1}^{N}\frac{\gamma_k'}{\gamma_k}\int \tilde{R}_{jx}\left(\frac{\tilde{R}_k}{4} + (x - y_k(t))\frac{\tilde{R}_{kx}}{2}\right) \\
&\qquad - \int\left((\tilde{w}_n + U(t)V + \tilde{R})^5 - \sum_{k=1}^{N}\tilde{R}_k^5\right)\tilde{R}_{jxx}.
\end{aligned}$$

First consider the three first terms : as $Q_{xx} = Q - Q^p$, we can express $\tilde{R}_{jxx}$ and $\tilde{R}_{jxxxx}$ in terms of powers of $\tilde{R}_j$. Therefore, the integral part of these terms is bounded by

$$\int |\tilde{w}_n(t)|(1 + |x - c_jt|)e^{-\sqrt{\sigma_0}|x - c_jt|} \leq C\left(\int |\tilde{w}_n(t)|^2 e^{-\sqrt{\sigma_0}|x - c_jt|}\right)^{1/2}.$$

For the fourth term, $\int |R_{jx}R_{kx}| \leq e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}$. This also apply to the fifth term, but for j-term, which vanishes :

$$\int \tilde{R}_{jx}\left(\frac{\tilde{R}_j}{4} + (x - y_j(t))\frac{\tilde{R}_{jx}}{2}\right) = 0.$$

And for the non-linear last term, when developing, the large terms cancel one another, so that we can control the rest by :

$$C\int (|\tilde{w}_n(t)| + |U(t)V|)e^{-\sqrt{\sigma_0}|x - c_jt|}.$$

Finally, we have altogether

$$
\begin{aligned}
|y_j'(t)-c_j| \le C\left(1+\left|\frac{\gamma_j'(t)}{\gamma_j(t)}\right|\right)\left(\int |\tilde{w}_n(t)|^2 e^{-\sqrt{\sigma_0}|x-c_jt|}\right)^{1/2} \\
+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\sum_{k,k\neq j}|y_k'(t)-c_k| + Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\sum_{k,k\neq j}\left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right| \\
+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C\|U(t)V\|_{L^2(1-\psi_0(t))}. \qquad (3.30)
\end{aligned}
$$

Now, we have to do the same kind of argument on γ_j. As

$$
\tilde{R}_{j_t} = -y_j'(t)\tilde{R}_{j_x} + \frac{\gamma_j'(t)}{\gamma_j(t)}\left(\frac{\tilde{R}_j(t)}{4} + (x-y_j(t))\frac{\tilde{R}_{j_x}(t)}{2} + \frac{\tilde{R}_j(t)}{2}\right),
$$

we have

$$
\int \tilde{w}_{nt}\tilde{R}_j^3 = -3\int \tilde{w}_n\tilde{R}_{j_t}\tilde{R}_j^2 = 3\int\left((y_j' - \frac{\gamma_j'}{2\gamma_j}(x-y_j)\right)\left(\tilde{w}_n\tilde{R}_{j_x}\tilde{R}_j^2\right).
$$

Let us multiply (3.29) by $\tilde{R}_j^3$. We obtain, after integration by parts $\int(x-y_j(t))\tilde{R}_j\tilde{R}_{j_x} = -\frac{1}{2}\int\tilde{R}_j^2$,

$$
\begin{aligned}
\frac{1}{4}\frac{\gamma_j'(t)}{\gamma_j(t)}\int\tilde{R}_j^4 = \frac{\gamma_j'(t)}{2\gamma_j(t)}\int\tilde{w}_n(t)(x-y_k(t))\tilde{R}_j^{\,3} - \int\tilde{w}_n(t)(\tilde{R}_j^3)_{xxx} \\
- \sum_{k=1}^{N}(c_k-y_k')\int\tilde{R}_j^3\tilde{R}_{kx} + \sum_{k\neq j}\frac{\gamma_k'}{\gamma_k}\int\tilde{R}_j^3\left(\frac{\tilde{R}_k}{4}+(x-y_k(t))\frac{\tilde{R}_{kx}}{2}\right) \\
- 3\int\left((\tilde{w}_n+U(t)V+\tilde{R})^5 - \sum_{k,k\neq j}\tilde{R}_k^5\right)\tilde{R}_{j_x}\tilde{R}_j^2.
\end{aligned}
$$

Let us notice again that the only possibly large term (in the first sum) is in fact 0 ($\int\tilde{R}_j^3\tilde{R}_{j_x} = 0$). If we argue like before, we get

$$
\begin{aligned}
\left|\frac{\gamma_j'(t)}{\gamma_j(t)}\right| \le C\left(1+\frac{\gamma_j'(t)|}{\gamma_j(t)}\right)\left(\int\tilde{w}_n^2(t)e^{-\sqrt{\sigma_0}|x-c_jt|}\right)^{1/2} \\
+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\sum_{k,k\neq j}|y_k'(t)-c_k| + e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\sum_{k,k\neq j}\left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right| \\
+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C\|U(t)V\|_{L^2(1-\psi_0(t))}. \qquad (3.31)
\end{aligned}
$$

We can now use our computations. Let us sum our $2N$ estimates (3.30) and (3.31) together :

$$
\begin{aligned}
\sum_{k=1}^{N}\left(|y_k'(t)-c_k| + \left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\right) \le C\left(1+\sum_{k=1}^{N}|y_k'(t)| + \sum_{k=1}^{N}\left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\right)\|\tilde{w}_n\|_{L^2} \\
+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\left(\sum_{k=1}^{N}|y_k'(t)| + \left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\right) + Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C\|U(t)V\|_{L^2(1-\psi_0(t))}.
\end{aligned}
$$

So that for ε_1 small enough, as $\|\tilde{w}_n\|_{L^2} \leq \varepsilon_0 \leq \varepsilon_1$, and $t \geq T_1$ large enough, we get

$$\sum_{k=1}^{N} |y_k'(t) - c_k| + \left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right| \leq C.$$

Let us now go back to (3.30) : we get exactly what we want on $|y_j'(t) - c_j|$. In the same way, as $\gamma_k > \sigma_0$ for ε_0 small enough (first estimate), we get the result for $|\gamma_j'(t)|$ by plugging in (3.31). □

Notice that

$$\|w_n(t) - \tilde{w}_n(t)\|_{H^s} = \|R(t) - \tilde{R}(t)\|_{H^s} \leq C(s)(|y_j(t) - c_j t - x_j| + |\gamma_j(t) - c_j t|). \quad (3.32)$$

We now turn to the extraction of the main terms in $\int u_n^2(t)$ and $E(u_n(t))$, which writes as follows : recall $\|Q_c\|_{L^2} = \|Q\|_{L^2}$ and $E(Q_c) = 0$). Let us denote for simplicity :

$$\tilde{v}_n(t) = \tilde{w}_n(t) + U(t)V = u_n(t) - \tilde{R}(t).$$

Lemma 3.4 (Main terms in M_j and E_j, $j \geq 1$)**.** *We have, for all $t \in [I_n, S_n]$,*

(1) $$\left|M_j(t) - \left(\int Q^2 + 2\int \tilde{v}_n(t)\tilde{R}_j(t) + \int \tilde{v}_n^2(t)\phi_j(t)\right)\right| \leq C_{12} e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t},$$

(2) $$\left|E_j(t) - \left[\frac{1}{2}\int (\tilde{v}_{nx}^{\,2}(t) - 5\tilde{R}_j^4(t)\tilde{v}_n^2(t))\phi_j(t) - \gamma_j(t)\int \tilde{v}_n(t)\tilde{R}_j(t)\right)\right| \leq C_{12} e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C_{12}\varepsilon_0 \int \tilde{v}_n^2(t)\phi_j(t),$$

(3) $$\left|\left(E_j(t) + \frac{\gamma_j(t)}{2}M_j(t)\right) - \frac{\gamma_j(t)}{2}\int Q^2 - \frac{1}{2}H_j(t)\right| \leq C_{12} e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C_{12}\varepsilon_0 \int \tilde{v}_n^2(t)\phi_j(t),$$

where $H_j(t) = \displaystyle\int (\tilde{v}_{nx}^{\,2}(t) - 5\tilde{R}_j^4(t)\tilde{v}_n^2(t) + \gamma_j(t)\tilde{v}_n^2(t))\phi_j(t)$.

Proof. (1) We compute ($u_n = \tilde{v}_n + \tilde{R}$) :

$$M_j(t) = \int u_n^2 \phi_j(t) = \int \left(\tilde{v}_n^2 + 2\tilde{v}_n \tilde{R}(t) + \sum_{k=1}^{N} \tilde{R}_k^2(t)\right) \phi_j(t).$$

As $\phi_j(t)$ localized in the interval $[m_{j-1}(t), m_j(t)]$, like $\tilde{R}_j(t)$ we get ($k \neq j$)

$$\left|\int \tilde{R}_j^2(t)\phi_j(t) - \int Q_{\gamma_j(t)}^2\right| \leq C e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}, \quad \int \tilde{R}_k^2(t)\phi_j(t) \leq C e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}.$$

(2) In the same way,

$$\begin{aligned} E_j(t) &= \int \left(\frac{1}{2}(\tilde{v}_{nx}^{\,2}(t) + 2\tilde{v}_{nx}(t)\tilde{R}_x + \tilde{R}_x^2) - \frac{1}{6}(\tilde{v}_n(t) + \tilde{R}(t))^6\right) \phi_j(t) \\ &= \int (\frac{1}{2}\tilde{v}_{nx}^{\,2}(t) - \frac{5}{2}\tilde{R}^4\tilde{v}_n^2(t))\phi_j + \int (\frac{1}{2}\tilde{R}_x^2 - \frac{1}{6}\tilde{R}^6)\phi_j(t) \end{aligned}$$

$$-\int \tilde{v}_n(t)(\tilde{R}_{xx}+\tilde{R}^5)\phi_j - \int \tilde{R}_x\tilde{v}_n(t)\phi_{j\,x}$$
$$+\int\left[\frac{(-(\tilde{v}_n(t)+\tilde{R})^6+\tilde{R}^6)}{6}+\tilde{v}_n(t)\tilde{R}^5+\frac{5}{2}\tilde{R}^4\tilde{v}_n^2(t)\right]\phi_j.$$

We keep the first integral untouched. The second one is $E(Q_{\gamma_j(t)})$ up to an exponential correction. For the third one, recall that $Q_{xx}+Q^5=Q$, so that again

$$\int \tilde{v}_n(t)(\tilde{R}_{xx}+\tilde{R}^5)\phi_j = \gamma_j(t)\int \tilde{v}_n(t)\tilde{R}_j(t)+O(e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}).$$

The fourth one is exponentially small (with $\tilde{R}$ and $\phi_{j\,x}$). Finally the fifth is of order at least 3 in v_n, so that we control it by

$$\int \tilde{v}(t)^k\phi_j(t) \le \|\tilde{v}(t)\|_{L^\infty}^{k-2}\int \tilde{v}(t)^2\phi_j(t).$$

This gives the desired result.

(3) is the sum of (1) and (2). Notice that the scalar product $\int \tilde{v}(t)\tilde{R}_j(t)$ vanishes in H_j : the linear combination has been constructed for this. □

Proposition 3.4 (Positivity of a quadratic form)**.** *There exists $\sigma_1>0$ small enough and $\lambda_1>0$ so that the following is true. For $\sigma_0\le\sigma_1$, there exists $T_1=T_1(\sigma_0)$, so that for all $t\ge T_1$, for all $j=1,\dots,N$, and for all $v\in H^1$,*

$$\int (v_x^2-5\tilde{R}_j(t)^4v^2+\gamma_j(t)v^2)\phi_j(t)$$
$$\ge \lambda_1\int (v_x^2+v^2)\phi_j(t)-\frac{1}{\lambda_1}\left(\left(\int v\tilde{R}_j^3(t)\right)^2+\left(\int v\tilde{R}_{j\,x}(t)\right)^2\right).$$

Proof. A similar result can be found in [29, Lemma 4] and [28, Appendix A]. For the sake of completeness, the complete proof is done in the Appendix. □

From now on and throughout the rest of the proof, $\sigma_0<\sigma_1$ is fixed.

3.5.2 Almost monotonicity properties and Abel transform

Lemma 3.5 (Monotocity formula [24])**.** *There exists T_1 large enough, ε_1 small enough, and $C_{13}>0$ such that for all $j=0,\dots,N$ and $t\in[I_n,S_n]$,*

$$\sum_{k=0}^{j}(M_k(S_n)-M_k(t))\ge -C_{13}e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t},$$
$$\sum_{k=0}^{j}(F_k(S_n)-F_k(t))\ge -C_{13}e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}.$$

Proof. This lemma is very similar to the monotonicity Lemma of [29] and [23]. The only difference is the presence of the term $U(t)V$: this will be taken care of essentially due to pointwise smallness of $U(t)V$ for $x\ge(\sigma_0/2)t$, that is (3.22).

Let us now do the computations. First notice

$$\sum_{k=0}^{j} M_k(t) = \int u_{nt}^2(t)\psi_j(t), \quad \sum_{k=0}^{j} E_k(t) = \int \left(\frac{1}{2}u_{nx}^2(t) - \frac{1}{6}u_n^6(t)\right)\psi_j(t).$$

For $j = N$, the result is the conservation of mass and energy. Otherwise we compute for $f(t,x) \in C^3$:

$$\begin{aligned}\frac{d}{dt}\int u_n^2 f - \int u_n^2 f_t = 2\int u_{nt}u_n f &= -2\int (u_{nxx} + u_n^5)_x u_n f \\ &= 2\int (u_{nxx} + u_n^p)(u_{nx}f + u_n f_x) \\ &= \int \left(-3u_{nx}^2 + \frac{5}{3}u_n^6\right) f_x - 2\int u_{nx}u_n f_{xx} \\ &= \int \left(-3u_{nx}^2 + \frac{5}{3}u_n^6\right) f_x + \int u_n^2 f_{xxx}.\end{aligned}$$

So that we get

$$\frac{d}{dt}\int u_n^2\psi_j(t) = -\int \left(3u_{nx}^2 + m_j'(t)u_n^2 - \frac{5}{3}u_n^6\right)\psi_{jx} + \int u_n^2\psi_{jxxx}.$$

But $m_j'(t) \geq \sigma_0$ so that by (3.18), and $\psi_{jx} \leq 0$:

$$\frac{d}{dt}\int u_n^2\psi_j(t) \geq -\int \left(3u_{nx}^2 + \frac{3}{4}\sigma_0 u_n^2 - \frac{5}{3}u_n^6\right)|\psi_{jx}|(t).$$

It remains to bound the third term. We consider two cases. When $x \in I_j(t) = [m_j t - \frac{\sigma_0}{2}t, m_j(t) + \frac{\sigma_0}{2}t]$, ψ_{jx} is big but $R(t)$ and $U(t)V$ are small (recall (3.22)) so that u_n too. More precisely, for $x \in I_{j1}(t)$, $x \geq m_{-1}(t)$, and

$$\begin{aligned}\left|\frac{5}{3}u_n^4(t,x)\right| &\leq C(\|w_n(t)\|_{L^\infty}^4 + \|U(t)V\|_{L^\infty(x \geq m_{-1}(t))}^{p-1} + |R(t,x)|^4) \\ &\leq C(K_0^2\varepsilon_0^2 + \|V\|_{L^2(1-\psi_{-1}(t/2))}\|V_x\|_{L^2(1-\psi_{-1}(t/2))} + e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}) \\ &\leq \frac{\sigma_0}{4},\end{aligned} \tag{3.33}$$

if T_1 is large enough ($t \geq T_1$), and ε_1 is small enough. On this interval, the second term is larger than the third :

$$\frac{5}{3}\int_{x \in I_j(t)} u_n^6|\psi_{jx}|(t) \leq \frac{\sigma_0}{4}\int u_n^2|\psi_{jx}|(t).$$

When $x \in \mathbb{R} \setminus I_j(t)$, then $x \notin [m_j(t) - \frac{\sigma_0}{2}t, m_j(t) + \frac{\sigma_0}{2}t]$, so that

$$|\psi_{jx}(t,x)| \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}.$$

Now by interpolation between L^2 and H^1, we have a uniform control $\int |u_n|^6 \leq C$:

$$\frac{5}{3}\int_{x \in \mathbb{R}\setminus I_j(t)} u_n^6|\psi_{jx}|(t) \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}.$$

So that finally

$$\frac{d}{dt}\int u_n^2\psi_j(t) \geq \int \left(3{u_n}_x^2 + \frac{\sigma_0}{2}u_n^2\right)|{\psi_j}_x(t)| - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} \geq -Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}. \tag{3.34}$$

We integrate this last estimate between t and S_n, and this gives the estimates on M_j.

For the estimates on F_j, we compute in a similar way

$$\begin{aligned}
&\frac{d}{dt}\int\left({u_n}_x^2 - \frac{1}{3}u_n^6\right)f - \int\left({u_n}_x^2 - \frac{1}{3}u_n^6\right)f_t \\
&\qquad = 2\int({u_n}_{xt}{u_n}_x - u_n^5{u_n}_t)f = -2\int {u_n}_t({u_n}_{xx} + u_n^p)f - 2\int {u_n}_t{u_n}_x f_x \\
&\qquad = -\int({u_n}_{xx} + u_n^5)^2 f_x + 2\int({u_n}_{xx} + u_n^5)_x{u_n}_x f_x \\
&\qquad = -\int\left(({u_n}_{xx} + u_n^5)^2 + 2{u_n}_{xx}^2 - 10{u_n}_x^2 u_n^{p-1}\right)f_x - 2\int {u_n}_{xx}{u_n}_x f_{xx} \\
&\qquad = -\int\left(({u_n}_{xx} + u_n^p)^2 + 2{u_n}_{xx}^2 - 10{u_n}_x^2 u_n^{p-1}\right)f_x + \int {u_n}_x^2 f_{xxx}.
\end{aligned}$$

So that

$$\begin{aligned}
&\frac{d}{dt}\int\left({u_n}_x^2 - \frac{1}{3}u_n^6\right)\psi_j(t) \\
&\quad = -\int\left(({u_n}_{xx} + u_n^5)^2 + 2{u_n}_{xx}^2 - 10{u_n}_x^2 u_n^4\right){\psi_j}_x(t) \\
&\qquad - m_j'(t)\int\left({u_n}_x^2 - \frac{1}{3}u_n^6\right){\psi_j}_x(t) + \int {u_n}_x^2{\psi_j}_{xxx}(t).
\end{aligned}$$

Again $m_j'(t) \geq \sigma_0$ and $|m_j'(t)| \leq c_N$, so that $\int {u_n}_x^2{\psi_j}_{xxx}(t) - \frac{\sigma_0}{4}\int {u_n}_x^2{\psi_j}_x(t) \geq 0$ and

$$\frac{d}{dt}\int\left({u_n}_x^2 - \frac{u_n^6}{3}\right)\psi_j(t) \geq \frac{3}{4}\sigma_0\int {u_n}_x^2|{\psi_j}_x(t)| - \int\left(10{u_n}_x^2|u_n|^4 - \frac{c_N}{3}|u_n|^6\right)|{\psi_j}_x(t)|. \tag{3.35}$$

To bound $10\int {u_n}_x^2|u_n|^{p-1}|{\psi_j}_x(t)|$, we proceed like before and get

$$10\int {u_n}_x^2|u_n|^4|{\psi_j}_x(t)| \geq -\frac{\sigma_0}{4}\int |{u_n}_x^2|{\psi_j}_x(t) - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}. \tag{3.36}$$

However for $\frac{c_N}{6}\int u_n^6|{\psi_j}_x(t)|$, some L^2 norm is needed (which is why we introduced F_j, as in [23]). Choosing ε_1 small enough and T_1 large enough, we can improve (3.33) to $\sigma_0/400$, and so obtain

$$\frac{c_N}{3}\int u_n^6 \geq -\frac{\sigma_0}{400}\int u_n^2|{\psi_j}_x(t)| - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}. \tag{3.37}$$

Now adding up (3.35) and 1/100·(3.34), and using (3.36) and (3.37), we get

$$\frac{d}{dt}\int\left({u_n}_x^2 - \frac{1}{3}u_n^6 + \frac{1}{200}u_n^2\right)\psi_j(t) \geq \frac{\sigma_0}{2}\int {u_n}_x^2|\psi_x(t)| - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}.$$

And the estimate on F_j comes by integration between t and S_n. □

***Remark* 3.5.** *Notice that it is possible to obtain almost monotonicity properties (on the left) related to other quantities than mass and energy. However, these are related to conservation laws : thus they translate to monotonicity properties on the right, and this is specially interesting and useful.*

We can now conclude the proof of Proposition 3.2.

Proof of Proposition 3.2. We do some estimates on $\tilde{w}_n(t)$ first. The key point is the following resummation argument, which will allow us to use the monotonicity property. We compute

$$\begin{aligned}\sum_{j=1}^{N}\frac{1}{\gamma_j^2(t)}\left(E_j+\frac{\gamma_j(t)}{2}M_j\right)&=\sum_{j=1}^{N-1}\left(\left(\frac{1}{\gamma_j^2(t)}-\frac{1}{\gamma_{j+1}^2(t)}\right)\sum_{k=1}^{j}F_k\right)\\&+\sum_{j=1}^{N-1}\left(\frac{1}{2}\left(\frac{1}{\gamma_j(t)}-\frac{1}{\gamma_{j+1}(t)}\right)\left(1-\frac{\sigma_0}{50}\left(\frac{1}{\gamma_j(t)}+\frac{1}{\gamma_{j+1}(t)}\right)\right)\sum_{k=1}^{j}M_k\right)\\&+\frac{1}{\gamma_N^2(t)}\sum_{k=1}^{N}F_k+\frac{1}{2\gamma_N(t)}\left(1-\frac{\sigma_0}{50c_N}\right)\sum_{j=1}^{N}M_k.\end{aligned}$$

All the terms in the right hand side are positives, so that we can apply Lemma 3.5 :

$$\sum_{j=1}^{N}\frac{1}{\gamma_j^2(t)}\left(E_j(t)+\frac{\gamma_j(t)}{2}M_j(t)\right)-\sum_{j=1}^{N}\frac{1}{\gamma_j^2(t)}\left(E_j(S_n)+\frac{\gamma_j(t)}{2}M_j(S_n)\right)\le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}.$$

Now we use fact 3. of Lemma 3.4 at time t and at time S_n (recall that $|\gamma_j(t)-c_j|\le C\varepsilon_0$, so that $c_N+\varepsilon_0\ge\gamma_j(t)\ge\sigma_0$) :

$$\begin{aligned}\sum_{j=1}^{N}\frac{1}{\gamma_j^2(t)}H_j(t)&\le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}+C\varepsilon_0\int\tilde{v}_n^2(t)\sum_{j=1}^{N}\phi_j(t)+C\varepsilon_0\int\tilde{v}_n^2(S_n)\sum_{j=1}^{N}\phi_j(S_n)\\&\le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}+C\varepsilon_0\|\tilde{v}_n(t)\|^2_{L^2(1-\psi_0(t))}+C\varepsilon_0\|U(S_n)V\|^2_{L^2(1-\psi_0(S_n))}.\end{aligned}\tag{3.38}$$

By Proposition 3.4, we have that for $j=1,\dots,N$,

$$H_j(t)\ge\lambda_1\int(\tilde{v}_n^2(t)+\tilde{v}_x^2(t))\phi_j(t)-\frac{1}{\lambda_1}\left(\left(\int\tilde{v}(t)\tilde{R}_j^3\right)^2+\left(\int\tilde{v}(t)\tilde{R}_{j_x}\right)^2\right).$$

So that if we sum up those N inequalities, there exists $\lambda_0>0$ neither depending on σ_0 nor ε_0) such that

$$\begin{aligned}\sum_{j=1}^{N}\frac{1}{\gamma_j^2(t)}H_j(t)&\ge\lambda_0\|\tilde{v}_n(t))\|^2_{H^1(1-\psi_0(t))}-\frac{1}{\lambda_0}\sum_{j=1}^{N}\left(\left(\int\tilde{v}_n(t)\tilde{R}_j^3(t)\right)^2+\left(\int\tilde{v}_n(t)\tilde{R}_{j_x}(t)\right)^2\right)\\&\ge\lambda_0\|\tilde{v}_n(t))\|^2_{H^1(1-\psi_0(t))}-\frac{1}{\lambda_0}\sum_{j=1}^{N}\left(\left(\int U(t)V\tilde{R}_j^3\right)^2+\left(\int U(t)V\tilde{R}_{j_x}\right)^2\right)\\&\ge\lambda_0\|\tilde{v}_n(t))\|^2_{H^1(1-\psi_0(t))}-\frac{C}{\lambda_0}\|U(t)V\|^2_{L^2(1-\psi_0(t))}.\end{aligned}\tag{3.39}$$

Combining (3.39) and (3.38), provided that ε_0 is small enough so that $C_3\varepsilon_0 < \lambda_0/2$, we deduce

$$\frac{1}{C}\|\tilde{v}_n(t))\|^2_{H^1(1-\psi_0(t))} \le e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} + \|U(t)V\|^2_{L^2(1-\psi_0(t))} + \|U(S_n)V\|^2_{L^2(1-\psi_0(S_n))}.$$

We will only use the obtained bound on $\|\tilde{v}_n(t))\|_{L^2(1-\psi_0(t))}$. Recall $\tilde{v}_n(t) = \tilde{w}_n(t) + U(t)V$, thus

$$\begin{aligned}\|\tilde{w}_n(t))\|^2_{L^2(1-\psi_0(t))} &\le 2\|\tilde{v}_n(t))\|^2_{L^2(1-\psi_0(t))} + 2\|U(t)V\|^2_{L^2(1-\psi_0(t))}\\ &\le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} + C\|U(t)V\|^2_{L^2(1-\psi_0(t))} + C\|U(S_n)V\|^2_{L^2(1-\psi_0(S_n))}. \quad (3.40)\end{aligned}$$

Relying on estimate (3.40), we only need to go back to $w_n(t) = \tilde{w}_n(t) + R(t) - \tilde{R}(t)$. As we noted in (3.32),

$$\begin{aligned}\|w_n(t)\|_{L^2(1-\psi_0(t))} &\le \|R(t) - \tilde{R}(t)\|_{L^2} + \|\tilde{w}_n(t))\|_{L^2(1-\psi_0(t))}\\ &\le C\sum_{k=1}^{N} |y_j(t) - x_j - c_jt| + |\gamma_j(t) - c_j| + Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\\ &\quad + C\|U(t)V\|_{L^2(1-\psi_0(t))} + C\|U(t)V\|_{L^2(1-\psi_0(S_n))}. \quad (3.41)\end{aligned}$$

Now, using the L^2_{loc} estimate of lemma 3.3, and (3.40) :

$$\begin{aligned}|y_j'(t) - c_j| + |\gamma_j'(t)| &\le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} + C\|U(t)V\|_{L^2(1-\psi_0(t))} + C\left(\int \tilde{w}_n^2(t)e^{-\sqrt{\sigma_0}|x-c_jt|}\right)^{1/2}\\ &\le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{8}t} + C\|U(t)V\|_{L^2(1-\psi_0(t))} + C\|U(S_n)V\|_{L^2(1-\psi_0(S_n))}.\end{aligned}$$

Let us integrate this between t and S_n. Recall the initial conditions $y_j(S_n) = x_j + c_jS_n$, $\gamma_j(S_n) = c_j$, we obtain

$$\begin{aligned}|y_j(t) - x_j - c_jt| + |\gamma_j(t) - c_j| &\le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} + C\int_t^{S_n} \|U(t)V\|_{L^2(1-\psi_0(t))}dt\\ &\quad + C(S_n - t)\|U(S_n)V\|_{L^2(1-\psi_0(S_n))}.\end{aligned}$$

This, together with (3.41), concludes the proof of Proposition 3.2. □

3.6 Control of the interaction of w_n with $U(t)V$: the linear theory

This section is devoted to the proof of Proposition 3.3.

However, let us first link Proposition 3.2, Proposition 3.3, Lemma 3.2 and Proposition 1' together. We compute the decay we obtained on $\|w_n(t)\|_{H^1(1-\psi_0(t))}$.

$(1 + x_+)^{2+\delta_0}V \in H^1$, so that from Lemma 3.2,

$$\|U(t)V\|_{L^2(1-\psi_0(t))} \le \frac{C}{t^{2+\delta_0}}. \quad (3.42)$$

From Proposition 3.2, we can then conclude that

$$\|w_n(t)\|_{L^2(1-\psi_0(t))} \le \frac{C}{t^{1+\delta_0}}. \quad (3.43)$$

This ensures that the assumptions of Proposition 3.3 are fulfilled.

Thus, Proposition 1' follows from Proposition 3.2, Lemma 3.2 and Proposition 3.3.

3.6.1 Preliminary lemmas

First recall the fundamental linear estimate.

Lemma 3.6 ([15]). *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $B \in \mathbb{R}$. The following inequalities hold, as long as the right-hand side is bounded :*

$$\left\| \partial_x \int_\tau^B U(t-s) f(s,x) ds \right\|_{L_x^5 L_\tau^{10}(\tau \in [t,B])} \leq \| f(\tau,x) \|_{L_x^1 L_\tau^2(\tau\in[t,B])}, \tag{3.44}$$

$$\sup_{\tau \in [t,B]} \left\| \partial_x \int_\tau^B U(t-s) f(s,x) ds \right\|_{L_x^2} \leq \| f(\tau,x) \|_{L_x^1 L_\tau^2(\tau\in[t,B])}. \tag{3.45}$$

Proof. In [15], the proof of the first estimate is done without restriction in time, that is

$$\left\| \partial_x \int_\tau^B U(t-s) f(s) ds \right\|_{L_x^5 L_\tau^{10}} \leq \| f \|_{L_x^1 L_\tau^2}.$$

Now as $s \in [\tau, B] \subset [t,B]$, we get

$$\begin{aligned}
\left\| \partial_x \int_\tau^B U(t-s) f(s) ds \right\|_{L_x^5 L_\tau^{10}(\tau\in[t,B])} &= \left\| \partial_x \int_\tau^B U(t-s)(f(s)\mathbb{1}_{s\in[\tau,B]}) ds \right\|_{L_x^5 L^{10}(\tau\in[t,B])} \\
&\leq \left\| \partial_x \int_\tau^B U(t-s)(f(s)\mathbb{1}_{s\in[\tau,B]}) ds \right\|_{L_x^5 L^{10}} \\
&\leq \| f(s)\mathbb{1}_{s\in[\tau,B]} \|_{L_x^1 L_t^2} = \| f \|_{L_x^1 L_\tau^2(\tau\in[t,B])}.
\end{aligned}$$

The proof of the second estimate with no restriction on time is done in [14] : the restricted one is done analogously. □

Now, let us prove a lemma which will handle the interference of the solitons when we will control the interaction of w_n with the linear term $U(t)V$.

Lemma 3.7 (Weak interference of solitons). *Let $A \geq 1$, $B \geq A$, $\delta_0 > 0$, and $f : [A,B]\times\mathbb{R} \to \mathbb{R}$. Suppose that*

$$f \in L_x^5 L_t^{10}(t \in [A,B]) \quad and \quad \forall t \in [A,B], \quad \| f(t) \|_{L^2(1-\psi_0(t))} \leq \frac{C}{t^{1+\delta_0}}.$$

Then there exists C (independent of A and B) such that

$$\forall t \in [A,B], \quad \| fR \|_{L_x^1 L_\tau^2(\tau\in[t,B])} \leq C e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} \| f \|_{L_x^5 L_\tau^{10}(\tau\in[t,B])} + \frac{C}{t^{\delta_0}}. \tag{3.46}$$

***Remark* 3.6.** *Observe that this result is almost optimal with respect to the decay rate required on $\| f(t) \|_{L^2(1-\psi_0(t))}$. Indeed, suppose that $f(t,x) = \frac{1}{t^\alpha} Q(x-t)$. Then*

$$\begin{aligned}
\| f(\tau,x) Q(x-t) \|_{L_x^1 L_\tau^2(\tau\in[t,B])} &= \int \left(\int_{\tau\in[t,B]} Q^3(x-t) \frac{d\tau}{\tau^{2\alpha}} \right)^{1/2} dx \\
&\sim \int_{x\in[t,B]} \frac{dx}{x^\alpha} \sim \frac{C}{t^{1-\alpha}}.
\end{aligned}$$

Thus, in order to have a decay estimate, we have to impose $\alpha > 1$, and we lose one order of decay.

Proof. First notice that it is enough to obtain the result for a single soliton. Indeed, to conclude for the N-soliton case, it suffices to see

$$\|fR\|_{L^1_x L^2_\tau(\tau\in[t,B])} \le \sum_{j=1}^N \|fR_j\|_{L^1_x L^2_\tau(\tau\in[t,B])}.$$

The idea is to split the double integral into two pieces, depending whether $|x - c_j\tau - x_j| \ge (x - x_j)/2$ or not. Denote

$$A(x) = \left\{\tau : \tau \in [t,B], |x - c_j\tau - x_j| \ge \frac{|x - x_j|}{2}\right\}, \quad \text{and} \quad B(x) = [t,B] \setminus A(x).$$

Then

$$\begin{aligned}
\|fR_j(s)\|_{L^1_x L^2_\tau(\tau\in[t,B])} &= \int_x \left(\int_{\tau\in[t,B]} |fR_j|^2(\tau,x)d\tau\right)^{1/2} dx \\
&= \int_x \left(\int_{\tau\in A(x)} |fR_j|^2(\tau,x)d\tau\right)^{1/2} dx \\
&\quad + \int_x \left(\int_{\tau\in B(x)} |fR_j|^2(\tau,x)d\tau\right)^{1/2} dx \\
&= I + II.
\end{aligned}$$

We estimate separately I and II.

For I, we are "away" from the soliton, and we use its decay to go from $L^1_x L^2_t$ to $L^5_x L^{10}_t$, with an exponentially small constant. Recall $R_j(t,x) = Q_{c_j}(x - c_j t - x_j)$. Remark that as $Q(y)$ is even and decreasing (to 0) for $y \ge 0$,

$$\sup_{\tau\in A(x)} |R_j(\tau,x)| = Q_{c_j}\left(\frac{x - x_j}{2}\right).$$

So that using Hölder's inequality in the τ integral with exponents $\frac{1}{2} = \frac{1}{10} + \frac{2}{5}$, we get

$$\begin{aligned}
I &\le \int_x Q_{c_j}^{1/2}\left(\frac{x - x_j}{2}\right)\left(\int_{\tau\in A(x)} |f|^2(\tau,x) Q_{c_j}(x - c_j\tau - x_j)d\tau\right)^{\frac{1}{2}} dx \\
&\le \int_x Q_{c_j}^{1/2}\left(\frac{x - x_j}{2}\right)\left(\int_{\tau\in A(x)} |f(\tau,x)|^{10} Q_{c_j}(x - c_j\tau - x_j)d\tau\right)^{\frac{1}{10}} \\
&\qquad\qquad \times \left(\int_{\tau\in A(x)} Q_{c_j}(x - c_j\tau - x_j)d\tau\right)^{\frac{2}{5}} dx.
\end{aligned}$$

Now let (τ, x) such that $\tau \in A(x)$. We claim that $|x - c_j\tau - x_j| \ge \frac{c_j}{3}\tau \ge \frac{c_j}{3}t$.

Indeed : first suppose $x - x_j \ge c_j\tau$. As $\tau \ge t \ge A \ge 0$, $x - x_j \ge 0$, and we have $x - c_j\tau - x_j \ge (x - x_i)/2$. Thus $x - x_j \ge 2c_j\tau$, so that $|x - c_j\tau - x_j| = x - c_j\tau - x_j \ge c_j\tau$.

Else if $x - x_j \le c_j\tau$: if $x - x_j \le 0$, as $\tau \ge 0$, we get $|x - c_j\tau - x_j| = c_j\tau - (x - x_j) \ge c_j\tau$. Else if $x - x_j \ge 0$, then $c_j\tau - (x - x_j) \ge (x - x_j)/2$, so that $\frac{2}{3}c_j\tau \ge (x - x_j)$. Thus, $c_j\tau - (x - x_j) \ge \frac{1}{3}c_j\tau$. This proves our claim.

And we get (with $y = x - c_j t - x_j$, $dy = c_j d\tau$)

$$\left(\int_{\tau\in A(x)} Q_{c_j}(x - c_j\tau - x_j)d\tau\right)^{2/5} \leq \left(\int_{|y|\geq c_j/3t} Q_{c_j}(y)\frac{dy}{c_j}\right)^{2/5}$$
$$\leq Ce^{-\frac{2c_j\sqrt{c_j}}{15}t}.$$

Applying again Hölder's inequality (in the x integral) with exponent $1 = \frac{4}{5} + \frac{1}{5}$, we can thus estimate

$$\begin{aligned}
I &\leq Ce^{-\frac{2c_j\sqrt{c_j}}{15}t}\int_x Q_{c_j}^{1/2}\left(\frac{x-x_j}{2}\right)\left(\int_{\tau\in A(x)}|f|^{10}(\tau,x)Q_{c_j}(x-c_j\tau-x_j)d\tau\right)^{\frac{1}{10}}dx \\
&\leq Ce^{-\frac{2c_j\sqrt{c_j}}{15}t}\left(\int_x Q_{c_j}^{5/8}\left(\frac{x-x_j}{2}\right)dx\right)^{4/5} \\
&\quad\times\left(\int_x\left(\int_{\tau\in A(x)}|f|^{10}(\tau,x)Q_{c_j}(x-c_j\tau-x_j)d\tau\right)^{1/2}dx\right)^{1/5} \\
&\leq Ce^{-\frac{2c_j\sqrt{c_j}}{15}t}\|Q_{c_j}^{1/2}\|_{L^{5/4}}\|fR_j^{1/10}\|_{L^5_xL^{10}_\tau(\tau\in[t,B])} \\
&\leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|f\|_{L^5_xL^{10}_\tau(\tau\in[t,B])}.
\end{aligned} \tag{3.47}$$

For II, we have the full bump of the soliton, but $x - x_j \geq \frac{1}{3}c_j\tau$, so we can use our decay on the right. This decay is in $L^\infty_t L^2_x$, so we have to interchange integrals : we will decompose the x integral in intervals $x \sim 2^j$, so that when applying the Cauchy-Schwarz inequality (to have $L^2_x L^2_t$, and then apply the Fubini-Tonelli Theorem), we don't pay too high a price.

Notice that for (τ, x) such that $\tau \in B(x)$, $|(x-x_j)-c_j\tau| \leq |x-x_j|/2$, so that $x-x_j \geq 0$ because $\tau \geq 0$. This implies that we can restrict ourselves to $x \geq x_j$ in the integral in x. Let $L_p = 2^p - 1 + x_j$. Then

$$II = \sum_{p\in\mathbb{N}}\int_{x=L_p}^{L_{p+1}}\left(\int_{\tau\in B(x)}|fR_j|^2(\tau,x)d\tau\right)^{1/2}dx,$$

and by Cauchy-Schwarz inequality,

$$\leq \sum_{p\in\mathbb{N}} 2^{p/2}\left(\int_{x=L_p}^{L_{p+1}}\int_{\tau\in B(x)}|fR_j|^2(\tau,x)d\tau dx\right)^{1/2}.$$

Now let (x,τ) such that $x \in [L_p, L_{p+1}$ and $\tau \in B(x)$. Then $|(x-x_j)-c_jt| \leq |x-x_j|/2$ and $x - x_j \geq 0$ implies that

$$c_j\tau \in [(x-x_j)-(x-x_j)/2, (x-x_j)+(x-x_j)/2] = [(x-x_j)/2, 3(x-x_j)/2],$$

so that $\tau \in [(2^p-1)/(2c_j), 3(2^{p+1}-1)/(2c_j)]$ and of course $\tau \in [t, B]$. This means that :

$$\{(\tau,x) : x\in[L_p,L_{p+1}], \tau\in B(x)\} \subset [L_p,L_{p+1}]\times\left(\left[\frac{2^p-1}{2c_j},\frac{3(2^{p+1}-1)}{2c_j}\right]\cap[t,B]\right),$$

which is a rectangle : thus we can interchange integrals.

$$\int_{x=L_p}^{L_{p+1}} \int_{\tau\in B(x)} |fR_j|^2(\tau,x)d\tau dx \leq \int_{x=L_p}^{L_{p+1}} \int_{\substack{\tau:\tau\in[t,B],\\ \tau\in\left[\frac{2^p-1}{2c_j},\frac{3(2^{p+1}-1)}{2c_j}\right]}} |fR_j|^2(\tau,x)d\tau dx$$
$$\leq \int_{\substack{\tau:\tau\in[t,B],\\ \tau\in\left[\frac{2^p-1}{2c_j},\frac{3(2^{p+1}-1)}{2c_j}\right]}} \int_{x=L_p}^{L_{p+1}} |fR_j|^2(\tau,x)dxd\tau.$$

Define K_1 the maximal index such that $(2^{K_1}-1)/(2c_j) \leq t$, and K_2 the maximal index such that $(2^{K_2}-1)/(2c_j) \leq B$. We can now use our decay estimate on $\|f(t)\|_{L^2(1-\psi_0(t))}$:

$$\begin{aligned} II \leq & \sum_{p=K_1}^{K_2} 2^{p/2}\left(\int_{\tau\in\left[\frac{2^p-1}{2c_j},\frac{3(2^{p+1}-1)}{2c_j}\right]}\int_x |fR_j|^2(\tau,x)dxd\tau\right)^{1/2} \\ & \qquad + 2^{(K_2+1)/2}\left(\int_{\tau:\tau\in\left[\frac{2^{K_2}-1}{2c_j},B\right]}\int_x |fR_j|^2(\tau,x)dxd\tau\right)^{1/2} \\ \leq & \sum_{p=K_1}^{K_2} 2^{p/2}\left(\int_{\tau=\frac{2^p-1}{2c_j}}^{\frac{3(2^{p+1}-1)}{2c_j}} \|f(\tau)\|^2_{L^2(1-\psi_0(\tau))}d\tau\right)^{1/2} \\ & \qquad + 2^{(K_2+1)/2}\left(\int_{\tau=\frac{2^{K_2}-1}{2c_j}}^{B} \|f(\tau)\|^2_{L^2(1-\psi_0(\tau))}d\tau\right)^{1/2} \\ \leq & C\sum_{p=K_1}^{K_2+1} 2^{p/2}\left(\int_{\tau=\frac{2^p-1}{2c_j}}^{\frac{3(2^{p+1}-1)}{2c_j}} \frac{d\tau}{\tau^{2+2\delta_0}}\right)^{1/2} \\ \leq & C\sum_{p=K_1}^{K_2+1} 2^{p/2}(2^{p-1}-1)^{-1/2-\delta_0}. \end{aligned}$$

As $2^{p/2}(2^{p-1}-1)^{-1/2-\delta_0} \leq C2^{-p\delta_0}$ and $(2^{K_1}-1)/(2c_j) \geq t/2$, which means $C2^{K_1} \geq t$, we get

$$II \leq C\sum_{p=K_1}^{K_2+1} 2^{-p\delta_0} \leq C2^{-K_1\delta_0} \leq Ct^{-\delta_0}. \tag{3.48}$$

Summing up (3.47) and (3.48) yields the result (3.46). □

Lemmas 3.6 and 3.7 will be used with $f = w_n + U(t)V$, $A = T_0$ and $B = S_n$.

3.6.2 Proof of Proposition 3.3

Proof of Proposition 3.3. From (3.9) and Duhamel formula, $w_n(t)$ satisfies the following integral formulation :

$$w_n(S_n) = U(S_n-t)w_n(t) + \partial_x \int_t^{S_n}\left((w_n(\tau)+U(\tau)V+R(\tau))^5 - \sum_j R_j^5(\tau)\right)d\tau.$$

Compose by $U(t-S_n)$, as recall that $w_n(S_n)=0$, so that

$$\begin{aligned} w_n(t) &= -\partial_x \int_t^{S_n} U(t-\tau)\Big((w_n(\tau)+U(\tau)V+R(\tau))^5 - \sum_j R_j^5(\tau)\Big)d\tau \\ &= -\sum_{k=1}^{5} C_5^k \partial_x \int_t^{S_n} U(t-\tau)\Big((w_n(\tau)+U(\tau)V)^k R^{5-k}(\tau)\Big)d\tau \\ &\quad - \partial_x \int_t^{S_n} U(t-\tau)\Big(R^5(\tau) - \sum_{j=1}^{N} R_j^5(\tau)\Big)ds. \end{aligned} \tag{3.49}$$

We now use the $L_x^5 L_t^{10}$ setting of [15]. According to (3.49), with estimates (3.44) and (3.45), we have

$$\begin{aligned} \|w_n(\tau,x)\|_{C^0([t,S_n],L_x^2)} &+ \|w_n(\tau,x)\|_{L_x^5 L_\tau^{10}(\tau\in[\tau,S_n])} \\ &\le \|(w_n+U(t)V)^5\|_{L_x^1 L_\tau^2(\tau\in[t,S_n])} \\ &\quad + C\sum_{k=1}^{4} C_5^k \|(w_n(\tau)+U(\tau)V)^k R^{5-k}(\tau)\|_{L_x^1 L_\tau^2(\tau\in[t,S_n])} \\ &\quad + C\Big\|R^5(\tau) - \sum_{j=1}^{N} R_j^5(\tau)\Big\|_{L_x^1 L_\tau^2(\tau\in[t,S_n])}. \end{aligned} \tag{3.50}$$

First consider the last term. Recall the simple inequality

$$|z-a|+|z-b| \ge 2\left|z-\frac{a+b}{2}\right| + \frac{|a-b|}{2}.$$

As $|R_j(t,x)| \le Ce^{-\frac{\sqrt{\sigma_0}}{2}|x-x_j-c_jt|}$, we get that for $i\neq j$,

$$|R_i(t,x)R_j(t,x)| \le Ce^{-\sqrt{\sigma_0}\left|x-\frac{x_i+x_j}{2}-\frac{c_i+c_j}{2}t\right|}e^{-\frac{\sqrt{\sigma_0}}{4}|c_i-c_j|t}.$$

As $|c_j-c_j|\ge 2\sigma_0$, we obtain

$$\Big\|R^5(\tau,x) - \sum_{j=1}^{N} R_j^5(\tau,x)\Big\|_{L_x^1 L_\tau^2(\tau\ge t)} \le C(\sigma_0)e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}. \tag{3.51}$$

Now consider the purely linear interaction ($k=5$), that is the first term in (3.50) :

$$\begin{aligned} \|(w_n(\tau)+U(\tau)V)^5\|_{L_x^1 L_\tau^2(\tau\in[t,S_n])} &= \|w_n(\tau)+U(\tau)V\|^5_{L_x^5 L_\tau^{10}(\tau\in[t,S_n])} \\ &\le C\|w\|^5_{\mathcal{N}([t,S_n])} + C\|U(\tau)V\|^5_{L_x^5 L_\tau^{10}(\tau\ge t)}. \end{aligned} \tag{3.52}$$

It remains to control in (3.50) the terms with an interaction between $w_n+U(t)V$ and the solitons.

From (3.42) and (3.43), Lemma 3.7 applies to all the remaining terms in (3.50) (i.e. $k=1,2,3,4$), to give

$$\|(w_n(\tau)+U(\tau)V)^k R^{5-k}(\tau)\|_{L_x^1 L_\tau^2(\tau\in[t,S_n])}$$

$$\leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|w_n(\tau)+U(\tau)V\|_{L^5_xL^{10}_\tau(\tau\in[t,S_n])}+\frac{C}{t^{\delta_0}}$$
$$\leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|w_n\|_{\mathcal{N}([t,S_n])}+Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|U(\tau)V\|_{L^5_xL^{10}_\tau(\tau\geq t)}+\frac{C}{t^{\delta_0}}. \qquad (3.53)$$

(recall that $\|w_n(t)\|_{H^1}$ is uniformly bounded, like $\|U(t)V\|_{H^1}$, so that

$$\|w_n(\tau)+U(\tau)V\|_{L^\infty_{x,\tau}(\tau\in[I_n,S_n])}\leq C$$

uniformly in n). Summing up (3.51), (3.52) and (3.53), and plugging it in (3.50), we obtain

$$\begin{aligned}\|w\|_{\mathcal{N}([t,S_n])} &= \|w_n(\tau,x)\|_{C^0([t,S_n],L^2_x)}+\|w_n(\tau,x)\|_{L^5_xL^{10}_\tau(\tau\in[t,S_n])}\\ &\leq C\|w_n\|^5_{\mathcal{N}([t,S_n]}+Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|w_n\|_{\mathcal{N}([t,S_n])}\\ &\quad +C\|U(\tau)V\|^5_{L^5_xL^{10}_\tau(\tau\geq t)}+Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\|U(\tau)V\|_{L^5_xL^{10}_\tau(\tau\geq t)}+\frac{C}{t^{\delta_0}}. \qquad (3.54)\end{aligned}$$

Then for ε_0 small enough so that $C\varepsilon_0^4\leq 1/3$, and T_0 large enough so that $Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}T_0}\leq 1/3$, we get that

$$\forall t\in[I_n,S_n],\quad \|w_n\|_{\mathcal{N}([t,S_n])}\leq\eta(t), \qquad (3.55)$$

where

$$\eta(t)=C\|U(\tau)V\|^5_{L^5_xL^{10}_\tau(\tau\geq t)}+Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|U(\tau)V\|_{L^5_xL^{10}_\tau(\tau\geq t)}+\frac{C}{t^{\delta_0}}$$

satisfies the conditions of Proposition 1'. □

Appendix

We state a version of the implicit function theorem, to be used in the proof of Lemma 3.3.

Implicit function Theorem with parameter. Let E,F,G,H be Banach spaces, and $f:E\times F\times G\to H$ a C^1 function. Let U be an open set in E. We suppose there exist C^1 functions $x_0,y_0:U\to F,G$ such that for all $t\in U$, $f(t,x_0(t),y_0(t))=0$, and that $\partial_y f(t,x_0(t),y_0(t))$ is invertible.

Furthermore, we suppose that there exist $\delta_0>0$, $\eta_0>0$ such that

$$\sup_{\substack{t\in U\\ x\in B(x_0(t),\delta_0)\\ y\in B(y_0(t),\eta_0)}}\|\partial_y^{-1}f(t,x_0(t),y_0(t))\|\|\partial_y f(t,x_0(t),y_0(t))-\partial_y f(t,x,y)\|=k<1,$$

and

$$\sup_{\substack{t\in U\\ x\in B(x_0(t),\delta_0)\\ y\in B(y_0(t),\eta_0)}}\|\partial_y^{-1}f(t,x_0(t),y_0(t))\|\|\partial_x f(t,x,y)\|=C<\infty.$$

Then there exist $\delta_1,\eta_1>0$ such that the following holds.

Define the tubular neighborhoods $V=\{(t,x)|t\in U,\ x\in B((x_0(t),\delta_1)\}$ and $W=\{(t,y)|t\in U,\ y\in B((y_0(t),\eta_1)\}$.

Then there exists a C^1 function $G:V\to W$, $G(t,x)=(t,g(t,x))$ such that

$$\forall t,\forall x,y\in B(x_0(t),\eta_1)\times B(y_0(t),\eta_1),\qquad f(t,x,y)=0\iff y=g(t,x).$$

Furthermore, $g(t,\cdot)$ is $(C+1)/(1-k)$-Lipschitz (in particular, we can choose $\eta_1 \leq (C+1)/(1-k)\cdot\delta_1$).

Let us conclude with the proof of Proposition 3.4.

Proof of Proposition 3.4. See [29, Lemma 4] and [28, Appendix A], for the proof of a very similar result. The main idea is to use localization arguments on a definite positive operator. Indeed, recall that there exists $\lambda > 0$ such that for all $v \in H^1$,

$$\int (v_x^2 - 5Q^4v^2 + v^2) \geq \lambda \|v\|_{H^1}^2 - \frac{1}{\lambda}\left(\left(\int vQ^3\right)^2 + \left(\int vQ_x\right)^2\right). \tag{3.56}$$

By translation, and denoting

$$\phi(t,x) = \psi(x-(m_{j-1}t-\gamma_j(t))) - \psi(x-(m_jt-\gamma_j(t))),$$

we want to prove that for all $v \in H^1$,

$$\int (v_x^2 - 5Q^4v^2 + v^2)\phi(t) \geq \lambda_1 \int (v_x^2+v^2)\phi(t) - \frac{1}{\lambda_1}\left(\left(\int vQ^3\right)^2 + \left(\int vQ_x\right)^2\right).$$

For simplicity, let us denote

$$H(v,w) = \int (v_xw_x - 5Q^4vw + vw), \quad H_\phi(v,w) = \int (v_xw_x - 5Q^4vw+vw)\phi(t).$$

We begin by the following perturbation lemma concerning (3.56).

Lemma 3.8. *There exists $\delta > 0$ (depending only on $\lambda > 0$) such that if $|(v|Q^3)| + (v|Q_x)| \leq \delta\|v\|_{H^1}$, then*

$$H(v,v) = \int v_x^2 - 5Q^4v^2 + v^2 \geq \frac{\lambda}{2}\int (v_x^2+v^2).$$

Proof. See [56], where an analoguous case is treated. Let us decompose in L^2 : $v = v_1 + aQ^3 + bQ_x = v_1 + v_2$, $(v_1|Q^3) = (v_1|Q_x) = 0$, so that by hypothesis,

$$|a| + |b| \leq \delta\|v\|_{H^1}.$$

If $\delta < 1/2$, we deduce that

$$\frac{\sqrt{3}}{2}\|v\|_{H^1} \leq \|v_1\|_{H^1} \leq 2\|v\|_{H^1}.$$

Now $H(v,v) = H(v_1,v_1) + H(v_2,v_2) + 2H(v_1,v_2)$. By (3.56),

$$H(v_1,v_1) \geq \lambda\|v_1\|_{H^1}^2 \geq \frac{3\lambda}{4}\|v\|_{H^1}^2.$$

If $C\delta^2 \leq \lambda/8$ (where C is the continuity norm of H),

$$|H(v_2,v_2)| \leq C(|a|^2+|b|^2) \leq C\delta^2\|v\|_{H^1}^2 \leq \frac{\lambda}{8}\|v\|_{H^1}^2,$$

and if $4C\delta \leq \lambda/8$,

$$2|H(v_1,v_2)| \leq C\|v_1\|_{H^1}\|v_2\|_{H^1} \leq 4C(|a|+|b|)\|v\|_{H^1} \leq 2C\delta\|v\|_{H^1}^2 \leq \frac{\lambda}{8}\|v\|_{H^1}^2.$$

Finally :

$$H(v,v) \geq \left(\frac{3}{4}-\frac{1}{8}-\frac{1}{8}\right)\lambda\|v\|_{H^1}^2 \geq \frac{\lambda}{2}\|v\|_{H^1}^2.$$

□

Lemma 3.9. *There exists $\sigma_1 > 0$ such that the following is true. Given $\sigma_0 < \sigma_1$, there exists $T_3 = T_3(\sigma)$, such that if $(v,Q^3) = (v,Q_x) = 0$, then*

$$H_\phi(v,v) \geq \frac{\lambda}{4}\int (v_x^2+v^2)(\phi).$$

Proof. Notice that $|(\phi)_x| \leq C\sqrt{\sigma_0}\phi$, the constant C not depending on t,x,σ_0 (it is a computation involving ψ). Let $v \in H^1$ such that $(v|Q^3) = (v|Q_x) = 0$. Now let us compute with $v\sqrt{\phi}$:

$$\begin{aligned} H_\phi(v,v) &= \int (v_x^2 - 5Q^4v^2 + v^2)\phi \\ &= \int ((v\sqrt{\phi})_x^2 - 5Q^4(v\sqrt{\phi})^2 + (v\sqrt{\phi})^2) - \int vv_x\phi_x - \int v^2(\sqrt{\phi})_x^2 \\ &= H(v\sqrt{\phi}, v\sqrt{\phi}) - \int vv_x\phi_x - \int v^2(\sqrt{\phi})_x^2. \end{aligned}$$

Thanks to the orthogonality properties on v,

$$\left|\int v\sqrt{\phi}Q^3\right| = \left|\int v(1-\sqrt{\phi})Q^3\right| \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|v\|_{L^2} \text{ and,}$$
$$\left|\int v\sqrt{\phi}Q_x\right| = \left|\int v(1-\sqrt{\phi})Q_x\right| \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|v\|_{L^2}.$$

Thus, as soon as $t \geq T_3 = T_3(\sigma_0)$ is large enough, we are in the setting of the previous lemma, and so

$$H(v\sqrt{\phi}) \geq \frac{\lambda}{2}\|v\sqrt{\phi}\|_{H^1}^2 = \frac{\lambda}{2}\left(\int (v_x^2+v^2)\phi + \int vv_x\phi_x + \int v^2(\sqrt{\phi})_x^2\right).$$

Thus :

$$H_\phi(v,v) \geq \frac{\lambda}{2}\int (v_x^2+v^2)\phi + \left(\frac{\lambda}{2}-1\right)\int vv_x\phi_x + \left(\frac{\lambda}{2}-1\right)\int v^2(\sqrt{\phi})_x^2.$$

We need to control the last two terms. First,

$$\left|\int vv_x\phi_x\right| \leq C\sqrt{\sigma_0}\int |vv_x|\phi \leq C\sqrt{\sigma_0}\int (v_x^2+v^2)\phi,$$

and in the same way, as $|(\sqrt{\phi})_x| \leq C\sqrt{\sigma_0}\sqrt{\phi}$,

$$\int v^2(\sqrt{\phi})_x^2 \leq C\sigma_0\int v^2\phi.$$

Choose $\sigma_1 \leq \frac{1}{8C}\frac{\lambda_1}{1-\lambda_1/2}$, then for $t \geq T(\sigma_0)$ large enough, we get

$$H_\phi(v,v) \geq \frac{\lambda}{4}\int (v_x^2+v^2)(\phi),$$

as claimed. □

We can now conclude the proof of Proposition 3.4. Let $v \in H^1$. Let us write the L^2 decomposition $v = v_1 + aQ + bQ_x$, and develop :

$$H_\phi(v,v) = H_\phi(v_1,v_1) + 2aH_\phi(v_1,Q) + 2bH_\phi(v_1,Q_x) \\ + a^2H_\phi(Q,Q) + 2abH_\phi(Q,Q_x) + b^2H_\phi(Q_x,Q_x).$$

By hypothesis, $H_\phi(v_1,v_1) \geq \lambda_2 \int (v_x^2+v^2)\phi(t)$. Then, notice that $|a| = |(v|Q)| \leq C\|v\|_{L^2}$ and $|b| \leq \|v\|_{L^2}$: thus,

$$\|v_1\|_{H^1} \leq C\|v\|_{H^1}.$$

Now $H(v_1,Q) = (v_1,\lambda_Q Q) = 0$ as Q (and Q_x) are eigenfunctions, and H_ϕ is symmetric : we deduce $H_\phi(v_1,aQ) = H_{1-\phi}(v_1,aQ)$ etc. And we get, by continuity of H,

$$|H_\phi(v_1,aQ)| + |H_\phi(bQ_x,aQ)| \leq C\|v\|_{H^1}^2\|(|Q_x|+Q)(1-\phi(t))\|_{L^\infty}.$$

And of course $\|(|Q_x|+Q)(1-\phi(t))\|_{L^\infty} \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}$. As $|\phi| \leq 1$, $|H_\phi(Q,Q)| \leq \|Q\|_{H^1}^2$ and $|H_\phi(Q_x,Q_x)| \leq \|Q_x\|_{H^1}^2$. This gives

$$H_\phi(v,v) \geq \lambda_2\int (v_{1x}^2+v_1^2)\phi(t) - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\|v\|_{H^1}^2 - C(a^2+b^2).$$

Now,

$$\int (v_{1x}^2+v_1^2)\phi(t) \geq \int (v_x^2+v^2)\phi(t) - C(a^2+b^2) \\ - 2a\int (v_xQ_x+vQ)\phi(t) - 2b\int (v_xQ_{xx}+vQ_x)\phi(t).$$

But

$$\left|a\int (v_xQ_x+vQ)\phi(t)\right| \leq C|a|\left(\int (v_x^2+v^2)\phi(t)\right)^{1/2} \leq C^2a^2 + \frac{1}{4}\int (v_x^2+v^2)\phi(t).$$

Doing the same for $\int (v_xQ_{xx}+vQ_x)\phi(t)$, we get

$$\int (v_{1x}^2+v_1^2)\phi(t) \geq \frac{1}{2}\int (v_x^2+v^2)\phi(t) - C(a^2+b^2),$$

so that finally,

$$H_\phi(v,v) \geq \lambda_2/2\int (v_x^2+v^2)\phi(t) - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\|v\|_{H^1}^2 - C(a^2+b^2).$$

Choosing T_1 so that $Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}T_1} \leq \lambda_2/4$, as $t \geq T_1$, this gives :

$$H_\phi(v,v) \geq \lambda_2/2\int (v_x^2+v^2)\phi(t) - C(a^2+b^2).$$

□

Chapitre 4

Construction of solutions to the subcritical gKdV equations with a given asymptotic behavior[1]

4.1 Introduction

4.1.1 General setting

We consider the following sub-critical generalized Korteweg-de Vries equation :

$$u_t + (u_{xx} + u^4)_x = 0, \quad t, x \in \mathbb{R}. \tag{4.1}$$

It is a special case of the generalized Korteweg-de Vries equation :

$$u_t + (u_{xx} + u^p)_x = 0, \quad t, x \in \mathbb{R}, \tag{4.2}$$

where $p \geq 2$. The case $p = 2$ corresponds to the original equation introduced by Korteweg and de Vries [19] in the context of shallow water waves. For both $p = 2$ and $p = 3$, this equation has many applications to Physics : see for example Miura [39], Lamb [22].

There are two formally conserved quantities for solutions to (4.2) :

$$\int u^2(t) = \int u^2(0) \qquad (L^2 \text{ mass}), \tag{4.3}$$

$$E(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) = E(u(0)) \qquad (\text{energy}). \tag{4.4}$$

The local Cauchy problem for (4.2) has been intensively studied by many authors. Kenig, Ponce and Vega [14] proved the following existence and uniqueness result in $H^1(\mathbb{R})$: for $u_0 \in H^1(\mathbb{R})$, there exist $T = T(\|u_0\|_{H^1}) > 0$ and a solution $u \in C([0,T], H^1(\mathbb{R}))$ to (4.1) satisfying $u(0) = u_0$, which is unique in some class $Y_T \subset C([0,T], H^1(\mathbb{R}))$. For such a solution, one has conservation of mass and energy. Moreover, if T_1 denotes the maximal time of existence for u, then either $T_1 = +\infty$ (global solution) or $T_1 < \infty$ and $\|u(t)\|_{H^1} \to \infty$ as $t \uparrow T_1$ (blow-up solution).

[1] Ce chapitre a fait l'objet d'une prépublication à paraître dans *Journal of Functional Analysis*.

In the case $2 \leq p < 5$, all solutions to (4.2) in H^1 are global and uniformly bounded thanks to the conservation laws and the Gagliardo-Nirenberg inequality :

$$\forall v \in H^1(\mathbb{R}), \quad \int |v|^{p+1} \leq C(p) \left(\int v^2\right)^{\frac{p+3}{4}} \left(\int v_x^2\right)^{\frac{p-1}{4}}. \tag{4.5}$$

The case $p = 5$ is L^2-critical, in the sense that the mass remains unaffected by scaling. If

$$u_t + (u_{xx} + u^5)_x = 0, \quad t, x, \in \mathbb{R}, \tag{4.6}$$

then $u_\lambda(t,x) = \lambda^{1/6} u(\lambda t, \lambda^{1/3} x)$ is also a solution to (4.6), and $\|u_\lambda\|_{L^2} = \|u\|_{L^2}$. In this case, the local existence result of [14] is improved to initial data in L^2 (instead of H^1). However, existence of finite time blow-up solutions was proved by Merle [32] and Martel and Merle [28]. Therefore $p = 5$ also appears as a critical exponent for the long time behavior of solutions to (4.2).

A fundamental property of (4.2) is the existence of a family of explicit traveling wave solutions. If Q denotes the only solution (up to translation) of :

$$Q > 0, \quad Q \in H^1(\mathbb{R}), \quad Q_{xx} + Q^p = Q, \quad \text{i.e.} \quad Q(x) = \left(\frac{p+1}{2\cosh^2(\frac{p-1}{2}x)}\right)^{\frac{1}{p-1}},$$

then for $c > 0$ the soliton

$$R_{c,x_0} = c^{\frac{1}{p-1}} Q(\sqrt{c}(x - x_0 - ct)) \text{ is a solution to (4.2).}$$

For $p = 2$ and $p = 3$, equation (4.2) is completely integrable, and thus has very special features. The inverse scattering transform method allows to solve the Cauchy problem in an appropriate space (for example if $u_0 \in H^4$ and $xu_0 \in L^1$) and the qualitative behaviour of solutions is well understood. For example, given u_0 smooth and with rapid decay, there exist N solitons R_{c_j,x_j} such that

$$\left\| u(t) - \sum_{j=1}^N R_{c_j,x_j}(t) \right\|_{L^\infty(x \geq -t^{1/3})} \leq \frac{C}{t^{1/3}} \quad (\text{as } t \to \infty).$$

See for example Schuur [43], Eckhaus and Schuur [9], Miura [39].

However, if $p \neq 2$ or 3, the inverse scattering transform method does not longer apply, and the description of solutions in the general, non-integrable case is an open problem. It can be decomposed in two types of problems.

Problem 1 : Asymptotic behaviour. Given an initial data u_0, does the out coming solution $u(t)$ to (4.2) exists for all time ? If it does (for example in the subcritical case), can its behavior be described, as $t \to \infty$? If blow up happens, can the blow up rate and profile be determined ?

Problem 2 : Non-linear wave operator. Given some reasonable behaviour as $t \to \infty$, can we find a solution $u(t)$ to (4.2) defined for large enough t, with this behaviour ? Is there uniqueness for $u(t)$?

4.1.2 Recent results on Problems 1 and 2

Let us now develop some recent results which will be the base to our result. We denote $U(t)$ the linear operator for KdV equation, i.e. $v(t) = U(t)V$ satisfies $v_t + v_{xxx} = 0$, $v(0) = V$.

The first result deals with scattering for small initial data, a problem studied by many authors (see for example [48], [40], [4], [13]). Let us recall the result of Hayashi and Naumkin [13]. Introduce the following weighted Sobolev spaces :

$$H^{s,m} = \{\phi \in \mathcal{S}' |\, \|\phi\|_{H^{s,m}} = \|(1+|x|^2)^{m/2}(1-\partial_x^2)^{s/2}\phi\|_{L^2} < \infty\}. \tag{4.7}$$

Scattering operator. Let $p > 3$. Given u_0 small enough in $H^{1,1}$, the solution $u(t)$ to (4.2) is global in time, and there is scattering, in the sense that there exists a function $V \in L^2$ so that :

$$\|u(t) - U(t)V\|_{L^2} \to 0 \quad \text{as} \quad t \to \infty.$$

Furthermore, $\|u(t)\|_{L^\infty} \leq Ct^{-1/3}$ (linear decay rate).

This is the description of solutions with initial data around 0 (in $H^{1,1}$), a result which can be understood as stability around 0.

The second type of results we want to focus on is that which describes the solutions around solitons or a sum of solitons. The following result of Martel, Merle, Tsai [29] solves the problem of stability in H^1 of a sum of N decoupled solitons (see also Martel and Merle [25]).

Stability of the sum of N solitons. Suppose $p = 2, 3$ or 4. Let $N \in \mathbb{N}$, and $0 < c_1 < \ldots < c_N$. There exist γ_0 and α_0 (small) and A, L_0 (large), so that the following is true. Assume that there exist $L \geq L_0$, $\alpha < \alpha_0$ and $x_1^0 < \ldots < x_N^0$ such that :

$$\left\| u(0) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j^0) \right\|_{H^1} \leq \alpha, \quad \text{with} \quad x_j^0 \geq x_{j-1}^0 + L, \quad \text{for } j = 2, \ldots, N.$$

Then there exist $x_1(t), \ldots, x_N(t) \in \mathbb{R}$ such that :

$$\forall t \geq 0, \quad \left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j(t) - c_j t) \right\|_{H^1} \leq A(\alpha + e^{-\gamma_0 L}).$$

These results are related to Problem 1. Let us now turn to results concerning Problem 2. First, Martel [23] proved the existence and uniqueness of N-solitons in the cases $p = 2, 3, 4$ or 5 :

Existence and uniqueness of the N-soliton. Let $p \in [2, 5]$. Let $N \in \mathbb{N}$, $0 < c_1 < \ldots < c_N$, and $x_1, \ldots, x_N \in \mathbb{R}$. There exist $T_0 \in \mathbb{R}$ and a unique function $u \in C([T_0, +\infty), H^1)$ solution to (4.1), and such that

$$\left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j - c_j t) \right\|_{H^1} \to 0 \quad \text{as} \quad t \to \infty.$$

Furthermore, $u \in C^\infty([T_0,\infty)\times\mathbb{R})$ and convergence takes place in H^s for all $s \geq 0$, with an exponential decay :

$$\exists \gamma > 0,\ \forall s \geq 0, \exists A_s \quad / \quad \left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j - c_j t) \right\|_{H^1} \leq A_s e^{-\gamma t}.$$

This result appears as a development of monotonicity properties and a dynamical argument, ideas which where used by Martel and Merle [25] and Martel, Merle and Tsai [29].

However, it is a surprise that the method could be adapted even to the critical case $p = 5$, although it is well known that solitons are unstable in $H^1(\mathbb{R})$: there is in fact blow-up for a large class of initial data and the blow-up profile is stable, see [26], [28], [32], [27]. Another surprise is uniqueness of the N-soliton.

Notice that in view of this result, the stability of a sum of N solitons can be interpreted as stability of the N-soliton (solution to (4.2)).

The last result solves the case of a linear behavior, that is the existence of a wave operator :

Large data wave operator. Let $p > 3$, and $V \in H^{2,2}$. There exist $T_0 \in \mathbb{R}$ and $u \in C([T_0,\infty), H^1)$ solution to (4.2) such that :

$$\|u(t) - U(t)V\|_{H^1} \to 0 \quad as \quad t \to \infty.$$

Furthermore u is unique in an adapted class.

In the same way that the result of Martel [23] was based on considerations of Martel, Merle and Tsai [29], this result strongly relies on the analysis of Hayashi and Naumkin [13].

4.1.3 Statement of the main result

Our goal is to construct solutions which behave like a sum of a linear term $U(t)V$, and of N solitons, in the subcritical $p < 5$ case. Notice that in [5] such solutions are constructed in the critical case $p = 5$. More precisely, given $0 < c_1 < \ldots < c_n$ and $x_1, \ldots x_N \in \mathbb{R}$, we would like to construct solutions $u(t)$ to (4.2), defined for large enough times and such that

$$\left\| u(t) - U(t)V - \sum_{j=1}^{N} R_{c_j,x_j}(t) \right\|_{H^1} \to 0 \quad \text{as} \quad t \to \infty.$$

In this article, we construct such solutions in the case $p = 4$ (that is, for equation (4.1)), provided that V is smooth enough, with sufficient decay on the right. From now on and throughout the rest of the article,

$$\text{we focus on the sub-critical case } p = 4. \tag{4.8}$$

Let us first recall the functional setting which will be used throughout the proofs. Fix once for all the three constants :

$$\gamma \in (0, 1/3), \quad \alpha = \frac{1}{2} - \gamma \in (0, 1/2) \ \text{ and } \ \delta = \frac{1-2\gamma}{3} > 0. \tag{4.9}$$

(γ is arbitrary). These constants are those of [13] in the case $p = 4$.

Again as in [13], we will use the notation $D = \partial_x = \frac{\partial}{\partial x}$ for the partial differentiation with respect to the space variable x, and

$$D^\alpha f = \mathcal{F}^{-1}\xi^\alpha e^{-(i\pi/2)(1+\alpha)}\hat{f},$$

along with the two following operators

$$J^t f = U(t)xU(-t)f = (x - 3t\partial_x^2)f, \text{ and } I^t\phi = x\phi + 3t\int_{-\infty}^{x}\partial_t f(t,y)dy.$$

We write J^t and I^t so as to emphasize that we will always consider norms at a fixed time t although J^t and I^t are space-time operators.

Our working spaces will be defined through the time dependent M_0^t norm :

$$\mathcal{H}_t = \{f \in L^2(\mathbb{R}) |\, M_0^t(f) = \|f\|_{H^1} + \|DJ^t f\|_{L^2} + \|D^\alpha J^t f\|_{L^2} < \infty\}.$$

J^t only appears in the norm, as it is convenient to do linear estimates (see [13], Lemma 2.3). But we introduced I^t because it is easier to handle when doing energy methods estimates. Notice that M_0^0 is very similar to $\|\cdot\|_{H^{1,1}}$.

We will finally use the following notation for weighted spaces : for a positive function h,

$$\|f\|^2_{H^s(h)} = \int |(Id - \Delta)^{s/2} f|^2(x)h(x)dx.$$

Following a usual convention, different positive constants might be denoted by the same letter C.

Our main result is the following.

Theorem 4.1 (Nonlinear wave operator)**.** *Let $V \in H^{5,1} \cap H^{2,2}$ be such that :*

$$x_+^{4/3}D^5V \in L^2, \qquad x_+^8 V \in H^1,$$

(where $x_+ = \max\{0, x\}$). Let $N \in \mathbb{N}$, $0 < c_1 < \ldots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. Denote $R_j(t,x) = Q_{c_j}(x - x_j - c_j t)$ N solitons.

Then there exists $u^* \in C([T_0, +\infty), H^4 \cap \mathcal{H}_0^t)$, for some $T_0 \in \mathbb{R}$, solution to (4.1), such that if we introduce :

$$w^*(t) = u^*(t) - U(t)V - \sum_{j=1}^{N} R_j(t),$$

we have

$$\|w^*(t)\|_{H^4} + M_0^t(w^*(t)) \to 0 \quad \text{as} \quad t \to \infty.$$

Furthermore, we have the following decay rate :

$$\|w^*(t)\|_{H^4} \leq Ct^{-1/3}, \qquad M_0^t(w^*(t)) \leq Ct^{-\delta}.$$

***Remark* 4.1.** *This result allows to work with large data (V large in L^2), which is both surprising and satisfactory. However, it deals with smooth and decaying data. A natural setting would be a result with $V \in H^1$, and some decay on the right to ensure low interaction with the solitons. Theorem 4.1 should be viewed as a step in the solving process of Problem 2.*

An important change in the method of proof when considering [23] is the following. Solitons have an exponential decay, and so integrability (in time) is always automatic. Here the linear term $U(t)V$ will interfere with the solitons to produce a polynomial decay in time, and this will require taking care of.

Similarly, when handling the linear term $U(t)V$ (following the framework of [8]), we will have to take care of the interference of the solitons.

***Remark* 4.2.** *This result is similar to [5], where a non-linear wave operator is constructed in the L^2 critical case $p = 5$.*

In both cases, the scheme of proof first dwells on the interaction with the solitons, and in a second step uses arguments from the linear scattering theory to control the interaction with the linear term (along with the results obtained in the first step). The argument for the soliton interaction is very similar in the case $p = 4$ and in the case $p = 5$. However, the second step is very different.

For $p = 5$, the linear scattering theory of Kenig, Ponce and Vega [15] is available : it is done in L^2, and so requires much less smoothness and decay on V. The main difficulty is to mix both approaches, as the soliton theory relies on an analysis in $C^0_t H^1_x$, and the natural space in the theory of [15] is $L^5_x L^{10}_t$: in particular, solitons do not belong to this space (nor to $L^5_x L^{10}_{t\geq T}$ for any T). The problem is then to separate the linear analysis from the non-linear one, and when considering the interference of one over the other, to be able to interchange integrals in time and in space in an adequate way. This can be done with a small loss in the decay, with respect to the optimal result one can expect using this method.

In the non-critical case, the scattering analysis of [15] is no longer available, and we have to relie on the theory of Naumkin and Hayashi [13]. Their method breaks down at some point, when taking care of the interference between the solitons and the linear term. However, we manage to recover the gap by energy method arguments, and this is why we have to reinforce the assumptions on V, and obtain a stronger convergence (H^4). Our method could be adapted also to the critical case, but would give a much less sharp result than what is obtained in [5].

The problem of the uniqueness of solutions behaving as the sum of a linear term and N soliton is an open question, in both the critical and sub-critical case. Recall that if $V = 0$, one has uniqueness in H^1 (see [23]) : this result is linked with very fast convergence of the constructed solution to its profile not only in H^1 but in H^4. However, it seems that one can not derive easily from this work a proof for $V \neq 0$.

***Remark* 4.3.** *Theorem 4.1 is valid only for $p = 4$ for two main reasons. First, it contains the existence of a scattering operator, so that $p > 3$. Second, it also contains the existence of a N-soliton, which is only true for $p \leq 5$. The fact that our setting only deals with integer p comes from our crucial use of the regularity of the non-linearity function $x \mapsto x^p$ and also from better integrability properties (if $p \geq 4$ instead of $p > 3$).*

However, one can prove an analoguous result for $p = 5$, but that one would be much less precise than we is stated in [5].

***Remark* 4.4.** *There are some related results for the (critical) non-linear Schrödinger equation (NLS) : due to the pseudo-conformal transform, the construction of a non-linear wave operator is equivalent to the construction of solutions which blow up with a given behavior. For (NLS), the results are expressed in the latter form, see Bourgain and Wang [3], Krieger and Schlag [21], Merle [30]. In this case also do conditions on the linear term regarding smoothness and low interaction with the solitons appear.*

In Section 2, we give a detailed outline of the proof of Theorem 4.1, decomposing it into steps : each of these step is summarized in a proposition. In Section 3, we give some preliminary results and each of the following sections is devoted to the proof of one of the propositions stated in Section 2.

4.2 Outline of the proof

Let $V \in H^{5,1} \cap H^{2,2}$ such that $x_+^8 D^5 V \in L^2$ and $x_+^{4/3} V \in H^1$. Let $0 < c_1 < \ldots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. Denote the soliton with speed c_j and shift x_j :

$$R_j(t,x) = Q_{c_j}(x - x_j - c_j t).$$

Define also $R(t) = \sum_{j=1}^N R_j(t)$.

Let S_n be an increasing sequence of time, so that $S_n \to \infty$ as $n \to \infty$. For $n > 0$, we define $u_n(t)$, the solution to

$$\begin{cases} u_{nt} + (u_{nxx} + u_n^4)_x = 0, \\ u_n(S_n) = U(S_n) + R(S_n). \end{cases} \tag{4.10}$$

Equivalently, we introduce the error term

$$w_n(t) = u_n(t) - U(t)V - R(t),$$

so that $w_n(t)$ satisfies the equation

$$\begin{cases} w_{nt} + w_{nxxx} + \left(u_n^4 - \sum_{j=1}^N R_j^4 \right)_x = 0, \\ w_n(S_n) = 0. \end{cases} \tag{4.11}$$

As $u_n(S_n) \in H^1$, $u_n \in C_b(\mathbb{R}, H^1)$; the same thing is true for $w_n(t)$.

The heart of the proof of Theorem 4.1 is the following result.

Proposition 4.1 (Uniform estimates)**.** *There exists T_0 such that for all n such that $S_n \geq T_0$, the solution $u_n(t)$ to (4.10) and the solution $w_n(t)$ to (4.11) belong to $C([T_0, S_n], \mathcal{H}_0^t \cap H^4)$. Furthermore, we have*

$$\forall t \in [T_0, S_n], \quad \|w_n(t)\|_{H^4} \leq C_0 t^{-1/3}, \quad M_0^t(w_n(t)) \leq C_0 t^{-\delta}, \tag{4.12}$$

for some constant C_0 not depending on n (recall $\delta > 0$ is introduced in (4.9)).

The proof of this proposition requires several steps.

The first remark allows us to further assume smallness on $w_n(t)$, in order to get the decay (4.12).

Proposition 1' (Reduction of proof)**.** *There exist $\varepsilon_0 > 0$, C_0, and $T_0 \geq 1$ with $2C_0 T_0^{-\delta} \leq \varepsilon_0$ such that the following is true, for all $n \in \mathbb{N}$. Suppose that there exists $I_n \in [T_0, S_n]$ such that*

$$\forall t \in [I_n, S_n], \quad \|w_n(t)\|_{H^4} + M_0^t(w_n(t)) \leq \varepsilon_0.$$

Then in fact

$$\forall t \in [I_n, S_n], \quad \|w_n(t)\|_{H^4} \leq C_0 t^{-1/3}, \quad M_0^t(w_n(t)) \leq C_0 t^{-\delta}.$$

Proof of Proposition 4.1 assuming Proposition 1'. Let $T_0 = \max\{1, C_0^{1/\delta}\varepsilon_0\}$, and define

$$I_n^* = \inf_{t^* \in [1,S_n]} \{t^* | \ \forall t \in [t^*, S_n], \quad \|w_n(t)\|_{H^4} + M_0^t(w_n(t)) \le \varepsilon_0\}.$$

As $w_n(S_n) = 0$, by upper semi-continuity of the norm of the flow (see [8, Appendix B]), we obtain that the set on which we do the infimum is non-empty, so that $I_n^* < S_n$.

Then of course, for all $t \in (I_n^*, S_n]$, $\|w_n(t)\|_{H^4} + M_0^t(w_n(t)) \le \varepsilon_0$. This allows us to apply Proposition 1' so that

$$\forall t \in (I_n^*, S_n], \qquad \|w_n(t)\|_{H^4} \le C_0 t^{-1/3}, \quad M_0^t(w_n(t)) \le C_0 t^{-\delta}. \tag{4.13}$$

If $I_n^* > 1$, we also get that $\limsup_{t \downarrow I_n^*} \|w_n(t)\|_{H^4} + M_0^t(w_n(t)) \ge \varepsilon_0$ (from the minimality of I_n^*). In particular, this gives

$$\varepsilon_0 \le \limsup_{t \downarrow I_n^*} \|w_n(t)\|_{H^4} + M_0^t(w_n(t)) \le 2C_0 {I_n^*}^{-\delta}.$$

So that $I_n^* \le \varepsilon_0/(2C_0))^{-1/\delta}$. In any case, we get that $I_n^* \le T_0$: (4.13) allows us to conclude. □

Thus, our goal is now to prove Proposition 1'.

Proof of Proposition 1'.
Step 1 : Monotonicity and non-linear tools. We obtain H^1 estimates on the right. Let us intoduce the cut-off speed

$$\sigma_0 \in (0, \min\{c_1, c_2 - c_1, \dots, c_N - c_{N-1}\}), \tag{4.14}$$

to be determined in the proof of the following Proposition 4.2, and the cut-off function

$$\psi(x) = \frac{2}{\pi} \arctan\left(\exp\left(-\frac{\sqrt{\sigma_0}}{2}x\right)\right), \qquad \psi_0(t,x) = \psi(x - \sigma_0 t - 2|x_1|). \tag{4.15}$$

$\psi_0(t)$ allows us to separate the solitons interaction from the $U(t)V$ interaction.

Proposition 4.2 (Interaction with the solitons)**.** *There exist $\sigma_1 > 0$, $\varepsilon_1 > 0$, C_1 and T_1 such that the following is true. If $\sigma_0 \le \sigma_1$, $\varepsilon_0 \le \varepsilon_1$ and $T_0 \ge T_1$, then, for all $n \in \mathbb{N}$ and all $t \in [I_n, S_n]$,*

$$\begin{aligned}\|w\|_{H^1(1-\psi_0(t))} \le{}& C_1 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} + C_1\|U(t)V\|_{H^1(1-\psi_0(t))} \\ &+ C_1(S_n - t + 1)\|U(t)V\|_{L^2(1-\psi_0(S_n))} + C_1\int_t^{S_n} \|U(t)V\|_{H^1(1-\psi_0(t))}dt.\end{aligned}$$

Observe that this proposition in fact holds for all $p \in [2,5]$; however, we will only do it for $p = 4$.

Essentially we obtain a polynomial decay on $\|w_n(t)\|_{H^1(1-\psi_0(t))}$ (instead of an exponential decay in the case of solely soliton). However the good point is that we can choose this polynomial decay to be as fast as we want by lowering the interaction of $U(t)V$ with the solitons, that is, by requiring sufficicient decay on the right for V.

Now we would like to complete the M_0^t estimate. The construction of [13] relies on a very nice cancellation involving the operators J^t and I^t, which allows a bootstrap in $\mathcal{H}_0^t$. Here, this nice clockwork breaks down because of the interaction with the solitons R_j (the precise term that arise will be treated in full detail in the proof of the final step 4). We therefore are forced to work in H^3 which is the more natural space where all the computations of [13] are done (of course in H^3, the bootstrap of [13] doesn't work anymore because of a lack of information).

We need a good control on the interaction with the soliton at the H^3 level : more precisely (this will be done in full detail in subsection, we need $t\|w_n\|_{H^3(1-\psi_0(t))}$ be integrable in time. This can not be achieved by improving Proposition 4.2 to H^3, as its proof is done through considerations at H^1 level. This is why we go up to H^4 : with a weak control on $\|w_n\|_{H^4}$, and a strong control on $\|w_n\|_{H^1(1-\psi_0(t))}$, we obtain by interpolation the desired control on $\|w_n\|_{H^3(1-\psi_0(t))}$. Indeed, we have the following corollary to Proposition 4.2, in which we estimate some quantities which we will need later on.

Corollary 4.1. *Suppose $V \in H^{5,1} \cap H^{2,2}$ is such that*

$$x_+^{4/3} D^5 V \in L^2, \quad \textit{and} \quad x_+^8 V \in H^1.$$

Then for some $C_1' > 0$, we have, for all $n \in \mathbb{N}$ and for all $t \in [I_n, S_n]$,

$$t\|w_n(t)\|_{H^3(1-\psi_0(t))} + t\|U(t)V\|_{H^2(1-\psi_0(t))} + \|U(t)V\|_{H^5(1-\psi_0(t))} + \|U(t)(xV_x)\|_{H^1(1-\psi_0(t))} \leq \frac{C_1'}{t^{4/3}}.$$

Proof. We combine the result of Proposition 4.2 and Lemma 4.3. First observe that from Lemma 4.3, our assumptions translate to

$$\|D^5 U(t)V\|_{L^2(1-\psi_0(t))} \leq Ct^{-4/3}, \tag{4.16}$$

$$\|U(t)V\|_{L^2(1-\psi_0(t))} + \|U(t)V_x\|_{L^2(1-\psi_0(t))} \leq Ct^{-8}. \tag{4.17}$$

So that by interpolation of (4.16) and (4.17),

$$\|U(t)V\|_{H^5(1-\psi_0(t))} \leq Ct^{-4/3}.$$

Again by interpolation, we obtain

$$\|U(t)V\|_{H^2(1-\psi_0(t))} \leq \|U(t)V\|_{H^1(1-\psi_0(t))}^{3/4} \|U(t)V\|_{H^5(1-\psi_0(t))}^{1/4} \leq \frac{C}{t^{\frac{3}{4}\cdot 8}} \cdot \frac{C}{t^{\frac{1}{4}\cdot\frac{4}{3}}} \leq \frac{C}{t^{19/3}} \leq \frac{C}{t^{7/3}}.$$

Now, by Proposition 4.2 and (4.17), we get

$$\|w_n(t)\|_{H^1(1-\psi_0(t))} \leq \frac{C}{t^7}.$$

Recall that $\|w_n(t)\|_{H^4} \leq \varepsilon_0$, so that by interpolation

$$\begin{aligned}\|w_n(t)\|_{H^3(1-\psi_0(t))} &\leq C\|w_n(t)\|_{H^1(1-\psi_0(t))}^{1/3} \|w_n(t)\|_{H^4(1-\psi_0(t))}^{2/3} \\ &\leq \frac{C}{t^{7/3}} \|w_n(t)\|_{H^4}^{2/3} \leq \frac{C}{t^{7/3}}.\end{aligned}$$

For the xV_x estimate : first notice that

$$\begin{aligned}\int V_{xx}^2(x)x_+^{14/3}dx &= \int_0^\infty V_{xx}^2 x^{14/3}dx \\ &= -\int_0^\infty V_{xxx}V_x x^{14/3}dx - \int_0^\infty V_{xx}V_x x^{11/3}dx \\ &\le \left(\int_0^\infty V_{xxx}^2 x^{8/3}dx \int_0^\infty V_x^2 x^{20/3}dx\right)^{1/2} \\ &\qquad + \left(\int_0^\infty V_{xx}^2 x^{8/3}dx \int_0^\infty V_x^2 x^{14/3}dx\right)^{1/2} \\ &\le \|x_+^{4/3}V_{xxx}\|_{L^2}\|x_+^{10/3}V_x\|_{L^2} + \|x_+^{4/3}V_{xx}\|_{L^2}\|x_+^{7/3}V_x\|_{L^2}.\end{aligned}$$

As $V \in H^{2,2}$, $xV_x \in H^1$, and moreover,

$$\int \left((xV_x)^2 + |D(xV_x)|^2\right) x_+^{8/3}dx \le \int \left(V_x^2 + V_{xx}^2\right)(1 + x_+^{14/3})dx,$$

so that

$$\|(1 + x_+^{7/3})(xV_x)\|_{H^1}^2 \le \|(1 + x_+^{10/3})V\|_{H^1}\|(1 + x_+^{4/3})V\|_{H^3}.$$

From our H^5 estimate and $(1 + x_+^8)V \in H^1$, we get

$$\|U(t)(xV_x)\|_{H^1(1-\psi_0(t))} \le Ct^{-7/3}\|(1 + x_+^{7/3})(xV_x)\|_{H^1} \le Ct^{-7/3}. \qquad \square$$

Step 2 : Energy method estimates. Now that we have assumed H^4 control, we have to obtain H^4 uniform decay.

Proposition 4.3 (Interaction with the linear term, H^4 bounds)**.** *There exists C_2 such that $\forall n \in \mathbb{N}$, $\forall t \in [I_n, S_n]$,*

$$\|w_n(t)\|_{H^4} \le \frac{C_2}{t^{1/3}}.$$

First consider L^2 and H^1 estimates. We want to control what happens in the zone $x < \sigma_0 t$, that is the interaction with the linear term $U(t)V$: we follow the framework of [8]. The crucial point is to use our a priori control on $M_0^t(w_n(t))$. We have

$$\begin{aligned}|w_n(t,x)| &\le \frac{C}{t^{1/3}}\left(1 + \frac{|x|}{\sqrt[3]{t}}\right)^{-1/4} M_0^t(w_n(t)), \\ |w_{nx}(t,x)| &\le \frac{C}{t^{2/3}}\left(1 + \frac{|x|}{\sqrt[3]{t}}\right)^{1/4} M_0^t(w_n(t)).\end{aligned}$$

These, along with Proposition 4.2, allow to obtain the H^1 decay estimate, in a very similar way to [8].

For the higher order estimates, i.e. H^2, H^3 and H^4, the pointwise control that we have on w_n and w_{nx} is not enough. If we wanted to improve our control to $M_0^t(w_{nx})$, we would always face the same problem for the higher order derivatives. The path that we will follow to avoid this is to use almost conservation quantities at level H^2 etc. For example, let u be a solution to (4.1), then

$$\frac{d}{dt}\left(\int u_{xx}^2 - \frac{20}{3}\int u_x^2 u^3\right) = 2\int u_x^5 + 80\int u_x^3 u^5.$$

Three elements are to be noticed. First, there is a corrective term $\int u_x^2 u^3$ to prevent the apparition of $\int u_{xx}^2 u_x u^2$, which we could not control, as noted in [23]. Second, $\int u_x^3 u^5$ comes from the corrective term, and will never be harmful, as it has a better integrability than the others (power 8 instead of 5). Third, $\int u_x^5$ has a more than quadratic term in u_x (when u_x appear less than twice, we can use directly our control on $\|u_x\|_{L^2}$ already obtained). To control this kind of terms, we use the Gagliardo-Nirenberg inequality :

$$\forall q \geq 2,\ \forall v \in H^1, \quad \|v\|_{L^q}^q \leq C(q) \|v\|_{L^2}^{\frac{q+2}{2}} \|v_x\|_{L^2}^{\frac{q-2}{2}}. \tag{4.5}$$

As the maximal exponent on the term with highest derivatives is 5 or less, exponent on $\|v_x\|_{L^2}$ will always be less than 2, which means that we will always be in the position to apply Lemma 4.4. Assume for now that, when estimating the derivative in time of the H^{s+1} norm (squared) of $w_n(t)$, all terms have appropriate control except for ($\beta \in [0,3]$)

$$\int |D^s w_n|^{2+\beta} |D^{s-1} w_n|^{3-\beta}.$$

Further assume that all previous estimates gave a decay $\|w_n\|_{H^s} \leq C t^{-1/3}$. Thus, as our term has power 5, from (4.5) we would get a control :

$$\left| \frac{d}{dt} \|w_n\|_{H^{s+1}}^2 \right| \leq \|w_n\|_{H^s}^{5-\beta} \|w_n\|_{H^{s+1}}^{\beta} \leq \frac{\|w_n\|_{H^{s+1}}^{\beta}}{t^{(5-\beta)/3}}.$$

With $\mu = \beta/2$, $\lambda = (5-\beta)/3$, Lemma 4.4 gives the decay $\|w\|_{H^{s+1}} \leq C t^{-\nu}$, with

$$\nu = \frac{1}{2} \frac{(5-\beta)/3 - 1}{1 - \beta/2} = \frac{1}{3}.$$

This means that the rate of decay $t^{-1/3}$ is likely to propagate as the regularity index s increases (in fact, for $p \geq 4$, similar computations show that the rate of decay $t^{-(p-3)/3}$ propagates). p integer is interesting regarding the regularity of the non-linearity function : to obtain the H^2 formula quoted, we already need a C^4 regularity, which translates to $p \geq 4$. In any case, our assumption $p = 4$ is now crucial. Of course we will need the estimate of Corollary 4.1 to handle some interaction terms.

Observe finally that this decay rate of $t^{-1/3}$ is the best one can expect, due to the slow decay of the linear term $U(t)V$.

Step 3 : Linear tools from scattering theory. We can now complete the decay estimate, by controling the remaining of the M_0^t norm.

Proposition 4.4 (Interaction with the linear term, M_0^t bound)**.** *There exists C_3 such that $\forall n \in \mathbb{N}$, $t \in [I_n, S_n]$*

$$M_0^t(w_n(t)) \leq \frac{C_3}{t^\delta}.$$

Recall that $M_0^t(w_n(t)) = \|w_n(t)\|_{H^1} + \|D^\alpha J^t w_n(t)\|_{L^2} + \|DJ^t w_n(t)\|_{L^2}$. $\|w_n(t)\|_{H^1}$ has already been estimated, so we only need to focus on the last two terms. We follow the framework of [13] and [8]. First, we estimate $\|D^\alpha I^t w_n(t)\|_{L^2}$ and $\|I^t w_{nx}(t)\|_{L^2}$. For this, we use the usual $\frac{1}{2}\frac{d}{dt}\|f\|_{L^2}^2 = (Lf, f)$, and plug in Lf the equation satisfied by f : here $f = D^\alpha I^t w_n(t)$ or $DI^t w_n(t)$.

When doing the computations on $(LI^t w_{nx}(t), I^t w_{nx}(t))$, we encounter a term of the type

$$\int (I^t w_{nx}(t))^2 R^2. \tag{4.18}$$

This is localized term in space, but in H^3 regularity instead of H^1 regularity. This fact explains that we needed to get decay for higher regularity norms than just H^1. Ideally, we would try to obtain directly H^3 on the right decay. However, this seems not to be possible. One easy way is to obtain low decay rate for the global space norms H^s, which we did up to H^4. Corollary 4.1 allows us to bound this troublesome term (4.18).

This explains how to obtain

$$\|D^\alpha I^t w_n(t)\|_{L^2} + \|I^t w_{nx}(t)\|_{L^2} \leq Ct^{-\delta}.$$

It remains to go back to J^t, which is done in a similar way as in [13] and [8], and does not raise more difficulties than those treated earlier.

This concludes the proof of Proposition 1', and thus of Proposition 4.1. □

We can now conclude :

Proof of Theorem 4.1.
Step 1 : A compactness result. From Proposition 4.1, we dispose of a sequence $u_n(t)$ defined on $[T_0, S_n]$, solution to (4.1), such that

$$u(S_n) = U(S_n)V + \sum_{j=1}^{N} R_j(S_n) = U(S_n)V + R(S_n),$$

and that the uniform estimates hold ($w_n(t) = u_n(t) - U(t)V - R(t)$) :

$$\exists T_0 \geq 1, \exists C_0 > 0, \quad \forall n \in \mathbb{N},\ \forall t \in [T_0, S_n], \qquad \|w_n(t)\|_{H^4} + M_0^t(w_n(t)) \leq \frac{C_0}{t^\delta}.$$

Let us prove the following compactness result on the sequence $u_n(T_0)$.

Claim. We have

$$\lim_{A\to\infty} \sup_{n\in\mathbb{N}} \int_{|x|\geq A} u_n^2(T_0, x)dx = 0.$$

Proof. Indeed, let $\varepsilon > 0$, and $T(\varepsilon)$ such that $C_0 T(\varepsilon)^{-\delta} \leq \sqrt{\varepsilon}$. Then

$$\int (u_n(T(\varepsilon)) - U(T(\varepsilon))V - R(T(\varepsilon))^2 \leq \varepsilon.$$

Let $A(\varepsilon)$ be such that $\int_{|x|\geq A(\varepsilon)} (U(T(\varepsilon))V + R(T(\varepsilon)))^2(x)dx \leq \varepsilon$; we get

$$\int_{|x|\geq A(\varepsilon)} u_n^2(T(\varepsilon), x)dx \leq 2\varepsilon.$$

Let $g \in C^3$ a function such that $g(x) = 0$ if $x \leq 0$, $g(x) = 1$ if $x \geq 2$, and furthermore $0 \leq g'(x) \leq 1$, $0 \leq g'''(x) \leq 1$.

Recall that if $f \in C^3$ does only depend on x, we have

$$\frac{d}{dt}\int u_n^2 f = -3\int {u_n}_x^2 f_x + \int u_n^2 f_{xxx} + \frac{8}{5}\int u_n^5 f_x.$$

(See Lemma 4.7 and its proof). For $C(\varepsilon)$ to be determined later, we then have :

$$\frac{d}{dt}\int u_n^2(t,x)g\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right) = -\frac{3}{C(\varepsilon)}\int {u_n}_x^2 g'\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right) + \frac{1}{C(\varepsilon)^3}\int u_n^2 g'''\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right) + \frac{8}{5C(\varepsilon)}\int u_n^5 g'\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right).$$

As $t \geq T_0 \geq 1$, u_n satisfies $\|u_n(t)\|_{H^1} \leq C_0 + \|V\|_{H^1} + \sum_{j=1}^N \|Q_{c_j}\|_{H^1} \leq C^0$. So that :

$$\left|\frac{d}{dt}\int u_n^2(t,x)g\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right)\right| \leq \frac{1}{C(\varepsilon)}\left(3\int {u_n}_x^2(t) + \int u_n^2(t) + \frac{8}{5}\|u_n\|_{L^\infty}^3 \int u_n^2(t)\right) \leq \frac{1}{C(\varepsilon)}\left(3{C^0}^2 + \frac{8}{5}2^{3/2}{C^0}^5\right).$$

Now choose $C(\varepsilon) = \max\left\{1, \frac{T(\varepsilon)-T_0}{\varepsilon}\left(3{C^0}^2 + \frac{8}{5}2^{3/2}{C^0}^5\right)\right\}$, from which we derive

$$\left|\frac{d}{dt}\int u_n^2(t,x)g\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right)\right| \leq \frac{\varepsilon}{T(\varepsilon)-T_0}.$$

And after integration in time between T_0 and $T(\varepsilon)$:

$$\int_{x\geq 2C(\varepsilon)+A(\varepsilon)} u_n^2(T_0,x) \leq \int u_n^2(T_0,x)g\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right) \leq 3\varepsilon.$$

Now considering $\frac{d}{dt}\int u_n^2(t,x)g\left(\frac{-A(\varepsilon)-x}{C(\varepsilon)}\right)$, we get in a similar way

$$\int_{x\leq -2C(\varepsilon)-A(\varepsilon)} u_n^2(T_0,x) \leq 3\varepsilon.$$

So that if we denote $A_\varepsilon = 2C(\varepsilon/6) + A(\varepsilon/6)$, we obtain :

$$\forall n \in \mathbb{N}, \qquad \int_{|x|\geq A_\varepsilon} u_n^2(T_0,x) \leq \varepsilon,$$

as claimed. □

Step 2 : Construction of u^*. $u_n(T_0)$ is a bounded sequence in $H^4 \cap \mathcal{H}_0^{T_0}$, so that it converges weakly to $\varphi_0 \in H^4(\mathbb{R}) \cap \mathcal{H}_0^{T_0}(\mathbb{R})$ (up to a subsequence). The previous compactness result ensures that the convergence is strong in $L^2(\mathbb{R})$. Indeed, let $\varepsilon > 0$, and A such that $\int_{|x|\geq A} \varphi_0^2(x)dx \leq \varepsilon$ and

$$\forall n \in \mathbb{N}, \qquad \int_{|x|\geq A} u_n^2(T_0,x) \leq \varepsilon.$$

As the embedding $H^1([-A,A]) \hookrightarrow L^2([-A,A])$ is compact, as $n \to \infty$, $\int_{|x|\leq A} |u_n(T_0,x) - \varphi_0(x)|^2 dx \to 0$. We thus derive that

$$\limsup_{n\in\mathbb{N}} \|u_n(T_0) - \varphi_0\|_{L^2(\mathbb{R})}^2 \leq 4\varepsilon.$$

As this is true for all $\varepsilon > 0$, $u_n(T_0) \to \varphi_0$ in $L^2(\mathbb{R})$. By interpolation, $u_n(T_0)$ converges strongly to φ_0 in H^3. Denote $u^*(t)$ the solution to

$$\begin{cases} u^*_t + (u^*_{xx} + u^{*4})_x = 0, \\ u^*(T_0) = \varphi_0. \end{cases}$$

The Cauchy problem (4.1) being globally well-posed in H^1, u^* is well defined. Now the flow is continuous in H^3, so that for all $t \in \mathbb{R}$, $u_n(t) \to u^*(t)$ in H^3, and we can pass to the limit in the H^3 estimates, to get

$$\forall t \in \mathbb{R}, \qquad \|u^*(t) - U(t)V - R(t)\|_{H^3} \leq C_0 t^{-1/3}.$$

Denote $w^*(t) = u^*(t) - U(t)V - R(t)$. $w_n(t) \to w^*(t)$ in H^1 so that $w^*(t)$ is the only possible weak limit of $w_n(t)$ in $H^4 \cap \mathcal{H}^t_0$. In particular, the convergence is strong in H^3 and

$$\|w^*(t)\|_{H^4} \leq \liminf_{n\to\infty} \|w_n(t)\|_{H^4} \leq \frac{C_0}{t^{1/3}}, \quad M^t_0(w^*(t)) \leq \liminf_{n\to\infty} M^t_0(w_n(t)) \leq \frac{C_0}{t^\delta}.$$

This completes the proof of Theorem 4.1. □

This scheme of proof is similar to that of [23], [8]. Steps 2, 3 and 4 of the proof of Proposition 1' remain to be completed.

In Section 3, we present some preliminary results. In Section 4, we prove Proposition 4.2. In Section 5, we prove Proposition 4.3. Finally, in Section 6, we prove Proposition 4.4. This completes the proof of Proposition 1', and thus, the proof of Theorem 4.1.

4.3 Preliminaries

4.3.1 Cut-off functions and notation for localized quantities

We already introduced $\sigma_0 \in (0, 1/2\min\{c_1, c_2 - c_1, \ldots, c_N - c_{N-1}\})$, and the cut off function

$$\psi(x) = \frac{2}{\pi}\arctan\left(\exp\left(-\frac{\sqrt{\sigma_0}}{2}x\right)\right). \tag{4.15}$$

We can check that $\lim_{+\infty}\psi = 0$, $\lim_{-\infty}\psi = 1$, ψ is decreasing. Furthermore, by direct computations :

$$\psi'(x) = -\frac{\sqrt{\sigma_0}}{2\pi\cosh\left(\frac{\sqrt{\sigma_0}}{2}x\right)}, \quad \psi''' = \frac{\sigma_0}{4}\psi'(x)\left(1 - \frac{2}{\cosh\left(\frac{\sqrt{\sigma_0}}{2}x\right)}\right),$$

so that

$$|\psi'''(x)| \leq -\frac{\sigma_0}{4}\psi'(x). \tag{4.19}$$

We introduce, for $j = 1, \ldots, N-1$,

$$m_j(t) = \frac{c_j + c_{j+1}}{2}t + \frac{x_j + x_{j+1}}{2}, \quad m_0(t) = \sigma_0 t - 2|x_1|.$$

So that we can define, for $j = 0, \ldots, N-1$,

$$\psi_j(t, x) = \psi(x - m_j(t)), \quad \psi_N(t, x) = 1.$$

Then we set, for $j = 1, \dots, N-1$,

$$\phi_0(t) = \psi_0(t), \quad \phi_j(t) = \psi_j(t) - \psi_{j-1}(t), \quad \phi_N(t) = 1 - \psi_{N-1}(t).$$

By construction, $\sum_{k=1}^{j} \phi_k = \psi_j$. Finally, we define some local quantities related to mass and energy :

$$M_j(t) = \int u_t^2(t)\phi_j(t), \quad E_j(t) = \int \left(\frac{1}{2}u_x^2(t) - \frac{1}{5}u^5(t)\right)\phi_j(t),$$
$$F_j(t) = E_j(t) + \frac{\sigma_0}{100}M_j(t).$$

4.3.2 $\mathcal{H}_0^t$ estimates

Recall our notations

$$\gamma \in \left(0, \frac{1}{3}\right), \quad \alpha = \frac{1}{2} - \gamma, \quad \delta = \frac{1-2\gamma}{3} > 0, \tag{4.9}$$

the operator $J^t f = xf - 3t\partial_x^2 f = U(t)xU(-t)f$, and our working norm

$$M_0^t(f) = \|f\|_{H^1} + \|D^\alpha J^t f\|_{L^2} + \|\partial J^t f\|_{L^2}.$$

First a few remarks on M_0^t. Of course $M_0^0(f) \le C\|f\|_{H^{1,1}}$. Second, note that $J^t U(t)V = U(t)xV$ (and $U(t)$ is a H^s isometry), so that if $V \in H^{1,1}$, we have the uniform control in t :

$$M_0^t(U(t)V) \le C\|V\|_{H^{1,1}}. \tag{4.20}$$

We now recall the linear results obtained in [13] (Lemma 2.2), in a slightly improved form.

Lemma 4.1 ([13])**.** *Let $t > 0$ and f be a function so that $M_0^t(f)$ is bounded. Then for $r > 4$,*

$$\|f\|_{L^r} \le \frac{C}{(1+t)^{1/3-1/(3r)}} M_0^t(f).$$

And one also has the point wise inequalities

$$|f(x)| \le \frac{CM_0^t(f)}{(1+t)^{1/3}}\left(1 + \left|\frac{x}{t^{1/3}}\right|\right)^{-\frac{1}{4}}, \quad |f_x(x)| \le \frac{CM_0^t(f)}{t^{2/3}}\left(1 + \left|\frac{x}{t^{1/3}}\right|\right)^{\frac{1}{4}}.$$

As a simple consequence, for $V \in H^{1,1}$, we have similar decay estimates on $U(t)V$.

Proof. See [13, Lemma 2.2] and its proof (especially inequalities (2.16), (2.17) and (2.18)). The proof of refinement can be found in [8], Appendix A. □

We will also need the polarized version of Lemma 2.3 of [13] (in the case $p = 4$) :

Lemma 4.2. *Let $p \ge 3$ and $g, h : \mathbb{R} \to \mathbb{R}$. Then the following inequalities are hold if their right-hand side is bounded :*

$$\|D^\alpha g^p\|_{L^2} \le C\|g^{p-1}\|_{L^2}(\|gg_x\|_{L^\infty}^{1/2} + \|g\|_{L^\infty}^{3\gamma}\|gg_x\|_{L^\infty}^{(1-3\gamma)/2}),$$
$$\|D^\alpha |g|^{p-1}h_x\|_{L^2} \le C(\|D^\alpha h\|_{L^2} + \|h_x\|_{L^2})(\|g\|_{L^\infty}^{p-3}\|gg_x\|_{L^\infty}$$
$$+ \|g\|_{L^\infty}^{p-3-2\gamma}\|g\|_{L^2}^{2\gamma}\|gg_x\|_{L^\infty} + \|g\|_{L^\infty}^{p-3+2\gamma}\|gg_x\|_{L^\infty}^{1-\gamma}).$$

Proof. See [13, Lemma 2.3] and its proof (case $\sigma = 0$). □

4.3.3 Estimates of $U(t)V$ on the right

Recall our definition of $\psi_0(t)$ (4.15), given $\sigma_0 > 0$. We will often need estimates of the type $\|U(t)V\|_{H^1(1-\psi_0(t))}$, as it is a measure of the interaction between the linear term $U(t)V$ and the solitons.

Let us denote $x_+ = \max\{x, 0\}$.

Lemma 4.3 ($U(t)V$ estimates on the right)**.** *Let $f \in L^2$, then*

$$\|U(t)f\|_{L^2(1-\psi_0(t))} \leq \|f\|_{L^2(1-\psi_0(t/2))} \to 0 \quad \text{as} \quad t \to \infty. \tag{4.21}$$

Assume in addition that $(1 + x_+^q)f(x) \in L^2$, for some $q > 0$. Then there exists a constant $C = C(\sigma_0, x_1)$ independent of f such that

$$\forall t \geq 1, \quad \|U(t)f\|_{L^2(1-\psi_0(t))} \leq \frac{C}{t^q}\|(1 + x_+^q)f(x)\|_{L^2}. \tag{4.22}$$

We will apply this result to V and its derivatives (see Corollary 4.1).

Proof. The key remark is that $U(t)$ "pushes" the L^2-mass on the left. We compute :

$$\begin{aligned}
&\frac{d}{d\tau}\int |U(2\tau - t)f|^2\psi_0(\tau) \\
&= 2\int (U(2\tau - t)f)_\tau U(2\tau - t)f\psi_0(\tau) + \int |U(2\tau - t)f|^2\psi_{0\tau}(\tau) \\
&= -4\int U(2\tau - t)f_{xxx}U(2\tau - t)f\psi_0(\tau) + \int |U(2\tau - t)f|^2\psi_{0\tau}(\tau) \\
&= 4\int U(2\tau - t)f_{xx}U(2\tau - t)f_x\psi_0(\tau) + 4\int U(2\tau - t)f_{xx}U(2\tau - t)f\psi_{0x}(\tau) \\
&\quad + \int |U(2\tau - t)f|^2\psi_{0\tau}(\tau) \\
&= -6\int |U(2\tau - t)f_x|^2\psi_{0x}(\tau) - 4\int U(2\tau - t)f_xU(2\tau - t)f\psi_{0xx}(\tau) \\
&\quad + \int |U(2\tau - t)f|^2\psi_{0\tau}(\tau) \\
&= -6\int |U(2\tau - t)f_x|^2\psi_{0x}(\tau) + \int |U(2\tau - t)f|^2(2\psi_{0xxx}(\tau) + \psi_{0\tau}(\tau)).
\end{aligned}$$

As $\psi_{xxx} \leq \frac{\sigma_0}{4}|\psi_x|$, $\psi_{0\tau} = -\sigma_0\psi_{0x}$, and $\psi_x < 0$, we have that,

$$\psi_{0x}(\tau) < 0 \quad \text{and} \quad 2\psi_{0xxx}(\tau) + \psi_{0\tau}(\tau) \geq 0.$$

So that $\tau \mapsto \int |U(2\tau - t)f|^2(x)\psi_0(\tau, x)dx$ is an increasing function of τ. In particular, when comparing for $\tau = t$ and $\tau = t/2$ ($t \geq 0$), we have :

$$\forall t \geq 0, \quad \int |U(t)f|^2\psi_0(t) \geq \int f^2\psi_0(t/2).$$

As the flow $U(t)$ preserves the L^2-mass, we get

$$\int |U(t)f|^2(x)(1 - \psi_0(t, x))dx \leq \int f^2(x)(1 - \psi_0(t/2, x))dx. \tag{4.23}$$

Suppose that for some $q > 0$, $(1 + x_+^q)f(x) \in L^2$. Then for $t \geq 1$,

$$\begin{aligned}\int f^2(1-\psi_0(t/2)) &= \int_{x\leq\sigma_0 t/4} f^2(1-\psi_0(t/2)) + \int_{x\geq\sigma_0 t/4} f^2(1-\psi_0(t/2)) \\ &\leq \sup_{x\leq\sigma_0 t/4}(1-\psi_0(t/2,x))\int f^2 + \left(\frac{\sigma_0 t}{4}\right)^{-2q}\int_{x\geq\sigma_0 t/4} x^{2q} f^2 \\ &\leq C(x_0)e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|f\|_{L^2}^2 + C(\sigma_0)t^{-2q}\|x_+^q f\|_{L^2}^2.\end{aligned}$$

And we get

$$\forall t \geq 1, \quad \|U(t)f\|_{L^2(1-\psi_0(t))} \leq \frac{C}{t^q}\|(1+x_+)^q f\|_{L^2},$$

which is (4.22). □

4.3.4 An ODE lemma

Lemma 4.4 (Booster). *Let $\kappa > 0$, $\lambda > 1$, $\mu \in (0,1)$, and $f \in L^\mu([a,b])$ $(0 < a < b < +\infty)$ be a non-negative upper semi-continuous function satisfying*

$$\forall t \in [a,b], \quad f(t) \leq \frac{C}{t^\kappa} + C\int_t^b \frac{f^\mu(\tau)}{\tau^\lambda}d\tau,$$

Define $\nu = \min\{\kappa, \frac{\lambda-1}{1-\mu}\}$. Then there exists $k = k(C,\kappa,\lambda,\mu)$ (neither depending on b or a) such that

$$\forall t \in [a,b], \quad f(t) \leq \frac{kC}{t^\nu}.$$

***Remark* 4.5.** *Of course, if instead we have*

$$f(t) \leq \frac{C}{t^\kappa} + \sum_{i=1}^{I} C_i \int_t^b \frac{f^{\mu_i}(\tau)}{\tau^{\lambda_i}}d\tau,$$

the final decay estimate is still valid, with $\nu = \min\{\kappa, (\frac{\lambda_i-1}{1-\mu_i})_i\}$ being the least favorable exponent.

Proof. Let $k > 1$ to be determined later. Let us consider

$$T = \inf\left\{\tau \geq a \,\middle|\, \forall t \in [\tau,b], \quad f(t) \leq \frac{kC}{t^\nu}\right\}.$$

Observe that T is in fact minimal for the property. As $b > 0$, $f(b) \leq \frac{C}{t^\nu} < \frac{kC}{t^\nu}$, so that by upper semi continuity, $T < b$. Then, if $t \in [T,b]$, we have $(t \geq a > 0)$

$$f(t) \leq \frac{C}{t^\nu} + \frac{C(kC)^\mu}{(\lambda - 1 + c\nu)t^{\lambda-1+\mu\nu}}.$$

If $\nu = \frac{\lambda-1}{1-\mu}$, $\lambda - 1 + \mu\nu = (\lambda-1)\left(1 + \frac{\mu}{1-\mu}\right) = \frac{\lambda-1}{1-\mu} = \nu$. Else $\nu = \kappa$, $\frac{\lambda-1}{1-\mu} \geq \kappa = \nu$ so that $\lambda - 1 \geq (1-\mu)\nu$ and $\lambda - 1 + \mu\nu \geq \nu$. In any case, we get

$$f(t) \leq C\frac{1 + \frac{(kC)^\mu}{\lambda-1+\mu\nu}}{t^\nu}.$$

Let us now choose k such that $2\left(1 + \frac{(kC)^\mu}{\lambda-1+\mu\nu}\right) \leq k$, which is possible as $\mu < 1$ (notice that $k > 2$). We get finally $f(t) \leq \frac{kC}{2t^\nu}$. By a standard continuity argument, we deduce that $T = a$. □

4.4 Estimates on the right : proof of Proposition 4.2

We follow the framework of [23]. The hypothesis we will use in this section is :

$$\forall t \in [I_n, S_n], \quad \|w_n(t)\|_{H^1} \leq \varepsilon_0.$$

4.4.1 Modulation close to asymptotic profile

Let us recall that $Q_c(x) = c^{\frac{1}{p-1}} Q(\sqrt{c}x)$.

Lemma 4.5 (Modulation of $w_n(t)$)**.** *There exist T_2 and ε_2 such that if $I_n \geq T_2$ and $\varepsilon_0 \leq \varepsilon_2$, the following is true. For all $t \in [I_n, S_n]$, there exist $y_j(t)$ and $\gamma_j(t)$ such that if we denote*

$$\tilde{R}_j(t,x) = Q_{\gamma_j(t)}(x - y_j(t)), \qquad \tilde{R}(t,x) = \sum_{j=1}^{N} \tilde{R}_j(t,x),$$
$$\tilde{w}_n(t) = u_n(t,x) - U(t)V - \tilde{R}(t,x),$$

we have for all $j = 1, \ldots, N$,

$$\int \tilde{w}_n(t,x) \tilde{R}_{j_x}(t,x) dx = 0 \quad \text{and} \quad \int \tilde{w}_n(t,x) \tilde{R}_j(t,x) dx = 0.$$

Moreover, there exists C_1^2 such that :

$$\|\tilde{w}_n(t)\|_{H^1} + \sum_{j=1}^{N} |\gamma_j(t) - c_j| + \sum_{j=1}^{N} |y_j(t) - x_j - c_j t| \leq C_1^2 \varepsilon_0,$$
$$|y_j'(t) - c_j| + |\gamma_j'(t)| \leq C_1^2 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C_1^2 \|U(t)V\|_{L^2(1-\psi_0(t))} + C_1^2 \left(\int \tilde{w}_n^2(t) e^{-\sqrt{\sigma_0}|x - c_j t|} \right)^{1/2}.$$

Proof. The construction of the modulated parameters (and the first estimate) essentially relies on the implicit function theorem by a standard arguement : we refer to [57] and [58].

Let us focus on the second estimate (local estimate). We begin by computing the equation satisfied by $\tilde{w}_n$. The equation satisfied by $\tilde{R}_k$ (using $-c_k R_{kx} + R_{kxxx} + R_k^4)_x = 0$) is

$$\begin{aligned}
\tilde{R}_{kt} + \tilde{R}_{kxxx} &= (-y_k'(t) + c_k)\tilde{R}_{kx} + \frac{\gamma_k'(t)}{\gamma_k(t)} \left(\frac{\tilde{R}_k(t)}{3} + (x - y_k(t)) \frac{\tilde{R}_{kx}(t)}{2} \right) \\
&\quad - c_k \tilde{R}_{kx} + \tilde{R}_{kxxx} \\
&= (-y_k'(t) + c_k)\tilde{R}_{kx} + \frac{\gamma_k'(t)}{\gamma_k(t)} \left(\frac{\tilde{R}_k(t)}{3} + (x - y_k(t)) \frac{\tilde{R}_{kx}(t)}{2} \right) - (\tilde{R}_k^4)_x.
\end{aligned}$$

So that $\tilde{w}_n = u_n(t) - U(t)V - \tilde{R}(t)$ satisfies

$$\tilde{w}_{nt} + \tilde{w}_{nxxx} = \sum_{k=1}^{N} (y_k'(t) - c_k)\tilde{R}_{kx} - \sum_{k=1}^{N} \frac{\gamma_k'}{\gamma_k} \left(\frac{\tilde{R}_k}{3} + (x - y_k(t)) \frac{\tilde{R}_{kx}}{2} \right)$$

$$- \left((\tilde{w}_n + U(t)V + \tilde{R})^4 - \sum_{k=1}^{N} \tilde{R}_k^4 \right)_x . \quad (4.24)$$

Now, if we express $\tilde{R}_j$ in terms of R_j :

$$\tilde{R}_{jxt} = -y_j'(t)\tilde{R}_{jxx} + \frac{\gamma_j'(t)}{\gamma_j(t)} \left(\frac{\tilde{R}_{jx}(t)}{3} + (x - y_j(t)) \frac{\tilde{R}_{jxx}(t)}{2} + \frac{\tilde{R}_{jx}(t)}{2} \right).$$

And keeping in mind that $\frac{d}{dt} \int \tilde{w}_n \tilde{R}_{jx} = \int \tilde{w}_n \tilde{R}_{jx} = 0$, we get

$$\int \tilde{w}_{nt} \tilde{R}_{jx} = - \int \tilde{w}_n \tilde{R}_{jxt} = \int \tilde{w}_n \left(y_j'(t) - \frac{\gamma_j'(t)}{\gamma_j(t)} \frac{x - y_j(t)}{2} \right) \tilde{R}_{jxx}.$$

We multiply (4.24) by $\tilde{R}_{jx}$ and integrate in x, and do integration by parts :

$$\begin{aligned} (y_j'(t) - c_j) \int \tilde{R}_{jx}^{\,2} = & -y_j'(t) \int \tilde{w}(t) \tilde{R}_{jxx} + \frac{\gamma_j'(t)}{2\gamma_j(t)} \int \tilde{w}_n(t)(x - y_k(t)) \tilde{R}_{jxx} \\ & - \int \tilde{w}_n(t) \tilde{R}_{jxxxx} - \sum_{k, k \neq j} (c_k - y_k'(t)) \int \tilde{R}_{jx} \tilde{R}_{kx} \\ & + \sum_{k=1}^{N} \frac{\gamma_k'}{\gamma_k} \int \tilde{R}_{jx} \left(\frac{\tilde{R}_k}{3} + (x - y_k(t)) \frac{\tilde{R}_{kx}}{2} \right) \\ & - \int \left((\tilde{w}_n + U(t)V + \tilde{R})^4 - \sum_{k=1}^{N} \tilde{R}_k^4 \right) \tilde{R}_{jxx}. \end{aligned}$$

First consider the 3 first terms. Recall that for all $j = 1, \ldots, N$:

$$|\tilde{R}_j(t,x)| + |\tilde{R}_{jx}(t,x)| \leq C e^{-\sqrt{\sigma_0}|x - c_j t|}.$$

Furthermore, as $Q_{xx} = Q - Q^4$, we can express $\tilde{R}_{jxx}$ and $\tilde{R}_{jxxxx}$ in terms of powers of $\tilde{R}_j$. Hence, the integral part of these term is bounded by

$$\int |\tilde{w}_n(t)|(1 + |x - c_j t|) e^{-\sqrt{\sigma_0}|x - c_j t|} \leq C \left(\int |\tilde{w}_n(t)|^2 e^{-\sqrt{\sigma_0}|x - c_j t|} \right)^{1/2}.$$

For the fourth term, $\int |\tilde{R}_{jx} R_{kx}| \leq e^{-\frac{\sigma_0 \sqrt{\sigma_0}}{2} t}$. This also apply to the fifth term for $k \neq j$, and for $k = j$:

$$\int \tilde{R}_{jx} \left(\frac{\tilde{R}_j}{3} + (x - y_j(t)) \frac{\tilde{R}_{jx}}{2} \right) = 0.$$

And for the non-linear last term, when developing, the large terms cancel one another, so that we can control the rest by

$$C \int (|\tilde{w}_n(t)| + |U(t)V|) e^{-\sqrt{\sigma_0}|x - c_j t|}.$$

Finally, we have altogether

$$|y_j'(t) - c_j| \leq C \left(1 + \left| \frac{\gamma_j'(t)}{\gamma_j(t)} \right| \right) \left(\int |\tilde{w}_n(t)|^2 e^{-\sqrt{\sigma_0}|x - c_j t|} \right)^{1/2}$$

$$+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} \sum_{k,k\neq j} |y_k'(t) - c_k| + Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} \sum_{k,k\neq j} \left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|$$
$$+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C\|U(t)V\|_{L^2(1-\psi_0(t))}. \tag{4.25}$$

Now, we have to do the same kind of argument on γ_j. Let us multiply (4.24) by $\tilde{R}_j$, using

$$\int \tilde{w}_{nt}(t)\tilde{R}_j = -\int \tilde{w}_n(t)\tilde{R}_{j\,t}(t) = -\frac{\gamma_j'(t)}{2\gamma_j(t)}\int (x - y_j(t))\tilde{w}_n \tilde{R}_{j\,x}.$$

We obtain (after an integration by parts $\int (x-y_j(t))\tilde{R}_j\tilde{R}_{j\,x} = -\frac{1}{2}\int \tilde{R}_j^2$) :

$$\frac{1}{12}\frac{\gamma_j'(t)}{\gamma_j(t)}\int \tilde{R}_j^2 = \frac{\gamma_j'(t)}{2\gamma_j(t)}\int \tilde{w}_n(t)(x - y_k(t))\tilde{R}_{j\,x} - \int \tilde{w}_n(t)\tilde{R}_{j\,xxx}$$
$$- \sum_{k=1}^{N}(c_k - y_k'(t))\int \tilde{R}_j\tilde{R}_{kx} + \sum_{k\neq j}\frac{\gamma_k'}{\gamma_k}\int \tilde{R}_j\left(\frac{\tilde{R}_k}{3} + (x - y_k(t))\frac{\tilde{R}_{kx}}{2}\right)$$
$$- \int \left((\tilde{w}_n + U(t)V + \tilde{R})^4 - \sum_{k,k\neq j}\tilde{R}_k^4\right)\tilde{R}_{j\,x}.$$

Let us notice again that the only possibly large term (in the first sum) is in fact $\int \tilde{R}_j\tilde{R}_{j\,x} = 0$. If we argue like before, we get

$$\left|\frac{\gamma_j'(t)}{\gamma_j(t)}\right| \le C\left(1 + \frac{\gamma_j'(t)|}{\gamma_j(t)}\right)\left(\int \tilde{w}_n^2(t)e^{-\sqrt{\sigma_0}|x-c_jt|}\right)^{1/2}$$
$$+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\sum_{k,k\neq j}|y_k'(t) - c_k| + e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\sum_{k,k\neq j}\left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|$$
$$+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C\|U(t)V\|_{L^2(1-\psi_0(t))}. \tag{4.26}$$

We can now do some computations. Let us sum our $2N$ estimates (4.25) and (4.26) together :

$$\sum_{k=1}^{N}\left(|y_k'(t) - c_k| + \left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\right) \le C\left(1 + \sum_{k=1}^{N}|y_k'(t)| + \sum_{k=1}^{N}\left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\right)\|\tilde{w}_n\|_{L^2}$$
$$+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\left(\sum_{k=1}^{N}|y_k'(t)| + \left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\right) + Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C\|U(t)V\|_{L^2(1-\psi_0(t))}.$$

So that for ε_0 small enough, as $\|\tilde{w}_n\|_{L^2} \le \varepsilon_0$, and t large enough, we get

$$\sum_{k=1}^{N}|y_k'(t) - c_k| + \left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right| \le C.$$

Let us now go back to (4.25) : we get exactly what we want on $|y_j'(t) - c_j|$. In the same way, as $\gamma_k > \sigma_0$ for ε_0 small enough (first estimate), we get the result for $|\gamma_j'(t)|$ (plugging in (4.26)). □

Let us recall that by construction

$$\tilde{w}(S_n) = w(S_n) = 0, \quad y_j(S_n) = x_j + c_j S_n, \quad \gamma_j(S_n) = c_j, \quad \tilde{R}_j(S_n) = R_j(S_n). \tag{4.27}$$

Naturally, the geometric parameters $y_j(t)$ and $\gamma_j(t)$ control the distance between $R_j(t)$ and $\tilde{R}_j(t)$:

$$\|\tilde{R}_j(t) - R_j(t)\|_{H^s}^2 \le C(s)(|y_j(t) - x_j - c_j t|^2 + |\gamma_j(t) - c_j(t)|^2).$$

For simplicity of notation, let us denote

$$\tilde{v}_n(t) = \tilde{w}_n(t) + U(t)V = u_n(t) - \tilde{R}(t).$$

Lemma 4.6 (Main terms in M_j and E_j, $j \ge 1$). *We have, for all $t \in [I_n, S_n]$,*

$$(1) \quad \left| M_j(t) - \left(\int Q_{\gamma_j(t)}^2 + 2\int \tilde{v}_n(t)\tilde{R}_j(t) + \int \tilde{v}_n^2(t)\phi_j(t) \right) \right| \le C_1^3 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t},$$

$$(2) \quad \left| E_j(t) - \left[\frac{1}{2}\int (\tilde{v}_{nx}^{\ 2}(t) - 4\tilde{R}_j^3(t)\tilde{v}_n^2(t))\phi_j(t) - \gamma_j(t)\int \tilde{v}_n(t)\tilde{R}_j(t) + E(Q_{\gamma_j(t)}) \right] \right|$$
$$\le C_1^3 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C_1^3\varepsilon_0 \int \tilde{v}_n^2(t)\phi_j(t),$$

$$(3) \quad \left| \left(E_j(t) + \frac{\gamma_j(t)}{2}M_j(t) \right) - \left(E(Q_{\gamma_j(t)}) + \frac{\gamma_j(t)}{2}\int Q_{\gamma_j(t)}^2 \right) - \frac{1}{2}H_j(t) \right|$$
$$\le C_1^3 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C_1^3\varepsilon_0 \int \tilde{v}_n^2(t)\phi_j(t),$$

where $H_j(t) = \int (\tilde{v}_{nx}^{\ 2}(t) - 4\tilde{R}_j^3(t)\tilde{v}_n^2(t) + \gamma_j(t)\tilde{v}_n^2(t))\phi_j(t)$.

Proof. (1) We compute ($u_n = \tilde{v}_n + \tilde{R}$) :

$$M_j(t) = \int u_n^2 \phi_j(t) = \int \left(\tilde{v}_n^2 + 2\tilde{v}_n \tilde{R}(t) + \sum_{k=1}^{N} \tilde{R}_k^2(t) \right) \phi_j(t).$$

As $\phi_j(t)$ is localized in the interval $[m_{j-1}(t), m_j(t)]$ like $\tilde{R}_j(t)$, we get for $k \neq j$

$$\int \tilde{R}_k^2(t)\phi_j(t) \le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}, \quad \text{and} \quad \left| \int \tilde{R}_j^2(t)\phi_j(t) - \int Q_{\gamma_j(t)}^2 \right| \le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}.$$

(2) In the same way,

$$\begin{aligned}
E_j(t) &= \int \left(\frac{1}{2}(\tilde{v}_{nx}^{\ 2}(t) + 2\tilde{v}_{nx}(t)\tilde{R}_x + \tilde{R}_x^2) - \frac{1}{5}(\tilde{v}_n(t) + \tilde{R}(t))^5 \right) \phi_j(t) \\
&= \int (\frac{1}{2}\tilde{v}_{nx}^{\ 2}(t) - 2\tilde{R}^3\tilde{v}_n^2(t))\phi_j + \int (\frac{1}{2}\tilde{R}_x^2 - \frac{1}{5}\tilde{R}^5)\phi_j(t) \\
&\quad - \int \tilde{v}_n(t)(\tilde{R}_{xx} + \tilde{R}^4)\phi_j - \int \tilde{R}_x \tilde{v}_n(t)\phi_{jx} \\
&\quad + \int \left[\frac{(-(\tilde{v}_n(t) + \tilde{R})^5 + \tilde{R}^5)}{5} + \tilde{v}_n(t)\tilde{R}^4 + 2\tilde{R}^3\tilde{v}_n^2(t) \right] \phi_j.
\end{aligned}$$

We keep the first integral untouched. The second one is $E(Q_{\gamma_j(t)})$ up to an exponential correction. For the third one, recall that $Q_{xx} + Q^4 = Q$, so that again

$$\int \tilde{v}_n(t)(\tilde{R}_{xx} + \tilde{R}^4)\phi_j = \gamma_j(t) \int \tilde{v}_n(t)\tilde{R}_j(t) + O(e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}).$$

The fourth one is exponentially small (with $\tilde{R}$ and ${\phi_j}_x$). Finally the fifth is of order at least 3 in v_n, so that we control it by

$$\int \tilde{v}_n(t)^k \phi_j(t) \leq \|\tilde{v}_n(t)\|_{L^\infty} \int \tilde{v}_n(t)^2 \phi_j(t).$$

This gives the desired result.

(3) is the sum of (1) and (2). Notice that the scalar product $\int \tilde{v}_n(t)\tilde{R}_j(t)$ cancels in H_j : the linear combination has been constructed for this. □

As usual, we now need definite positiveness on the quadratic form linked to the linearized operator of (4.1) around the soliton R_j.

Proposition 4.5 (Positivity of a quadratic form, sub-critical case)**.** *There exists $\sigma_1 > 0$ small enough so that the following is true. For $\sigma_0 \leq \sigma_1$, there exist $T_3 = T_3(\sigma_0)$ and $\lambda_1 > 0$ (not depending on σ_0), so that for all $t \geq T_3$, for all $j = 1, \dots, N$, and for all $v \in H^1$,*

$$\int (v_x^2 - 4\tilde{R}_j(t)^3 v^2 + \gamma_j(t) v^2)\phi_j(t) \geq \lambda_1 \int (v_x^2 + v^2)\phi_j(t) - \frac{1}{\lambda_1}\left(\left(\int v\tilde{R}_j(t)\right)^2 + \left(\int v{\tilde{R}_j}_x(t)\right)^2\right).$$

Proof. A similar result can be found in [29, Lemma 4], [28, Appendix A] and [5, Appendix], to which we refer for the proof. □

From now on and throughout the rest of the proof, $\sigma_0 < \sigma_1$ is fixed.

4.4.2 Monotonicity properties

The next step is a surprising and crucial almost-monotonicity lemma.

Lemma 4.7 (Monotonicity property [24])**.** *There exists $C_1^1 > 0$ such that for all $j = 0, \dots, N$ and $t \in [I_n, S_n]$,*

$$\sum_{k=0}^{j} (M_k(S_n) - M_k(t)) \geq -C_1^1 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t},$$

$$\sum_{k=0}^{j} (F_k(S_n) - F_k(t)) \geq -C_1^1 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}.$$

Proof. This lemma is very similar to the monotonicity Lemma of [29] and [23]. The only difference is the presence of the term $U(t)V$: this will be taken care of essentially due to the smallness of $\|U(t)V\|_{L^\infty}$. Let us now do the computations. First notice that

$$\sum_{k=0}^{j} M_k(t) = \int {u_n}_t^2(t)\psi_j(t), \quad \sum_{k=0}^{j} E_k(t) = \int \left(\frac{1}{2}{u_n}_x^2(t) - \frac{1}{5}u_n^5(t)\right)\psi_j(t).$$

For $j = N$, the result is the conservation of mass and energy. Otherwise we compute for $f(t,x) \in C^3$:

$$\begin{aligned}
\frac{d}{dt}\int u_n^2 f - \int u_n^2 f_t &= 2\int u_{nt}u_n f = -2\int (u_{nxx} + u_n^4)_x u_n f \\
&= 2\int (u_{nxx} + u_n^4)(u_{nx} f + u_n f_x) \\
&= \int \left(-3u_{nx}^2 + \frac{8}{5}u_n^5\right) f_x - 2\int u_{nx}u_n f_{xx} \\
&= \int \left(-3u_{nx}^2 + \frac{8}{5}u_n^5\right) f_x + \int u_n^2 f_{xxx}.
\end{aligned}$$

So that we get

$$\frac{d}{dt}\int u_n^2 \psi_j(t) = -\int \left(3u_{nx}^2 + m_j'(t)u_n^2 - \frac{8}{5}u_n^5\right)\psi_{jx} + \int u_n^2 \psi_{jxxx}.$$

But $m_j'(t) \geq \sigma_0$ so that by (4.19), and $\psi_{jx} \leq 0$:

$$\frac{d}{dt}\int u_n^2 \psi_j(t) \geq \int \left(3u_{nx}^2 + \frac{3\sigma_0}{4} - \frac{8}{5}u_n^5\right)|\psi_{jx}(t)|$$

It remains to bound the third term. We consider two cases : let $R_0 > 0$ be chosen later. When $x \in [c_j t + x_j + R_0, c_{j+1}t + x_{j+1} - R_0]$, ψ_{jx} is big but $R(t)$ is small so that u_n too. More precisely,

$$\begin{aligned}
\left|\frac{8}{5}u_n^3(t,x)\right| &\leq C(\|w_n(t)\|_{L^\infty}^3 + \|U(t)V\|_{L^\infty}^3 + |R(t,x)|^3) \\
&\leq C(\varepsilon_0^3 + t^{-1} + e^{-\sqrt{\sigma_0}R_0}) \leq \frac{\sigma_0}{4},
\end{aligned} \tag{4.28}$$

if R_0 and T_0 are large enough, and ε_0 is small enough. On this interval, the second term is larger than the third.

When x is not on the previously considered interval, then $x \notin [m_j(t) - \sigma_0 t, m_j(t) + \sigma_0 t]$ (for T_0 large enough), so that

$$|\psi_{jx}(t,x)| \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}.$$

Now by interpolation between L^2 and H^1, we have a uniform control $\int |u_n|^5 \leq C$. So that finally

$$\frac{d}{dt}\int u_n^2\psi_j(t) \geq \int \left(3u_{nx}^2 + \frac{\sigma_0}{2}u_n^2\right)|\psi_{jx}(t)| - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} \geq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}. \tag{4.29}$$

We integrate this last estimate between t and S_n, and this gives the estimates on M_j.

For the estimates on F_j, we compute in a similar way

$$\begin{aligned}
&\frac{d}{dt}\int \left(u_{nx}^2 - \frac{2}{5}u_n^5\right) f - \int \left(u_{nx}^2 - \frac{2}{5}u_n^5\right) f_t \\
&= 2\int (u_{nxt}u_{nx} - u_n^4 u_{nt}) f = -2\int u_{nt}(u_{nxx} + u_n^4) f - 2\int u_{nt}u_{nx} f_x
\end{aligned}$$

$$= -\int (u_{nxx} + u_n^4)^2 f_x + 2\int (u_{nxx} + u_n^4)_x u_{nx} f_x$$
$$= -\int \left((u_{nxx} + u_n^4)^2 + 2u_{nxx}^2 - 8u_{nx}^2 u_n^3\right) f_x - 2\int u_{nxx} u_{nx} f_{xx}$$
$$= -\int \left((u_{nxx} + u_n^4)^2 + 2u_{nxx}^2 - 8u_{nx}^2 u_n^3\right) f_x + \int u_{nx}^2 f_{xxx}.$$

So that

$$\frac{d}{dt}\int \left(u_{nx}^2 - \frac{2}{5}u_n^5\right)\psi_j(t)$$
$$= -\int \left((u_{nxx} + u_n^4)^2 + 2u_{nxx}^2 - 8u_{nx}^2 u_n^3\right)\psi_{jx}(t)$$
$$- m_j'(t)\int \left(u_{nx}^2 - \frac{2}{5}u_n^5\right)\psi_{jx}(t) + \int u_{nx}^2 \psi_{jxxx}(t).$$

Again $m_j'(t) \geq \sigma_0$ and $|m_j'(t)| \leq c_N$, so that $\int u_{nx}^2\psi_{jxxx}(t) - \frac{\sigma_0}{4}\int u_{nx}^2\psi_{jx}(t) \geq 0$ and

$$\frac{d}{dt}\int \left(u_{nx}^2 - \frac{2u_n^5}{5}\right)\psi_j(t) \geq \frac{3\sigma_0}{4}\int u_{nx}^2|\psi_{jx}(t)| - \int \left(8u_{nx}^2|u_n|^3 - \frac{2c_N}{5}|u_n|^5\right)|\psi_{jx}(t)|. \tag{4.30}$$

To bound $\int u_{nx}^2|u_n|^3|\psi_{jx}(t)|$, we proceed like before and get

$$8\int u_{nx}^2|u_n|^3|\psi_{jx}(t)| \geq -\frac{\sigma_0}{2}\int |u_{nx}^2|\psi_{jx}(t) - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}. \tag{4.31}$$

However for $\frac{2c_N}{5}\int u_n^5|\psi_{jx}(t)|$, some L^2 norm is needed (which is why we introduced F_j, as in [23]). Choosing ε_1 small enough and R_0 large enough, we can improve (4.28) to $\sigma_0/400$, and so obtain :

$$\frac{2c_N}{5}\int u_n^5 \geq -\frac{\sigma_0}{100}\int u_n^2|\psi_{jx}(t)| - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}. \tag{4.32}$$

Now adding up (4.30) and 1/50·(4.29), and using (4.31) and (4.32), we get

$$\frac{d}{dt}\int \left(u_{nx}^2 - \frac{2}{5}u_n^5 + \frac{1}{50}u_n^2\right)\psi_j(t) \geq \frac{\sigma_0}{2}\int u_{nx}^2|\psi_x(t)| - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}.$$

And the estimate on F_j comes by integration between t and S_n. □

4.4.3 Abel transform and conclusion of the proof of Proposition 4.2

Proof of Proposition 4.2. We can now conclude the H^1 estimates on the right for w_n. First let us obtain some estimates for $\tilde{w}_n(t)$. We compute

$$\sum_{j=1}^{N}\frac{1}{\gamma_j^2(t)}\left(E_j(\tau) + \frac{\gamma_j(t)}{2}M_j(\tau)\right) = \sum_{j=1}^{N-1}\left(\left(\frac{1}{\gamma_j^2(t)} - \frac{1}{\gamma_{j+1}^2(t)}\right)\sum_{k=1}^{j}F_k(\tau)\right)$$
$$+ \sum_{j=1}^{N-1}\left(\frac{1}{2}\left(\frac{1}{\gamma_j(t)} - \frac{1}{\gamma_{j+1}(t)}\right)\left(1 - \frac{\sigma_0}{50}\left(\frac{1}{\gamma_j(t)} + \frac{1}{\gamma_{j+1}(t)}\right)\right)\sum_{k=1}^{j}M_k(\tau)\right)$$
$$+ \frac{1}{\gamma_N^2(t)}\sum_{k=1}^{N}F_k(\tau) + \frac{1}{2\gamma_N(t)}\left(1 - \frac{\sigma_0}{50c_N}\right)\sum_{j=1}^{N}M_k(\tau).$$

All the terms in the right hand side are positive, so that we can apply Lemma 4.7 (between $\tau = t$ and $\tau = S_n$) :

$$\sum_{j=1}^{N} \frac{1}{\gamma_j^2(t)} \left(E_j(t) + \frac{\gamma_j(t)}{2} M_j(t) \right) - \sum_{j=1}^{N} \frac{1}{\gamma_j^2(t)} \left(E_j(S_n) + \frac{\gamma_j(t)}{2} M_j(S_n) \right) \leq C e^{-\frac{\sigma_0 \sqrt{\sigma_0}}{2} t}.$$

Now we use fact 3. of Lemma 4.6 at time t and at time S_n (recall that $|\gamma_j(t) - c_j| \leq C\varepsilon_0$, so that $c_N + \varepsilon_0 \geq \gamma_j(t) \geq \sigma_0$)

$$\begin{aligned} \sum_{j=1}^{N} \frac{1}{\gamma_j^2(t)} H_j(t) &\leq C e^{-\frac{\sigma_0 \sqrt{\sigma_0}}{2} t} + C_3 \varepsilon_0 \int \left(\tilde{v}_n^2(t) + \tilde{v}_n^2(S_n) \right) \sum_{j=1}^{N} \phi_j(t) \\ &\leq C e^{-\frac{\sigma_0 \sqrt{\sigma_0}}{2} t} + C\varepsilon_0 \|\tilde{v}_n(t)\|^2_{L^2(1-\psi_0(t))} + C\varepsilon_0 \|U(S_n)V\|^2_{L^2(1-\psi_0(S_n))}. \end{aligned} \tag{4.33}$$

By Proposition 4.5, we have for $j = 1, \ldots, N$,

$$H_j(t) \geq \lambda_1 \int (\tilde{v}_n^2(t) + \tilde{v}_{nx}^2(t)) \phi_j(t) - \frac{1}{\lambda_1} \left(\left(\int \tilde{v}_n(t) Q \right)^2 + \left(\int \tilde{v}_n(t) Q_x \right)^2 \right).$$

So that if we sum up those N inequalities, there exists $\lambda_0 > 0$, neither depending on σ_0 nor ε_0, such that

$$\begin{aligned} \sum_{j=1}^{N} \frac{1}{\gamma_j^2(t)} H_j(t) &\geq \lambda_0 \|\tilde{v}_n(t))\|^2_{H^1(1-\psi_0(t))} - \frac{1}{\lambda_0} \sum_{j=1}^{N} \left(\left(\int \tilde{v}_n(t) \tilde{R}_j(t) \right)^2 + \left(\int \tilde{v}_n(t) \tilde{R}_{j_x}(t) \right)^2 \right) \\ &\geq \lambda_0 \|\tilde{v}_n(t))\|^2_{H^1(1-\psi_0(t))} - \frac{1}{\lambda_0} \sum_{j=1}^{N} \left(\left(\int U(t) V \tilde{R}_j \right)^2 + \left(\int \tilde{U}(t) V Q_x \right)^2 \right) \\ &\geq \lambda_0 \|\tilde{v}_n(t))\|^2_{H^1(1-\psi_0(t))} - \frac{C}{\lambda_0} \|U(t)V\|^2_{L^2(1-\psi_0(t))}. \end{aligned} \tag{4.34}$$

Note that our control is only on the right because we summed up for $j \geq 1$, which is coherent : we do not expect to obtain some control in the domain $x < \sigma_0 t$, where $U(t)V$ has its L^2-mass.

Combining (4.34) and (4.33), provided that ε_0 is small enough so that $C_3 \varepsilon_0 < \lambda_0/2$, we deduce :

$$\frac{1}{C} \|\tilde{v}_n(t))\|^2_{H^1(1-\psi_0(t))} \leq e^{-\frac{\sigma_0 \sqrt{\sigma_0}}{2} t} + \|U(t)V\|^2_{L^2(1-\psi_0(t))} + \|U(S_n)V\|^2_{L^2(1-\psi_0(S_n))}.$$

Finally, recall $\tilde{v}_n(t) = \tilde{w}_n(t) + U(t)V$, thus

$$\begin{aligned} \|\tilde{w}_n(t))\|^2_{H^1(1-\psi_0(t))} &\leq 2\|\tilde{v}_n(t))\|^2_{H^1(1-\psi_0(t))} + 2\|U(t)V\|^2_{H^1(1-\psi_0(t))} \\ &\leq C e^{-\frac{\sigma_0 \sqrt{\sigma_0}}{2} t} + C\|U(t)V\|^2_{L^2(1-\psi_0(t))} + C\|U(S_n)V\|^2_{H^1(1-\psi_0(S_n))}. \end{aligned} \tag{4.35}$$

Now that we have an appropriate estimate on $\|\tilde{w}_n(t)\|_{H^1(1-\psi_0(t)}$, we have only to go back to $w_n(t) = \tilde{w}_n(t) + R(t) - \tilde{R}(t)$. As we noticed after the proof of Lemma 4.5 :

$$\|w_n(t)\|_{H^1(1-\psi_0(t))} \leq \|R(t) - \tilde{R}(t)\|_{H^1} + \|\tilde{w}_n(t))\|_{H^1(1-\psi_0(t))}$$

$$\leq C\sum_{j=1}^{N}|y_j(t)-x_j-c_jt|+|\gamma_j(t)-c_j|+Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}$$
$$+C\|U(t)V\|_{H^1(1-\psi_0(t))}+C\|U(t)V\|_{L^2(1-\psi_0(S_n))}. \quad (4.36)$$

Now, using the L^2_{loc} estimate of Lemma 4.5, and then the estimate 4.35 :

$$|y_j'(t)-c_j|+|\gamma_j'(t)|\leq C_2e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}+C_2\|U(t)V\|_{L^2(1-\psi_0(t))}+C_2\left(\int\tilde{w}_n^2(t)e^{-\sqrt{\sigma_0}|x-c_jt|}\right)^{1/2}$$
$$\leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}+C\|U(t)V\|_{H^1(1-\psi_0(t))}+C\|U(S_n)V\|_{L^2(1-\psi_0(S_n))}.$$

Let us integrate this between t and S_n. Recall the initial conditions $y_j(S_n)=x_j+c_jS_n$, $\gamma_j(S_n)=c_j$, we obtain

$$|y_j(t)-x_j-c_jt|+|\gamma_j(t)-c_j|\leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}+C\int_t^{S_n}\|U(t)V\|_{H^1(1-\psi_0(t))}dt$$
$$+C(S_n-t)\|U(S_n)V\|_{L^2(1-\psi_0(S_n))}.$$

This, together with (4.36), concludes the proof of Proposition 4.2. □

4.5 Global estimates : proof of Proposition 4.3

We now want to control what happens in the zone $x<\sigma_0t$, that is the interaction with the linear term $U(t)V$. We follow the path of [8]. As our a priori estimates only concern w_n, we cannot use $\tilde{w}_n$, which has a better H^1 decay on the right : we don't have any available control on $M_0^t(\tilde{w}_n)$. The second point is that it appears to be difficult to control only $\|w_n\|_{H^s(\psi_0(t))}$, and this is why we do computation on the whole space, to obtain the decay estimate :

$$\|w_n(t)\|_{H^4}\leq\frac{C}{t^{1/3}}.$$

(Some terms that appear in the integration by part behave badly, but vanish when integrating on the whole space).

Recall our pointwise estimates on $w_n(t)$ ($M_0^t(w_n(t)\leq\varepsilon_0)$: we have

$$|w_n(t,x)|\leq\frac{C}{t^{1/3}}\left(1+\frac{|x|}{\sqrt[3]{t}}\right)^{-1/4}M_0^t(w_n(t)),$$
$$|w_{nx}(t,x)|\leq\frac{C}{t^{2/3}}\left(1+\frac{|x|}{\sqrt[3]{t}}\right)^{1/4}M_0^t(w_n(t)).$$

We proceed in two subsections : one for the H^1 estimate, which is very similar to that of [13] or [8], and one for H^s, $s>1$, which requires high integrability and smoothness of the non-linearity ($p\geq4$).

4.5.1 H^1 estimate

Proof of Proposition 4.3, H^1 estimate. L^2 estimate.

Here, no monotonicity is involved (it is essentially a linear theory). We bound the absolute value of the derivative in time of the L^2 norm of $w_n(t)$, and then integrate our estimate backward in time, with $w_n(S_n)=0$.

We use the equation for w_n

$$w_{nt} + w_{nxxx} + \left(u_n^4 - \sum_{j=1}^N R_j^4\right)_x = 0. \tag{4.11}$$

We multiply by w_n, and integrate in x. After an integration by part, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int w_n^2 &= \int \left(u_n^4 - \sum_{j=1}^N R_j^4\right) w_{nx} \\ &= \int (w_n + U(t)V + R)^4 - R(t)^4 - (w_n + U(t)V)^4) w_{nx} \\ &\quad - \int \left(R^4 - \sum_{j=1}^N R_j^4\right)_x w_n - \int ((w_n + U(t)V)^4)_x w_n.\end{aligned}$$

Let us develop

$$(w_n + U(t)V + R)^4 - R^4 - (w_n + U(t)V)^4 = \sum_{k=1}^3 C_4^k (w_n + U(t)V)^k R^{4-k}.$$

So that

$$\begin{aligned}&\left|\int ((w_n + U(t)V + R)^4 - R^4 - (w_n + U(t)V)^4) w_{nx}\right| \\ &\leq \sum_{k=1}^3 C_4^k \int |w_n + U(t)V|^k R^{4-k} |w_{nx}| \\ &\leq C\|w_{nx}\|_{L^2} \|w_n + U(t)V\|_{L^2(1-\psi_0(t))} \sum_{k=1}^3 \|(w_n + U(t)V)^{k-1} R^{3-k}\|_{L^\infty} \\ &\leq C\varepsilon_0 \|w_n + U(t)V\|_{L^2(1-\psi_0(t))}.\end{aligned}$$

Note that our control is essentially $\|w_{nx}\|_{L^2}\|w_n + U(t)V\|_{L^2(1-\psi_0(t))}$, and so relies on a priori estimate on $\|w_n\|_{H^1}$ to control the L^2 level. In fact this problem will only be acute for H^4, but let us explain now how to avoid it. We need to fully develop the term $(w_n + U(t)V + R)^4$. We do integration by part in this way :

$$\int w_n^i U(t) V^j R^{4-i-j} w_{nx} = -\frac{1}{i+1}\int w_n^{i+1} (U(t)V^j R^{4-i-j})_x,$$

so that all derivatives go on R or on $U(t)V$. It is then clear that in the L^2 case, our control improves to :

$$C\|w_n\|^2_{L^2(1-\psi_0(t)} + C\|w_n\|_{L^2}\|U(t)V\|_{H^1(1-\psi_0)}.$$

The point being that the estimate only involves $\|w_n\|_{L^2}$, and we are safe if we assume enough regularity and decay on V. For now, the direct method is simpler, so we will use it up to the H^3 estimate. Let us now go back to rest of the terms.

Of course, the purely solitons-interaction is exponentially small :

$$\left|\int \left(R^4 - \sum_{j=1}^N R_j^4\right)_x w_n\right| \leq \left\|R^4 - \sum_{j=1}^N R_j^4\right\|_{H^1} \|w_n\|_{L^2} \leq Ce^{-\frac{-\sigma_0\sqrt{\sigma_0}}{4}t}\|w_n(t)\|_{L^2}.$$

And to complete, we have to treat the purely linear interaction, which we control as in [8] :

$$\begin{aligned}\left|4\int (w_n+U(t)V)^3(w_n+U(t)V)_x w_n\right| \\ &\leq \|(w_n+U(t)V)_x(w_n+U(t)V)\|_{L^\infty}\|w_n+U(t)V\|_{L^\infty}\|w_n+U(t)V\|_{L^2}\|w_n\|_{L^2} \\ &\leq \frac{C}{t^{4/3}}\|w_n\|_{L^2}.\end{aligned}$$

So that we get

$$\frac{d}{dt}\|w_n(t)\|_{L^2}^2 \leq \left(\frac{C}{t^{4/3}}+e^{-\frac{-\sigma_0\sqrt{\sigma_0}}{4}t}\right)\|w_n(t)\|_{L^2}+C\|w_n(t)+U(t)V\|_{L^2(1-\psi_0(t))}.$$

We integrate between t and S_n, and obtain $(w_n(S_n)=0)$

$$\|w_n(t)\|_{L^2} \leq \frac{C}{t^{1/3}}. \tag{4.37}$$

as soon as $\|w_n(t)+U(t)V\|_{L^2(1-\psi_0(t))} \leq Ct^{-5/3}$.

$\dot{H}^1$ *estimate.*

We differentiate (4.11) with respect to x :

$$w_{nxt}+w_{nxxxx}+\left(u_n^4-\sum_{j=1}^N R_j^4\right)_{xx}=0.$$

Now we multiply by w_{nx} and integrate in x. After an integration by parts, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int {w_{nx}}^2 &= \int \left(u_n^4-\sum_{j=1}^N R_j^4\right)_x w_{nxx} \\ &= \int ((w_n+U(t)V+R)^4-R^4-(w_n+U(t)V)^4)_x w_{nxx} \\ &\quad -\int \left(R^4-\sum_{j=1}^N R_j^4\right)_{xx} w_{nx} - \int ((w_n+U(t)V)^4)_x w_{nxx}.\end{aligned}$$

Let us first treat the second line. As in the L^2 case,

$$\left|\int \left(R^4-\sum_{j=1}^N R_j^4\right)_{xx} w_{nx}\right| \leq C\varepsilon^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|w_{nx}\|_{L^2}.$$

And for the purely linear interaction term,

$$\begin{aligned}\int ((w_n+U(t)V)^4)_x w_{nxx} &= 4\int w_{nxx}(w_{nx}+U(t)V_x)(w_n+U(t)V)^3 \\ &= -6\int {w_{nx}}^2(w_n+U(t)V)_x(w_n+U(t)V)^2 \\ &\quad -4\int w_{nx}U(t)V_{xx}(w_n+U(t)V)^3\end{aligned}$$

$$-12\int w_{nx}(w_n+U(t)V)_x^2(w_n+U(t)V)^2.$$

Now, we control each of the terms with $\|w_{nx}\|_{L^2}$, $\|(w_n+U(t)V)_x\|_{L^2}$ (first and third term), $\|w_n+U(t)V\|_{L^2}$ (second term) and the rest in L^∞, noticing for the second term that

$$|U(t)V_{xx}(x)|\leq \frac{C}{t^{2/3}}\left(1+\frac{|x|}{\sqrt[3]{t}}\right)^{1/4}\|V_x\|_{H^{1,1}}.$$

So that as previously

$$\left|\int((w_n+U(t)V)^4)_x w_{nxx}\right|\leq \frac{C}{t^{4/3}}\|w_{nx}\|_{L^2}.$$

We now turn to

$$\int((w_n+U(t)V+R)^4-R^4-(w_n+U(t)V)^4)_x w_{nxx}$$
$$=\sum_{k=1}^{3}kC_4^k\int(w_{nx}+U(t)V_x)(w_n+U(t)V)^{k-1}R^{4-k}w_{nxx}$$
$$+\sum_{k=1}^{3}(4-k)\int(w_n+U(t)V)^kR_xR^{3-k}w_{nxx}.$$

Hence this interaction term is controlled by

$$C\|w_n+U(t)V\|_{H^1(1-\psi_0(t))}\|w_{xx}\|_{L^2}\leq C\|w_n\|_{H^1(1-\psi_0(t))}.$$

(recall that $w, U(t)V, R, R_x\in L^\infty$). Again, we obtain

$$\frac{d}{dt}\|w_{nx}\|_{L^2}^2\leq C\left(\frac{1}{t^{4/3}}+e^{-\frac{-\sigma_0\sqrt{\sigma_0}}{4}t}\right)\|w_{nx}\|_{L^2}+C\|w_n\|_{H^1(1-\psi_0(t))}.$$

We integrate between t and S_n, and derive ($w_n(S_n)=0$)

$$\|w_{nx}(t)\|_{L^2}\leq\frac{C}{t^{1/3}},\tag{4.38}$$

as soon as $\|w_n(t)\|_{H^1(1-\psi_0(t))}\leq Ct^{-5/3}$. □

Notice that this proof extends to the case $p>3$.

4.5.2 H^4 estimate

Proof of Proposition 4.3. We only present here the proof for the H^2 estimate, as the higher estimate will be treated in the same way, and will raise in fact less difficulties. The H^3 and H^4 estimates are done in full detail in the Appendix.

The proof goes in two steps : the first step is to derive a satisfactory H^2 type relation, the second step is to do the appropriate estimates on this relation

Step 1. Obtaining the relation (4.40). We now derive a satisfactory relation involving $\frac{d}{dt}\int w_{nxx}^2$. As before, we use (4.11), twice differentiated :

$$w_{nxxt}+w_{nxxxxx}+\left(u_n^4-\sum_{j=1}^{N}R_j^4\right)_{xxx}=0.$$

We multiply it by w_{nxx}, and do an integration by parts, to obtain

$$\frac{1}{2}\frac{d}{dt}\int {w_{nxx}}^2 = \int \left(u_n^4 - \sum_{j=1}^N R_j^4\right)_{xx} w_{nxxx}$$
$$= \int \left(u_n^4 - R^4\right)_{xx} w_{nxxx} + \int \left(R^4 - \sum_{j=1}^N R_j^4\right)_{xxx} w_{nxx}.$$

The second integral is harmless. Let us develop the first term :

$$\begin{aligned}\left(u_n^4 - R^4\right)_{xx} &= 4(u_{nxx}u_n^3 - R_{xx}R^3) + 12({u_{nx}}^2 u_n^2 - R_x^2 R^2) \\ &= 4w_{nxx}u_n^3 + 4((U(t)V+R)_{xx}u_n^3 - R_{xx}R^3) \\ &\quad + 12({u_{nx}}^2u_n^2 - R_x^2R^2).\end{aligned}$$

We put in front the factor w_{nxx}, in view of an integration by parts. Indeed, we want to get rid of the 3 derivative term w_{nxxx}. We compute :

$$\begin{aligned}&\int \left(u_n^4 - R^4\right)_{xx} w_{nxxx} = -6\int {w_{nxx}}^2 u_{nx}u_n^2 \\ &\quad - 4\int ((U(t)V+R)_{xx}u_n^3 - R_{xx}R^3)_x w_{nxx} - 12\int ({u_{nx}}^2u_n^2 - R_x^2R^2)_x w_{nxx} \\ &= -6\int {w_{nxx}}^2 u_{nx}u_n^2 - 4\int ((U(t)V+R)_{xxx}u_n^3 - R_{xxx}R^3)w_{nxx} \\ &\quad - 12\int ((U(t)V+R)_{xx}u_{nx}u_n^2 - R_{xx}R_xR^2)w_{nxx} \\ &\quad - 24\int (u_{nxx}u_{nx}u_n^2 - R_{xx}R_xR^2)w_{nxx} - 24\int ({u_{nx}}^3u_n - R_x^3R)w_{nxx}.\end{aligned}$$

Let us focus on the first term on the last line, to get :

$$\begin{aligned}&= -30\int {w_{nxx}}^2 u_{nx}u_n^2 - 4\int ((U(t)V+R)_{xxx}u_n^3 - R_{xxx}R^3)w_{nxx} \\ &\quad - 36\int ((U(t)V+R)_{xx}u_{nx}u_n^2 - R_{xx}R_xR^2)w_{nxx} \\ &\quad - 24\int ({u_{nx}}^3u_n - R_x^3R)w_{nxx}.\end{aligned}$$

The first term $\int {w_{nxx}}^2 u_{nx}u_n^2$ is troublesome, as when developing it contains $\int {w_{nxx}}^2 R_xR^2$, which we do not control yet. This is why we will correct this by considering :

$$\begin{aligned}\frac{d}{dt}\int {w_{nx}}^2u_n^3 &= 2\int w_{nxt}w_{nx}u_n^3 + 3\int {w_{nx}}^2u_{nt}u_n^2 \\ &= -2\int w_{nxxxx}w_{nx}u_n^3 - \int \left(u_n^4 - \sum_{j=1}^N R_j^4\right)_{xx} w_{nx}u_n^3 \\ &\quad - 3\int {w_{nx}}^2u_{nxxx}u_n^2 - 12\int {w_{nx}}^2u_{nx}u_n^5.\end{aligned}$$

Remark that :

$$-\int\left(u_n^4-\sum_{j=1}^n R_j^4\right)_{xx} w_{nx}u_n^3=-\int(u_n^4-R^4)_{xx}w_{nx}u_n^3-\int\left(R^4-\sum_{j=1}^n R_j^4\right)_{xx} w_{nx}u_n^3,$$

where the second integral will be treated as usual. Two terms are to be rearranged in the previous expression : those with high derivative. The first one is

$$\begin{aligned}-2\int w_{nxxxx}w_{nx}u_n^3 &= 2\int w_{nxxx}w_{nxx}u_n^3+6\int w_{nxxx}w_{nx}u_{nx}u_n^2\\ &= -9\int w_{nxx}^2u_{nx}u_n^4-6\int w_{nxx}w_{nx}u_{nxx}u_n^2-12\int w_{nxx}w_{nx}u_{nx}^2u_n\\ &= -15\int w_{nxx}^2u_{nx}u_n^2+6\int w_{nxx}^2(U(t)V+R(t))_xu_n^2\\ &\quad -6\int w_{nxx}w_{nx}(U(t)V+R)_{xx}u_n^2-6\int w_{nxx}w_{nx}u_{nx}^2u_n,\end{aligned}$$

and the second one

$$\begin{aligned}-3\int w_{nx}^2u_{nxxx}u_n^2 &= 6\int w_{nxx}w_{nx}u_{nxx}u_n^4+6\int w_{nx}^2u_{nxx}u_n^2\\ &= 6\int w_{nxx}^2u_{nx}u_n^2-6\int w_{nxx}^2(U(t)V+R)_xu_n^2\\ &\quad +6\int w_{nxx}w_{nx}(U(t)V+R)_{xx}u_n^2-12\int w_{nxx}w_{nx}u_{nx}^2u_n.\end{aligned}$$

So that we get

$$\begin{aligned}\frac{d}{dt}\int w_{nx}^2u_n^3 &= -9\int w_{nxx}^2u_{nx}u_n^2-24\int w_{nxx}w_{nx}u_{nx}^2u_n\\ &\quad -\int(u_n^4-R^4)_{xx}w_{nx}u_n^3-\int\left(R^4-\sum_{j=1}^n R_j^4\right)_{xx}w_{nx}u_n^3-12\int w_{nx}^2u_{nx}u_n^5.\end{aligned} \tag{4.39}$$

If we put everything together, we obtain the desired equality, on which we will do all our estimates :

$$\begin{aligned}&\frac{d}{dt}\left(\frac12\int w_{nxx}^2-\frac{20}{3}\int w_{nx}^2u_n^3\right)\\ &= -4\int((U(t)V+R)_{xxx}u_n^3-R_{xxx}R^3)w_{nxx}\\ &\quad -36\int((U(t)V+R)_{xx}u_{nx}u_n^2-R_{xx}R_xR^2)w_{nxx}\\ &\quad -24\int(u_{nx}^3u_n-R_x^3R)w_{nxx}+40\int w_{nxx}w_{nx}u_{nx}^2u_n\\ &\quad +\frac{20}{3}\int(u_n^4-R^4)_{xx}w_{nx}u_n^3+80\int w_{nx}^2u_{nx}u_n^5\\ &\quad +\int\left(R^4-\sum_{j=1}^N R_j^4\right)_{xxx}w_{nxx}+\frac{20}{3}\int\left(R^4-\sum_{j=1}^n R_j^4\right)_{xx}w_{nx}u_n^3.\end{aligned} \tag{4.40}$$

Step 2. Estimating terms in (4.40). We now estimate separately every term appearing in the right hand side of (4.40).

• First let us bound the 2 terms of (4.40) with $R^4 - \sum_j R_j^4$.

$$\left| \int \left(R^4 - \sum_{j=1}^{N} R_j^4 \right)_{xxx} w_{nxx} \right| \leq C e^{-\frac{\sigma_0 \sqrt{\sigma_0}}{4} t} \| w_{nxx} \|_{L^2}. \tag{4.41}$$

• And (recall $\|u_n\|_{L^\infty} \leq C$) :

$$\left| \int (R^4 - \sum_{j=1}^{n} R_j^4)_{xx} w_{nx} u_n^3 \right| \leq C e^{-\frac{\sigma_0 \sqrt{\sigma_0}}{4} t} \| w_{nx} \|_{L^2}. \tag{4.42}$$

• Let us now consider the terms with exponent 8 in (4.40).

$$\int (u_n^4 - R^4)_{xx} w_{nx} u_n^3 = \int w_{nx} \left(\sum_{k=1}^{4} C_4^k (w_n + U(t)V)^k R^{4-k} \right)_{xx} u_n^3.$$

So that all terms are at least quadratic in w or $(w + U(t)V)$. We do an integration by parts on the (unique) term with $w_{nxx} w_{nx}$. Thus, all the terms with at least one R are controlled by

$$C \|w\|^2_{H^1(1-\psi_0(t))} + C \|U(t)V\|^2_{H^2(1-\psi_0(t))}.$$

It remains to treat

$$\int ((w_n + U(t)V)^4)_{xx} w_{nx} (w_n + U(t)V)^3.$$

Again, the term containing w_{nxx} is treated with an integration by parts, to have 3 terms with 1 derivative. The term with $U(t)V_{xx}$ is in some sense the worst, although the fact $V \in H^{2,2}$ allows to bound it (this is similar to what happens in the purely linear case [8]) :

$$\|U(t)V_{xx}(w_n + U(t)V)\|_{L^\infty} \|w_{nx}(w_n + U(t)V)\|_{L^\infty} \|w_n + U(t)V\|^4_{L^4} \leq \frac{C}{t^{8/3}}.$$

For the terms with three terms with one derivative, one of these is controlled in L^2, which gives the same decay rate $Ct^{-8/3}$. Finally, we get

$$\left| \int (u_n^4 - R^4)_{xx} w_{nx} u_n^3 \right| \leq C \|w\|^2_{H^1(1-\psi_0(t))} + C \|U(t)V\|^2_{H^2(1-\psi_0(t))} + \frac{C}{t^{8/3}}. \tag{4.43}$$

Arguing similarly allows us to bound the second term

$$\left| \int w_{nx}^2 u_{nx} u_n^5 \right| \leq \|w_n\|^2_{H^1(1-\psi_0(t))} + \frac{C}{t^{8/3}}. \tag{4.44}$$

We will now consider each of the 4 remaining terms of (4.40) separately. However, one constant in the treatment will be that the term w_{nxx} always appear exactly once, and will be controlled in L^2. The second point will be that all terms where only w_n and $U(t)V$ appear (not R) will be controlled by

$$\frac{C}{t^{4/3}} \|w_{nxx}\|_{L^2}.$$

All the terms that do not fall in this category will be bounded by a control of the type "estimates on the right", as they contain both R and $w_n + U(t)V$ (there is no term with only R).

To do this, we develop each term in a "purely linear" part and a "linear-non linear" interaction part.

- $\int ((U(t)V+R)_{xxx}u_n^3 - R_{xxx}R^3)w_{nxx}$. We develop our main term :

$$\begin{aligned}(U(t)V+R)_{xxx}u_n^3 - R_{xxx}R^3 \\ &= U(t)V_{xxx}(w_n+U(t)V)^3 + U(t)V_{xxx}R\cdot\sum_{k=0}^{2} C_3^k(w_n+U(t)V)^kR^{2-k} \\ &+ R_{xxx}(w+U(t)V)\cdot\sum_{k=1}^{3} C_3^k(w_n+U(t)V)^{k-1}R^{3-k}.\end{aligned}$$

Remember $V\in H^{3,1}$ so that $V_{xx}\in H^{1,1}$, and we get

$$\begin{aligned}&\left|\int ((U(t)V+R)_{xxx}u_n^3 - R_{xxx}R^3)w_{nxx}\right| \\ &\le C\Big(\|U(t)V_{xxx}(w+U(t)V)\|_{L^\infty}\|w_n+U(t)V\|_{L^\infty}\|w_n+U(t)V\|_{L^2} \\ &\quad + \|U(t)V\|_{H^3(1-\psi_0(t))} + \|w\|_{H^1(1-\psi_0(t))}\Big)\|w_{nxx}\|_{L^2} \\ &\le C\left(\frac{1}{t^{4/3}} + \|U(t)V\|_{H^3(1-\psi_0(t))} + \|w\|_{H^1(1-\psi_0(t))}\right)\|w_{nxx}\|_{L^2}. \end{aligned} \tag{4.45}$$

- $\int ((U(t)V+R)_{xx}u_{nx}u_n^2 - R_{xx}R_xR^2)w_{nxx}$. We develop as before

$$\begin{aligned}&(U(t)V+R)_{xx}u_{nx}u_n^2 - R_{xx}R_xR^2 \\ &= U(t)V_{xx}(w_n+U(t)V)_x(w_n+U(t)V)^2 + U(t)V_{xx}R_xu_n^2 \\ &\quad + R_{xx}(w+U(t)V)_xu_n^2 + U(t)V_{xx}(w_n+U(t)V)_x(2(w_n+U(t)V)+R)R \\ &\quad + R_{xx}R_x(w_n+U(t)V)(w_n+U(t)V+2R).\end{aligned}$$

So that :

$$\begin{aligned}&\left|\int (U(t)V+R)_{xx}u_{nx}u_n^2 - R_{xx}R_xR^2w_{nxx}\right| \\ &\le \Big(\|U(t)V_{xx}\|_{L^2}\|(w_n+U(t)V)_x(w_n+U(t)V)\|_{L^\infty}\|w_n+U(t)V\|_{L^\infty} \\ &\quad + C\|U(t)V\|_{H^2(1-\psi_0(t))} + \|w_n\|_{H^1(1-\psi_0(t))}\Big)\|w_{nxx}\|_{L^2}.\end{aligned} \tag{4.46}$$

The last two terms are the hardest : the assumption of high integrability ($p\ge 4$) is crucially used. Indeed, these terms contain the information on $\int u_x^5u^{p-4} = -4\int u_{xx}u_x^3u^{p-3}$.

- $\int ({u_n}_x^3u_n - R_x^3R)w_{nxx}$. We develop as usual

$${u_n}_x^3u_n - R_x^3R = (w_n+U(t)V)_x^3(w_n+U(t)V) + (w_n+U(t)V)_x^3R + R_x^3(w_n+U(t)V)$$

$$+ (w_n + U(t)V)_x R_x \cdot \left(\sum_{k=1}^{2} C_3^k (w_n + U(t)V)_x^{k-1} R^{2-k} \right) \cdot u_n.$$

First let us forget the first term with no soliton term, and focus on the last three. Recall that $w_{nx}, U(t)V_x \in L^\infty$. All these term have R and $w_n + U(t)V$ (with at most 1 derivative) in factor, so that they are bounded by

$$C\|w_n\|_{H^1(1-\psi_0(t))}\|w_{nxx}\|_{L^2}.$$

Let us now turn to the remaining term

$$\begin{aligned}
&\int (w_n + U(t)V)_x^3 (w_n + U(t)V) w_{nxx} \\
&\qquad = \int (w_n + U(t)V)_x^2 (w_n + U(t)V) w_{nx} w_{nxx} \\
&\qquad\quad + \int U(t)V_x (w_n + U(t)V)_x (w_n + U(t)V)(w_n + U(t)V)_x w_{nxx}.
\end{aligned}$$

We use our previously obtained decay $\|w_{nx}\|_{L^2} \leq Ct^{-1/3}$, and the a priori bound $\|w_{nx}\|_{L^\infty} \leq \varepsilon_0$ in the first integral, and $\|U(t)V_x\|_{L^\infty} \leq Ct^{-1/3}$ (as $V_x \in H^{1,1}$) for the second integral, to get the estimate

$$\begin{aligned}
&\left| \int (w_n + U(t)V)_x^3 (w_n + U(t)V) w_{nxx} \right| \\
&\leq \|w_{xx}\|_{L^2} \|w_x\|_{L^2} \|(w_n + U(t)V)_x\|_{L^\infty} \|(w_n + U(t)V)_x (w_n + U(t)V)\|_{L^\infty} \\
&\quad + \|w_{xx}\|_{L^2} \|w_{nx} + U(t)V_x\|_{L^2} \|(w_n + U(t)V)_x (w_n + U(t)V)\|_{L^\infty} \|U(t)V_x\|_{L^\infty} \\
&\leq \frac{C}{t^{4/3}} \|w_{nxx}\|_{L^2}.
\end{aligned}$$

So that we obtain in the end

$$\left| \int (u_{nx}^3 u_n - R_x^3 R) w_{nxx} \right| \leq C \left(\frac{1}{t^{4/3}} + \|w_n\|_{H^1(1-\psi_0(t))} \right) \|w_{nxx}\|_{L^2}. \tag{4.47}$$

- $\int w_{nxx} w_{nx} u_{nx}^2 u_n$. We develop as usual

$$\begin{aligned}
\int w_{nxx} w_{nx} u_{nx}^2 u_n = &\int w_{nxx} w_{nx} (w_n + U(t)V)_x^2 (w_n + U(t)V) \\
&+ \int w_{nxx} w_{nx} (w_n + U(t)V)_x^2 R + \int w_{nxx} w_{nx} R_x u_{nx} u_n.
\end{aligned}$$

The last two terms are clearly controlled as in the previous case by

$$\|w_n\|_{H^1(1-\psi_0(t))}\|w_{nxx}\|_{L^2}.$$

And for the term on the first line :

$$\begin{aligned}
&\left| \int w_{nxx} w_{nx} (w_n + U(t)V)_x^2 (w_n + U(t)V) \right| \\
&\leq \|w_{nxx}\|_{L^2} \|w_{nx}\|_{L^2} \|(w_n + U(t)V)_x\|_{L^\infty} \|(w_n + U(t)V)_x (w_n + U(t)V)\|_{L^\infty}
\end{aligned}$$

$$\leq \frac{C}{t^{4/3}}\|w_{nxx}\|_{L^2}.$$

And we get for this last term :

$$\left|\int w_{nxx}w_{nx}u_{nx}^2u_n\right| \leq C\left(\frac{1}{t^{4/3}}+\|w_n\|_{H^1(1-\psi_0(t))}\right)\|w_{nxx}\|_{L^2}. \tag{4.48}$$

Step 3. Conclusion of the H^2 bound. All the terms on the right hand side in (4.40) were estimated. As we would like to have a bound on $\|w_{nxx}\|_{L^2}$ (without the corrective term), we have to use an integral form for these bounds, and we have to estimate the corrective term $\int {w_{nx}}^2u_n^3$. When developing u_n^3, treating the term with R on one side and the purely "linear" term on the other side, we get

$$\begin{aligned}\left|\int {w_{nx}}^2u_n^3\right| &\leq \|w_n\|^2_{H^1(1-\psi_0(t)}+\int {w_{nx}}^2|w_n+U(t)V|^3\\ &\leq \|w_n\|^2_{H^1(1-\psi_0(t)}+\frac{C}{t^{5/3}}.\end{aligned}$$

If we put everything together, for this H^2 estimate, starting from the equation (4.40), and the bounds for each term (4.41), (4.42), (4.43), (4.44), (4.45), (4.46), (4.47), and (4.48), we get

$$\begin{aligned}&\left|\frac{d}{dt}\left(\frac{1}{2}\int {w_{nxx}}^2-\int {w_{nx}}^2u_n^3\right)\right|\\ &\leq C\Bigg(\frac{\|w_n\|_{H^2}}{e^{\frac{\sigma_0\sqrt{\sigma_0}}{4}t}}+\|w_n\|^2_{H^1(1-\psi_0(t))}+\|U(t)V\|^2_{H^2(1-\psi_0(t))}+\frac{(1+\|V\|_{H^{2,2}})}{t^{8/3}}\\ &\quad+\left(\frac{1+\|V\|_{H^{3,1}}}{t^{4/3}}+\|w_n\|_{H^1(1-\psi_0(t))}+\|U(t)V\|_{H^3(1-\psi_0(t))}\right)\|w_{nxx}\|_{L^2}\Bigg).\end{aligned}$$

Let us integrate in time between t and S_n, so that as soon as

$$\|w_n\|_{H^1(1-\psi_0(t))}+\|U(t)V\|_{H^3(1-\psi_0(t))}\leq\frac{C}{t^{4/3}},$$

we get, for all $t\in[I_n,S_n]$,

$$\|w_{nxx}(t)\|^2_{L^2}\leq\frac{C}{t^{5/3}}+\int_t^{S_n}\frac{\|w_{nxx}(\tau)\|_{L^2}}{\tau^{4/3}}d\tau.$$

With Lemma 4.4, we derive :

$$\forall t\in[I_n,S_n],\qquad \|w_{nxx}(t)\|_{L^2}\leq\frac{C}{t^{1/3}}.$$ □

4.6 M_0^t estimate : proof of Proposition 4.4

We now want to conclude the proof of Proposition 1', that is to prove that for $t\in[I_n,S_n]$,

$$M_0^t(w_n(t))=\|w_n(t)\|_{H^1}+\|D^\alpha Jw_n(t)\|_{L^2}+\|DJw_n(t)\|_{L^2}\leq\frac{C}{t^\delta}.$$

As $\delta < \frac{1}{3}$, it only remains to estimate $\|D^\alpha J^t w_n\|_{L^2}$ and $\|DJ^t w_n\|_{L^2}$. As in [13] and [8], we do the computations on the dilation operator

$$I^t f = xf + 3t \int_{-\infty}^{x} f_t dx,$$

as it is easier to compute with. So we will proceed in two lemmas, one concerning $I^t w_n$, and then coming back from $I^t w_n$ to $J^t w_n$. Let us first do a short reminder of commutation properties of these operators. Let us note $L = \partial_t + \partial_{xxx}$ the linear KdV operator. Then

$$I^t f - J^t f = 3t \int_{-\infty}^{x} Lf dx.$$

We have the following commutation relations :

$$[L, J^t] = 0, \quad [L, I^t]f = 3 \int_{-\infty}^{x} Lf dx, \quad [J^t, \partial_x] = [I^t, \partial_x] = -Id.$$

Notice that $I^t U(t)V - J^t U(t)V = 3t \int_{-\infty}^{x} LU(t)V dx = 0$, hence

$$\|D^\alpha I^t U(t)V\|_{L^2} + \|DI^t U(t)V\|_{L^2} \leq C\|V\|_{H^{1,1}}.$$

4.6.1 $I^t w_n$ estimates

Let f so that the following has a sense and $\Phi : \mathbb{R} \to \mathbb{R}$ a C^1 function. Then we have the chain rule relation :

$$I^t(\Phi(f)_x) = x\Phi(f)_x + 3t\Phi(f)_t = x\Phi'(f)f_x + 3t\Phi'(f)f_t = \Phi'(f)I^t f_x. \tag{4.49}$$

We will use this formula for $\Phi(x) = x^4$ and $f = u_n$ or $f = R$.

Let us start with $\|I^t w_{nx}\|_{L^2}$ as the result obtained will then be used for $\|D^\alpha I w_n\|_{L^2}$. We proceed in a very analoguous way as for the H^2 estimate, in 3 similar steps.

$\|I^t w_{nx}\|_{L^2}$ estimate.

Step 1. Notice that $(LI^t f, f) = \frac{1}{2}\frac{d}{dt}\|f(t)\|_{L^2}$, which is why we compute :

$$\begin{aligned} LI^t w_{nx} &= I^t L w_{nx} + L w_n = -I^t\left(\left(u_n^4 - \sum_{j=1}^{N} R_j^4\right)_{xx}\right) - \left(u_n^4 - \sum_{j=1}^{N} R_j^4\right)_x \\ &= -I^t(u_n^4 - R^4)_{xx} - (u_n^4 - R^4)_x + I^t\left(R^4 - \sum_{j=1}^{N} R_j^4\right)_{xx} - \left(R^4 - \sum_{j=1}^{N} R_j^4\right)_x. \end{aligned} \tag{4.50}$$

Let us can focus on

$$\begin{aligned} -I^t(u_n^4 - R^4)_{xx} - (u_n^4 - R^4)_x &= (I^t(u_n^4 - R^4)_x)_x - 2(u_n^4 - R^4)_x \\ &= -4(u_n^3 I^t u_{nx} - R^3 I^t R_x)_x - 2(u_n^4 - R^4)_x \\ &= -12(u_{nx}u_n^2 I^t u_{nx} - R_x R^2 I^t R_x) \\ &\quad - 4(u_n^3(I^t u_{nx})_x - R^3(I^t R_x)_x) - 8(u_{nx}u_n^3 - R_x R^3). \end{aligned}$$

So that

$$LI^t w_{nx} = -12(u_{nx}u_n^2 I^t u_{nx} - R_x R^2 I^t R_x)$$

$$
\begin{aligned}
&- 4(u_n^3(I^t u_{nx})_x - R^3(I^t R_x)_x) - 8(u_{nx}u_n^3 - R_x R^3)\\
&+ I^t\Big(R^4 - \sum_{j=1}^N R_j^4\Big)_{xx} - \Big(R^4 - \sum_{j=1}^N R_j^4\Big)_x. \qquad (4.51)
\end{aligned}
$$

This expression of $LI^t w_{nx}$ is the one will develop.

Step 2. As previously, for every term in (4.51), we take the "purely linear" term apart. All the remaining terms contain both $w_n + U(t)V$ and R, and so will be bounded using estimates "on the right" obtained in Section 4.

- Of course the terms on the last line will be negligible :

$$
\left|\int \Big(I^t\Big(R^4 - \sum_{j=1}^N R_j^4\Big)_{xx} - \Big(R^4 - \sum_{j=1}^N R_j^4\Big)_x\Big) I^t w_{nx}\right| \le Cte^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|I^t w_{nx}\|_{L^2}.
$$

- Then consider

$$
\begin{aligned}
u_{nx}u_n^3 - R_x R^3 &= (w_n + U(t)V)_x (w_n + U(t)V)^3\\
&+ (w_n + U(t)V)_x R \cdot \Big(\sum_{k=0}^2 C_3^k (w_n + U(t)V)^k R^{2-k}\Big)\\
&+ R_x (w_n + U(t)V) \cdot \Big(\sum_{k=1}^3 C_3^k (w_n + U(t)V)^{k-1} R^{3-k}\Big).
\end{aligned}
$$

The last two lines have both a localizing term R or R_x, and $w_n + U(t)V$ with at most 1 derivative ; for the first term we use the argument of the linear case, the L^2 norm going on one $(w_n + U(t)V)$, so that

$$
\begin{aligned}
&\left|\int (u_{nx}u_n^3 - R_x R^3) I^t w_{nx}\right|\\
&\le \Big(\|(w_n + U(t)V)_x (w_n + U(t)V)\|_{L^\infty}\|w_n + U(t)V\|_{L^\infty}\|w_n + U(t)V\|_{L^2}\\
&\quad + C\|w_n + U(t)V\|_{H^1(1-\psi_0(t))}\Big)\|I^t w_{nx}\|_{L^2}\\
&\le C\Big(\frac{1}{t^{4/3}} + \|w_n\|_{H^1(1-\psi_0(t))}\Big)\|I^t w_{nx}\|_{L^2}. \qquad (4.52)
\end{aligned}
$$

For the two other terms, we have to be a little more careful.

- We develop

$$
\begin{aligned}
u_n^3(I^t u_{nx})_x - R^3(I^t R_x)_x &= (w_n + U(t)V)^3((I^t w_{nx})_x + (I^t U(t)V_x)_x)\\
&+ (w_n + U(t)V)^3(I^t R_x)_x + R^3((I^t w_{nx})_x + (I^t U(t)V_x)_x)\\
&+ 3(w_n + U(t)V)R(w + U(t)V + R)((I^t w_{nx})_x + (I^t(U(t)V + R)_x)_x).
\end{aligned}
$$

First, split all the terms between those containing $(Iw_{nx})_x$ and those with $(IU(t)V_x)_x$ or $(IU(t)V_x)_x$. Now multiply all by Iw_{nx}, and integrate in x. For the terms containing $(Iw_{nx})_x$, further integrate by parts. We get

$$
\int (u_n^3(I^t u_{nx})_x - R^3(I^t R_x)_x) I^t w_{nx}
$$

$$
\begin{aligned}
&= -\frac{3}{2}\int (w_n+U(t)V)_x(w_n+U(t)V)^2(I^t w_{nx})^2\\
&\quad + \int (w_n+U(t)V)^3(I^tU(t)V_x)_x I^t w_{nx} + \int (w_n+U(t)V)^3(I^tR_x)_x I^t w_{nx}\\
&\quad - \frac{1}{2}\int (R^3)_x(I^t w_{nx})^2 + \int R^3(I^tU(t)V_x)_x I^t w_{nx}\\
&\quad - \frac{1}{2}\int A_x(I^t w_{nx})^2 + \int A(I^t(U(t)V+R)_x)_x)I^t w_{nx},
\end{aligned}
$$

where $A = 3(w_n+U(t)V)R(w_n+U(t)V+R)$. Then the first line is bounded as a regular "linear" term by

$$\frac{C}{t^{4/3}}\|I^t w_{nx}\|_{L^2}^2 \leq \frac{C}{t^{4/3}}\|I^t w_{nx}\|_{L^2}.$$

Observe that $(I^tU(t)V_x)_x = (J^tU(t)V_x)_x = (U(t)xV_x)_x$. As $V \in H^{2,2}$, $xV_x \in H^{1,1}$ and $(U(t)xV_x)_x$ has the "almost $t^{-2/3}$" decay of Lemma 4.1. So that the first term of the second line is bounded by

$$\frac{C}{t^{4/3}}\|I^t w_{nx}\|_{L^2}.$$

Notice that uniformly for $t \geq 1$,

$$|I^tR|(x) \leq Ct(1-\psi_0(t,x)). \tag{4.53}$$

And the same is true with derivatives on R etc. So that the second term of the second line is bounded by

$$Ct\|w+U(t)V\|_{L^2(1-\psi_0(t))}\|I^t w_{nx}\|_{L^2}.$$

We now have to bound $\int R^3(I^t w_{nx})^2$. This is the key point where we need some result on a H^3 decay on the right for w_n. Indeed, recall that by definition

$$I^t w_{nx} = xw_{nx} + 3tw_{nt} = xw_{nx} - 3tw_{nxxx} - 3t\left(u_n^4 - \sum_{j=1}^N R_j^4\right)_x.$$

Proceeding as previously, we naturally obtain ($t \geq I_n \geq 1$)

$$\|RI^t w_{nx}\|_{L^2} \leq Ct\|w_n\|_{H^3(1-\psi_0(t))}. \tag{4.54}$$

So that :

$$\left|\int R^3(I^t w_{nx})^2\right| \leq Ct\|w_n\|_{H^3(1-\psi_0(t))}\|I^t w_{nx}\|_{L^2}.$$

We go on treating our terms :

$$\left|\int R^3(I^tU(t)V_x)_x I^t w_{nx}\right| \leq \|U(t)(xV_x)_x\|_{L^2(1-\psi_0(t))}\|I^t w_{nx}\|_{L^2}.$$

And for the last line, we have the bound

$$
\begin{aligned}
&C\|(w_n+U(t)V)R\|_{W^{1,\infty}}\|I^t w_{nx}\|_{L^2}^2\\
&\qquad + C\|(w_n+U(t)V)R\|_{L^\infty}(\|(I^tU(t)V_x)_x\|_{L^2} + \|(I^tR_x)_x\|_{L^2})\|I^t w_{nx}\|_{L^2}.
\end{aligned}
$$

But $\|(I^tU(t)V_x)_x\|_{L^2} = \|U(t)(xV_x)_x\|_{L^2} = \|(xV_x)_x\|_{L^2}$, and $\|(I^tR_x)_x\|_{L^2} \leq Ct$. And of course

$$\|(w+U(t)V)R\|_{W^{1,\infty}} \leq C\|w_n\|_{H^2(1-\psi_0(t))} + \|U(t)V\|_{H^2(1-\psi_0(t))},$$

so that our bound for this last line rewrites

$$C(\|w_n\|_{H^2(1-\psi_0(t))} + \|U(t)V\|_{H^2(1-\psi_0(t))})(\|V\|_{H^{2,2}} + t + 1)\|I^tw_{nx}\|_{L^2}.$$

And for the second term of our main expression, we get ($t \geq 1$)

$$\begin{aligned} &\left|\int (u_n^3(I^tu_{nx})_x - R^3(I^tR_x)_x)I^tw_{nx}\right| \\ &\leq C\left(\frac{1}{t^{4/3}} + t\|w_n\|_{H^3(1-\psi_0(t))} + t\|U(t)V\|_{H^2(1-\psi_0(t))}\right)\|I^tw_{nx}\|_{L^2}. \end{aligned} \tag{4.55}$$

• We can now turn to the last term :

$$\begin{aligned} &u_{nx}u_n^2I^tu_{nx} - R_xR^2I^tR_x \\ &= (w_n + U(t)V)_x(w_n+U(t)V)^2I^t(w_n+U(t)V)_x \\ &\quad + (w_n+U(t)V)_x(w_n+U(t)V)^2I^tR_x \\ &\quad + (w_n+U(t)V)_xR(2(w_n+U(t)V)R)\cdot I^tu_{nx} \\ &\quad + R_xR^2I^t(w_n+U(t)V)_x + R_x(w_n+U(t)V)(2(w_n+U(t)V)+R)\cdot I^tu_{nx}. \end{aligned}$$

Multiply by I^tw_{nx}, and integrate in x. Remember that $\|I^tw_{nx}\|_{L^2} \leq \varepsilon$ by assumption on $[I_n, S_n]$, $\|I^tU(t)V_x\|_{L^2} \leq C\|V\|_{H^{1,1}}$ and $I^tR_x\|_{L^2} \leq Ct$, so that $\|I^tu_{nx}\|_{L^2} \leq Ct$. We obtain

$$\begin{aligned} &\int (u_{nx}u_n^2I^tu_{nx} - R_xR^2I^tR_x)I^tw_{nx} \\ &\qquad \leq \|(w_n+U(t)V)_x(w_n+U(t)V)\|_{L^\infty}\|w_n+U(t)V\|_{L^\infty} \\ &\qquad\qquad \times \|I^t(w_n+U(t)V)_x\|_{L^2}\|I^tw_{nx}\|_{L^2} \\ &\qquad + \|w_n+U(t)V\|_{L^\infty}^2\|(w_n+U(t)V)_xI^tR_x\|_{L^2}\|I^tw_{nx}\|_{L^2} \\ &\qquad + C\|(w_n+U(t)V)_xR\|_{L^\infty}\|I^tu_{nx}\|_{L^2}\|I^tw_{nx}\|_{L^2} \\ &\qquad + \|R_xR^2I^t(w_n+U(t)V)_x\|_{L^2}\|I^tw_{nx}\|_{L^2} \\ &\qquad + C\|(w_n+U(t)V)R_x\|_{L^2}\|I^tu_{nx}\|_{L^2}\|I^tw_{nx}\|_{L^2}. \end{aligned}$$

The only non straightforward term is $R_xR^2I^t(w_n+U(t)V)_x$. Now, analogously to (4.54), we have

$$\|R_xR^2I^tw_{nx}\|_{L^2} \leq Ct\|w_n\|_{H^3(1-\psi_0(t))}.$$

And we directly get

$$\|R_xR^2I^tU(t)V_x\|_{L^2} \leq C\|U(t)(xV_x)_x\|_{L^2(1-\psi_0(t))}.$$

So that when rewriting the previous estimate, we obtain

$$\begin{aligned} &\int (u_{nx}u_n^2I^tu_{nx} - R_xR^2I^tR_x)I^tw_{nx} \\ &\qquad \leq \frac{C}{t^{p/3}}\|I^tw_{nx}\|_{L^2} + Ct^{1/3}\|w_n\|_{H^1(1-\psi_0(t))}\|I^tw_{nx}\|_{L^2} \end{aligned}$$

$$+ Ct(\|w_n\|_{H^2(1-\psi_0(t))} + \|U(t)V\|_{H^2(1-\psi_0(t))})\|I^t w_{nx}\|_{L^2}$$
$$+ C(t\|w_n\|_{H^3(1-\psi_0(t))} + \|U(t)(xV_x)_x\|_{L^2(1-\psi_0(t))})\|I^t w_{nx}\|_{L^2}$$
$$+ Ct\|w_n + U(t)V\|_{L^2(1-\psi_0(t))}\|I^t w_{nx}\|_{L^2}. \tag{4.56}$$

Step 3. Let us now conclude the $I^t w_{nx}$ estimate : we add up the results of (4.52), (4.55), and (4.56), plug them in (4.51), and get

$$\left|\frac{1}{2}\frac{d}{dt}\|I^t w_{nx}\|_{L^2}^2\right| \leq C\Bigg(\frac{1}{t^{4/3}} + t\|w_n\|_{H^3(1-\psi_0(t))} + t\|U(t)V\|_{H^2(1-\psi_0(t))}$$
$$+ \|U(t)(xV_x)_x\|_{L^2(1-\psi_0(t))}\Bigg)\|I^t w_{nx}\|_{L^2}.$$

So that after integration in time between t and S_n, we have

$$\|I^t w_{nx}\|_{L^2} \leq \frac{C}{t^{4/3}}, \tag{4.57}$$

as soon as

$$t\|w_n\|_{H^3(1-\psi_0(t))} + t\|U(t)V\|_{H^2(1-\psi_0(t))} + \|U(t)(xV_x)_x\|_{L^2(1-\psi_0(t))} \leq \frac{C}{t^{4/3}}.$$

Notice that thanks to $DI^t w_n = I^t w_{nx} + w_{nx}$, we also have

$$\|DI^t w_n\|_{L^2} \leq \frac{C}{t^{1/3}}. \tag{4.58}$$

This will be useful for the following of the proof.

$\|D^\alpha I^t w_n\|_{L^2}$ *estimate.*
Step 1 and 2. Let us compute

$$LI^t w_n = I^t L w_n + 3\int L w_n = -I^t\left(u_n^4 - \sum_{j=1}^N R_j^4\right)_x - 3\left(u_n^4 - \sum_{j=1}^N R_j^4\right)$$
$$= -4(u_n^3 I^t u_{nx} - R^3 I^t R_x) - 3(u_n^4 - R^4)$$
$$- 4\left(R^3 I^t R_x - \sum_{j=1}^N R_j^3 I^t R_{j_x}\right) - 3\left(R^4 - \sum_{j=1}^N R_j^4\right). \tag{4.59}$$

What we want is then to apply operator D^α to our equation, multiply both sides by $D^\alpha I^t w_n$ and integrate in x : we get $\frac{1}{2}\|D^\alpha I^t w_n\|_{L^2}^2$ on the left hand side, and we are to do some estimations on the right hand side. As we already have an estimate on $DI^t w_n$ we can avoid a discussion on the behavior of D^α with respect to a product of functions. Indeed, apart from the purely "linear term" which is treated as in [8], we will use

$$|(D^\alpha h, D^\alpha I^t w_n)| = |(h, D^{2\alpha} I^t w_n)| \leq \|h\|_{L^2}(\|D^\alpha I^t w_n\| + \|DI^t w_n\|_{L^2}). \tag{4.60}$$

(as $\alpha < 1/2$). Now, let us bound the terms in (4.59).
• First :

$$\left\| -4\Bigg(R^3 I^t R_x - \sum_{j=1}^N R_j^3 I^t R_{j_x}\Bigg) - 3\Bigg(R^4 - \sum_{j=1}^N R_j^4\Bigg)\right\|_{L^2} \leq Cte^{-\frac{\sigma\sqrt{\sigma_0}}{4}t}. \tag{4.61}$$

• Second :

$$u_n^4 - R^4 = (w_n + U(t)V)^4 + (w_n + U(t)V)R \cdot \left(\sum_{k=1}^{3} C_4^k (w_n + U(t)V)^{k-1} R^{3-k}\right).$$

From this we get (using (4.60) on the second term)

$$\begin{aligned}|(D^\alpha(u_n^4 - R^4), D^\alpha I^t w_n)| &\le |(D^\alpha(w_n + U(t)V)^4, D^\alpha I^t w_n)| \\ &\quad + \|(w + U(t)V)\|_{L^2(1-\psi_0(t))}(\|D^\alpha I^t w_n\|_{L^2} + \|DI^t w_n\|_{L^2}).\end{aligned}$$

Now, thanks to the first estimate of Lemma 4.2 with $g = w_n + U(t)V$, we get

$$\begin{aligned}\|D^\alpha(w_n + U(t)V)^4\|_{L^2} &\le \|g\|_{L^6}^3 \left(\|g_x g\|_{L^\infty}^{1/2} + \|g\|_{L^\infty}^{3\gamma} \|g_x g\|_{L^\infty}^{(1-3\gamma)/2}\right) \\ &\le \frac{C}{t^{1-\frac{1}{6}}}\left(\frac{1}{t^{\frac{1}{2}}} + \frac{1}{t^\gamma} \cdot \frac{1}{t^{\frac{1-3\gamma}{2}}}\right) \le \frac{C}{t^{4/3-\gamma/2}}.\end{aligned}$$

So that

$$\begin{aligned}|(D^\alpha(u_n^4 - R^4), D^\alpha I^t w_n)| &\le \frac{C}{t^{1/3}} \|w\|_{H^1(1-\psi_0(t))} \\ &\quad + \left(\frac{C}{t^{4/3-\gamma/2}} + \|w\|_{H^1(1-\psi_0(t))}\right)\|D^\alpha I^t w_n\|_{L^2}. \quad (4.62)\end{aligned}$$

• And for the last remaining term (the first in the expression of $LI^t w_n$),

$$\begin{aligned}(u_n^3 I u_{nx} - R^3 I^t R_x) &= (w_n + U(t)V)^3 I^t (w_n + U(t)V)_x + (w_n + U(t)V)^3 I^t R_x \\ &\quad + R u_n^2 I^t (w_n + U(t)V)_x + R(w_n + U(t)V)(w_n + U(t)V + 2R) I^t R_x.\end{aligned}$$

Consider the fist term of the right hand side. Using the second estimate of Lemma 4.2 in an analogous way as for (4.62), with $g = w_n + U(t)V$ and $h = I^t w_n + U(t)V$, we have

$$\begin{aligned}&\|D^\alpha (w_n + U(t)V)^3 I^t (w_n + U(t)V)_x\|_{L^2} \\ &\qquad \le C\left(\frac{1}{t^{1/3}} \cdot \frac{1}{t} + \frac{1}{t^{(1-2\gamma)/3}} \cdot \frac{1}{t} + \frac{1}{t^{(1+2\gamma)/3}} \cdot \frac{1}{t^{1-\gamma}}\right) \le \frac{C}{t^{4/3-2\gamma/3}}.\end{aligned}$$

For all the other terms, we use (4.60), so that we are looking for an L^2 control.

$$\begin{aligned}&\|(w_n + U(t)V)^3 I^t R_x\|_{L^2} \le Ct\|w + U(t)V\|_{L^2(1-\psi_0(t))}, \\ &\left\|R \cdot \left(\sum_{k=0}^{2} C_2^k (w_n + U(t)V)^k R^{2-k}\right) \cdot I^t (w_n + U(t)V)_x\right\|_{L^2} \\ &\qquad \le C\|w_n\|_{H^3(1-\psi_0(t))} + C\|U(t)xV_x\|_{L^2(1-\psi_0(t))}, \\ &\left\|R(w_n + U(t)V) \cdot \left(\sum_{k=1}^{2} C_2^k (w_n + U(t)V)^{k-1} R^{2-k}\right) \cdot I^t R_x \cdot\right\|_{L^2} \\ &\qquad \le Ct\|(w + U(t)V\|_{L^2(1-\psi_0(t))}.\end{aligned}$$

And for this last term, we get (using 4.57))

$$|(D^\alpha(u_n^3 I u_{nx} - R^3 I^t R_x), D^\alpha I^t w_{nn})|$$

$$\leq C\left(\frac{1}{t^{4/3-2\gamma/3}}+t\|w_n\|_{H^1(1-\psi_0(t)}+\|w_n\|_{H^3(1-\psi_0(t))}+\|U(t)xV_x\|_{L^2(1-\psi_0(t))}\right)$$
$$\times\left(\frac{1}{t^{1/3}}+\|D^\alpha I^t w_{nn}\|_{L^2}\right). \tag{4.63}$$

Step 3. We can now sum up the results of (4.61), (4.62) and (4.63), and obtain

$$\left|\frac{d}{dt}\|D^\alpha I^t w_n\|_{L^2}^2\right| \leq C\left(\frac{1}{t^{4/3-2\gamma/3}}+t\|w_n\|_{H^1(1-\psi_0(t))}+\|w_n\|_{H^3(1-\psi_0(t))}\right.$$
$$\left.+\|U(t)xV_x\|_{L^2(1-\psi_0(t))}\right)\left(\frac{1}{t^{1/3}}+\|D^\alpha I^t w_n\|_{L^2}\right).$$

So that after integration in time between t and S_n, we get

$$\|D^\alpha I^t w_n\|_{L^2} \leq \frac{C}{t^{(1-2\gamma)/3}}=\frac{C}{t^\delta}, \tag{4.64}$$

as soon as

$$t\|w_n\|_{H^3(1-\psi_0(t))}+t\|U(t)V\|_{H^2(1-\psi_0(t))}+\|U(t)(xV_x)\|_{H^1(1-\psi_0(t))}\leq\frac{C}{t^{p/3}}.$$

(condition for both estimates (4.57) and (4.64)).

4.6.2 $J^t w_n$ estimates

We only need to go from our previous estimates (4.64) and (4.57) to estimates on $J^t w_n$. First recall that $I^t f(x) - J^t f(x) = 3t\int_{-\infty}^x Lf$. Thus

$$\|D^\alpha J^t w_n\|_{L^2}+\|DJ^t w_n\|_{L^2}\leq\|D^\alpha I^t w_n\|_{L^2}+\|I^t w_{nx}\|_{L^2}+t\|D^\alpha u_n^4-D^\alpha R^4\|_{L^2}$$
$$+t\|Du_n^4-DR^4\|_{L^2}+t\left\|D^\alpha\left(R^4-\sum_{j=1}^N R_j^4\right)\right\|_{L^2}+t\left\|D\left(R^4-\sum_{j=1}^N R_j^4\right)\right\|_{L^2}.$$

From (4.57) and (4.64), we have

$$\|D^\alpha I^t w_n\|_{L^2}+\|I^t w_{nx}\|_{L^2}\leq Ct^{-\delta}.$$

Obviously, we also have

$$t\left\|D^\alpha\left(R^4-\sum_{j=1}^N R_j^4\right)\right\|_{L^2}+t\left\|D\left(R^4-\sum_{j=1}^N R_j^4\right)\right\|_{L^2}\leq Cte^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}.$$

Now consider

$$t\|D^\alpha u_n^4-D^\alpha R^4\|_{L^2}+t\|Du_n^4-DR^4\|_{L^2}\leq t\|u_n^4-(w_n+U(t)V)^4-R^4\|_{H^1}$$
$$+t\|D^\alpha(w_n+U(t)V)^4\|_{L^2}+4t\|(w_n+U(t)V)_x(w_n+U(t)V)^3\|_{L^2}.$$

Using again the first estimate of Lemma 4.2 with $g=w_n+U(t)V$ (see (4.62)) :

$$t\|D^\alpha(w_n+U(t)V)^4\|_{L^2}\leq t\frac{C}{t^{4/3-\gamma/2}}\leq\frac{C}{t^{1/3-\gamma/2}}\leq\frac{C}{t^\delta}.$$

And also,

$$t\|(w_n+U(t)V)_x(w_n+U(t)V)^3\|_{L^2}\le Ct\|(w_n+U(t)V)_x(w_n+U(t)V)\|_{L^\infty}$$
$$\times\|w_n+U(t)V\|_{L^\infty}\|w_n+U(t)V\|_{L^2}\le\frac{C}{t^{1/3}}.$$

Finally

$$\begin{aligned}u_n^4-(w_n+U(t)V)^4-R^4&=(w_n+U(t)V)R\cdot\left(\sum_{k=1}^{3}C_4^k(w_n+U(t)V)^{k-1}R^{3-k}\right)\\&=(w_n+U(t)V)RA,\end{aligned}$$

where $\|A\|_{H^1}\le C$. As H^1 is an algebra,

$$\|u_n^4-(w_n+U(t)V)^4-R^4\|_{L^2}\le\|(w_n+U(t)V)R\|_{H^1}\|A\|_{H^1}\le C\|w_n\|_{H^1(1-\psi_0(t))}.$$

And we are done as soon as $\|w_n\|_{H^1(1-\psi_0(t))}\le Ct^{-4/3}$.

Finally we obtained

$$\|D^\alpha J^t w_n\|_{L^2}+\|DJ^t w_n\|_{L^2}\le Ct^{-\delta}.$$

This concludes the proof of Proposition 1', and thus of Proposition 4.1.

Appendix. H^3 and H^4 uniform decay estimates on $w_n(t)$

We complete the proof of 4.3, by giving the detailed proof of the H^3 and H^4 estimates.

Proof of Proposition 4.3, H^3 and H^4 cases. $\dot H^3$ estimate.

Step 1 : deriving the H^3 almost conservation law. Let us differentiate (4.11) three times :

$$w_{nxxxt}+w_{nxxxxxx}+\left(u_n^4-\sum_{j=1}^N R_j^4\right)_{xxxx}=0.$$

We multiply it by w_{nxxx}, and do an integration by parts, to obtain

$$\begin{aligned}\frac12\frac{d}{dt}\int w_{nxxx}^2&=\int\left(u_n^4-\sum_{j=1}^N R_j^4\right)_{xxx}w_{nxxxx}\\&=\int\left(u_n^4-R^4\right)_{xxx}w_{nxxxx}+\int\left(R^4-\frac{C}{t^a}\sum_{j=1}^N R_j^4\right)_{xxxx}w_{nxxx}.\end{aligned}$$

The second integral is harmless. Let us develop the first term :

$$\begin{aligned}\left(u_n^4-R^4\right)_{xxx}&=4(u_{nxxx}u_n^3-R_{xxx}R^3)+36(u_{nxx}u_{nx}u_n^2-R_{xx}R_xR^2)\\&\quad+24(u_{nx}^3u_n-R_x^3R)\\&=4w_{nxxx}u_n^3+4((U(t)V+R)_{xx}u_n^3-R_{xx}R^3)\\&\quad+36(u_{nxx}u_{nx}u_n^2-R_{xx}R_xR^2)+24(u_{nx}^3u_n-R_x^3R).\end{aligned}$$

We try to get rid of the w_{nxxxx} terms, by integration by parts.

$$\begin{aligned}
&\int \left(u_n^4 - R^4\right)_{xxx} w_{nxxxx} \\
&= -6 \int w_{nxxx}^2 u_{nx} u_n^2 - 4 \int w_{nxxx}((U(t)V+R)_{xxxx} u_n^3 - R_{xxxx} R^3) \\
&\quad - 12 \int w_{nxxx}((U(t)V+R)_{xxx} u_{nx} u_n^2 - R_{xxx} R_x R^2) - 36 \int w_{nxxx}^2 u_{nx} u_n^2 \\
&\quad - 36 \int w_{nxxx}((U(t)V+R)_{xxx} u_{nx} u_n^2 - R_{xxx} R_x R^2) \\
&\quad - 36 \int w_{nxxx}(u_{nxx}^2 u_n^2 - R_{xx}^2 R^2) - 144 \int w_{nxxx}(u_{nxx} u_{nx}^2 u_n - R_{xx} R_x^2 R) \\
&\quad - 24 \int w_{nxxx}(u_{nx}^4 - R_x^4).
\end{aligned}$$

We now get the troublesome term $-42 \int w_{nxxx}^2 u_{nx} u_n^2$. We thus introduce

$$\begin{aligned}
\frac{d}{dt} \int w_{nxx}^2 u_n^3 &= 2 \int w_{nxxt} w_{nxx} u_n^3 + 3 \int w_{nxx}^2 u_{nt} u_n^2 \\
&= -2 \int w_{nxxxxx} w_{nxx} u_n^3 - \int \left(u_n^4 - \sum_{j=1}^n R_j^4\right)_{xxx} w_{nxx} u_n^3 \\
&\quad - 3 \int w_{nxx}^2 u_{nxxx} u_n^2 - 12 \int w_{nxx}^2 u_{nx} u_n^5.
\end{aligned}$$

First :

$$\begin{aligned}
&- \int \left(u_n^4 - \sum_{j=1}^n R_j^4\right)_{xxx} w_{nxx} u_n^3 \\
&\qquad = - \int (u_n^4 - R^4)_{xxx} w_{nxx} u_n^3 - \int \left(R^4 - \sum_{j=1}^n R_j^4\right)_{xxx} w_{nxx} u_n^3,
\end{aligned}$$

where the second integral will be treated as usual. Now we rearrange the term with high derivatives (more than 3) through integrations by parts.

$$\begin{aligned}
&-2 \int w_{nxxxxx} w_{nxx} u_n^3 \\
&= 2 \int w_{nxxxx} w_{nxxx} u_n^3 + 6 \int w_{nxxxx} w_{nxx} u_{nx} u_n^2 \\
&= -9 \int w_{nxxx}^2 u_{nx} u_n^2 - 6 \int w_{nxxx} w_{nxx} u_{nxx} u_n^2 - 12 \int w_{nxxx} w_{nxx} u_{nx}^2 u_n.
\end{aligned}$$

So that we get

$$\begin{aligned}
\frac{d}{dt} \int w_{nx}^2 u_n^3 &= -9 \int w_{nxx}^2 u_{nx} u_n^2 - 24 \int w_{nxx} w_{nx} u_{nx}^2 u_n - \int (u_n^4 - R^4)_{xx} w_{nx} u_n^3 \\
&\quad - \int \left(R^4 - \sum_{j=1}^n R_j^4\right)_{xx} w_{nx} u_n^3 - 12 \int w_{nx}^2 u_{nx} u_n^5.
\end{aligned}$$

We derived the desired relation on w_{nn} at level $\dot H^3$:

$$\begin{aligned}
&\frac{d}{dt}\left(\frac{1}{2}\int {w_{n}}_{xxx}^2 - \frac{28}{3}\int {w_n}_{xx}^2 u_n^3\right)\\
&\quad = -4\int w_{nxxx}((U(t)V+R)_{xxxx}u_n^3 - R_{xxxx}R^3)\\
&\quad - 48\int w_{nxxx}((U(t)V+R)_{xxx}u_{nx}u_n^2 - R_{xxx}R_xR^2)\\
&\quad - 36\int w_{nxxx}({u_n}_{xx}^2u_n^2 - R_{xx}^2R^2) - 144\int w_{nxxx}(u_{nxx}{u_n}_x^2u_n - R_{xx}R_x^2R)\\
&\quad - 24\int w_{nxxx}({u_n}_x^4 - R_x^4) + 52\int w_{nxxx}w_{nxx}u_{nxx}u_n^2\\
&\quad + 104\int w_{nxxx}w_{nxx}{u_n}_x^2u_n + 28\int {w_n}_{xx}^2u_{nxxx}u_n^2\\
&\quad - 112\int {w_n}_{xx}^2u_{nx}u_n^5 + \frac{28}{3}\int (u_n^4-R^4)_{xxx}w_{nxx}u_n^3\\
&\quad - \frac{28}{3}\int\left(R^4-\sum_{j=1}^n R_j^4\right)_{xxx} w_{nxx}u_n^3 + \int\left(R^4-\sum_{j=1}^N R_j^4\right)_{xxxx} w_{nxxx}. \qquad (4.65)
\end{aligned}$$

Step 2. Estimating terms in (4.65). There are 10 lines to consider. From now on, A_i, A_i', A_i'', ... will denote a polynomial in w_n, $U(t)V$, R and their derivatives (involved in the term on line i), defining a function whose properties are given right after we introduced it.

- $\int w_{nxxx}((U(t)V+R)_{xxx}u_n^3 - R_{xxx}R^3)$.

$$\begin{aligned}
\|(U(t)V+R)_{xxxx}u_n^3 - R_{xxxx}R^3\|_{L^2} &\le \|U(t)V_{xxxx}(w_n+U(t)V)^3\|_{L^2}\\
&\quad + \|U(t)V_{xxxx}RA_1\|_{L^2} + \|R_{xxxx}(w_n+U(t)V)A_1'\|_{L^2}.
\end{aligned}$$

with $\|A_1\|_{L^\infty} + \|A_1'\|_{L^\infty} \le C$. Using that $V\in H^{4,1}$, that is $V_{xxx}\in H^{1,1}$, we get

$$\begin{aligned}
&\left|\int w_{nxxx}((U(t)V+R)_{xxxx}u_n^3 - R_{xxxx}R^3)\right|\\
&\qquad \le C\left(\frac{1}{t^{4/3}} + \|U(t)V\|_{H^4(1-\psi_0(t))} + \|w_n\|_{L^2(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2}. \qquad (4.66)
\end{aligned}$$

- $\int w_{nxxx}((U(t)V+R)_{xxx}u_{nx}u_n^2 - R_{xxx}R_xR^2)$.

$$\begin{aligned}
&(U(t)V+R)_{xxx}u_{nx}u_n^2 - R_{xxx}R_xR^2\\
&= U(t)V_{xxx}(w_n+U(t)V)_x(w_n+U(t)V)^2 + U(t)V_{xxx}(w_n+U(t)V)_xRA_2\\
&\quad + U(t)V_{xxx}R_xu_n^2 + R_{xxx}(w_n+U(t)V)_xu_n^2 + R_{xxx}R_x(w_n+U(t)V)A_2',
\end{aligned}$$

with $\|A_1\|_{L^\infty} + \|A_1'\|_{L^\infty} \le C$. For the "linear" term, we bound $U(t)V_{xxx}$ in L^2 and the rest using the pointwise estimates of Lemma 4.1, and obtain

$$\left|\int w_{nxxx}((U(t)V+R)_{xxx}u_{nx}u_n^2 - R_{xxx}R_xR^2\right| \le C\left(\frac{1}{t^{4/3}} + \|U(t)V\|_{H^3(1-\psi_0(t))} + \|w_n\|_{H^1(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2}. \quad (4.67)$$

- $\int w_{nxxx}(u_{nxx}^2 u_n^2 - R_{xx}^2 R^2)$.

$$u_{nxx}^2 u_n^2 - R_{xx}^2 R^2 = (w_n+U(t)V)_{xx}^2(w_n+U(t)V)^2 + (w_n+U(t)V)_{xx}^2 RA_3 + 2(w_n+U(t)V)_{xx}R_{xx}u_n^2 + R_{xx}^2(w_n+U(t)V)A_3',$$

with $\|A_3\|_{L^\infty} + \|A_3'\|_{L^\infty} \le C$. The L^2 norm of the second line is bounded by $\|w_n + U(t)V\|_{H^2(1-\psi_0(t))}$. The first term needs some attention and the use of estimate $\|w_{nxx}\|_{L^2} \le Ct^{-1/3}$ obtained earlier.

$$\begin{aligned}\left|\int w_{nxxx}(w_n+U(t)V)_{xx}^2(w_n+U(t)V)^2\right| &\le C\|w_{nxxx}\|_{L^2}\|w_{xx}\|_{L^4}^2\|w_n+U(t)V\|_{L^\infty}^2 \\ &\quad + C\|w_{nxxx}\|_{L^2}\|U(t)V_{xx}\|_{L^2}\|U(t)V_{xx}(w_n+U(t)V)\|_{L^\infty}\|w_n+U(t)V\|_{L^\infty} \\ &\le C\|w_{nxxx}\|_{L^2}\left(\|w_{nxx}\|_{L^2}^{3/2}\|w_{xxx}\|_{L^2}^{1/2}\frac{1}{t^{2/3}} + \frac{1}{t^{4/3}}\right) \\ &\le \frac{C}{t^{4/3}}\|w_{nxxx}\|_{L^2} + \frac{C}{t^{7/6}}\|w_{nxxx}\|_{L^2}^{3/2}.\end{aligned}$$

(we used $V \in H^{3,1}$). And for this term :

$$\left|\int w_{nxxx}(u_{nxx}^2 u_n^2 - R_{xx}^2 R^2)\right| \le C\left(\frac{1}{t^{4/3}} + \|w_n+U(t)V\|_{H^2(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2} + \frac{C}{t^{7/6}}\|w_{nxxx}\|_{L^2}^{3/2}. \quad (4.68)$$

- $\int w_{nxxx}(u_{nxx}u_{nx}^2u_n - R_{xx}R_x^2R)$.

$$\begin{aligned}&u_{nxx}u_{nx}^2u_n - R_{xx}R_x^2R \\ &= (w_n+U(t)V)_{xx}(w_n+U(t)V)_x^2(w_n+U(t)V) \\ &\quad + (w_n+U(t)V)_{xx}(w_n+U(t)V)_x^2RA_4 \\ &\quad + (w_n+U(t)V)_{xx}R_x(2(w_n+U(t)V)_x+R_x)u_n + R_{xx}R_x^2(w_n+U(t)V)A_4' \\ &\quad + R_{xx}(w_n+U(t)V)_x(2(w_n+U(t)V)_x+R_x)u_n,\end{aligned}$$

with $\|A_4\|_{L^\infty} + \|A_4'\|_{L^\infty} \le C$. Let aside the first term, all the others are bounded in L^2 norm by $\|w_n + U(t)V)_x\|_{H^2(1-\psi_0(t))}$. Now for the remaining first term

$$\|(w_n+U(t)V)_{xx}(w_n+U(t)V)_x^2(w_n+U(t)V)\|_{L^2}$$

$$\leq \|(w+U(t)V)_{xx}\|_{L^2}\|(w_n+U(t)V)_x\|_{L^\infty}\|(w_n+U(t)V)_x(w_n+U(t)V)\|_{L^\infty}.$$

Now $\|w_{nx}\|_{L^\infty} \leq Ct^{1/3}$ by interpolation, and as $V \in H^{2,2}$, $V_x \in L^1$ so that $\|U(t)V_x\|_{L^\infty} \leq Ct^{-1/3}$. So that our term bounded by

$$Ct^{-1/3}t^{-1} \leq Ct^{-4/3},$$

and we get

$$\left|\int w_{nxxx}(u_{nxx}u_{nx}^2u_n - R_{xx}R_x^2R)\right| \leq C\left(\frac{1}{t^{4/3}} + \|w_n+U(t)V\|_{H^2(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2}. \tag{4.69}$$

- $\displaystyle\int w_{nxxx}(u_{nx}^4 - R_x^4)$.

$$u_{nx}^4 - R_x^4 = (w_n+U(t)V)_x^4 + (w_n+U(t)V)_x^4R_xA_5,$$

where A_5 has factors with 1 derivative. As $\|(w_n+U(t)V)_x\|_{L^\infty} \leq \|w_n+U(t)V\|_{H^2} \leq C$, $\|A_5\|_{L^\infty} \leq C$. With the same estimate, we get that the last two terms are bounded in L^2 norm by $\|w_n+U(t)V\|_{H^1(1-\psi_0(t))}$. For the very first term, notice that

$$\|(w_n+U(t)V)_x^4\|_{L^2} \leq C\|w_{nx}\|_{L^8}^4 + \|U(t)V_x\|_{L^2}\|U(t)V_x\|_{L^\infty}^3 \leq \frac{C}{t^{4/3}} + \frac{C}{t^{3/2}}.$$

Indeed, we interpolate $\|w_{nx}\|_{L^8}$ between $\|w_{nx}\|_{L^2}$ and $\|w_{nxx}\|_{L^2}$, which both get decay rate of $Ct^{-1/3}$, so that $\|w_{nx}\|_{L^8} \leq Ct^{-1/3}$. Furthermore,

$$\|U(t)V_x^2\|_{L^\infty} \leq \frac{C}{t}M_0^t(U(t)V)M_0^t(U(t)V_x) \leq \frac{C}{t}\|V\|_{H^{2,2}},$$

hence the second estimate. And we have

$$\left|\int w_{nxxx}(u_{nx}^4 - R_x^4)\right| \leq C\left(\frac{1}{t^{4/3}} + \|w_n+U(t)V\|_{H^2(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2}. \tag{4.70}$$

- $\displaystyle\int w_{nxxx}w_{nxx}u_{nxx}u_n^2$.

$$u_{nxx}u_n^2 = w_{nxx}(w_n+U(t)V)^2 + U(t)V_{xx}(w_n+U(t)V)^2 + (w_n+U(t)V)_{xx}RA_6 + R_{xx}u_n^2,$$

with $\|A_6\|_{L^\infty} \leq C$. Then we compute :

$$\left|\int w_{nxxx}w_{nxx}^2(w_n+U(t)V)^2\right| \leq \|w_{nxxx}\|_{L^2}\|w_{nxx}\|_{L^4}^2\|w_n+U(t)V\|_{L^\infty}^2$$
$$\leq \frac{C}{t^{1/3\cdot 3/2+2/3}}\|w_{nxxx}\|_{L^2}^{3/2} \leq \frac{C}{t^{7/6}}\|w_{nxxx}\|_{L^2}^{3/2}.$$

($\|w_{nxx}\|_{L^4} \leq \|w_{nxx}\|_{L^2}^{3/4}\|w_{nxxx}\|_{L^2}^{1/4}$). For the second term, as $V_x \in H^{1,1}$,

$$\left|\int w_{nxxx}w_{nxx}U(t)V_{xx}(w_n+U(t)V)^2\right|$$

$$\leq \|w_{nxxx}\|_{L^2}\|w_{nxx}\|_{L^2}\|U(t)V_{xx}(w_n+U(t)V)\|_{L^\infty}\|w_n+U(t)V\|_{L^\infty}$$
$$\leq \frac{C}{t^{5/3}}\|w_{nxxx}\|_{L^2}.$$

And for the last two terms, as $\|w_{nxx}\|_{L^\infty} \leq \|w_{nxx}\|_{L^2}^{1/2}\|w_{nxxx}\|_{L^2}^{1/2}$,

$$\left|\int w_{nxxx}w_{nxx}(w_n+U(t)V)_{xx}RA_6\right| + \left|\int w_{nxxx}w_{nxx}R_{xx}u_n^2\right|$$
$$\leq \|w_{nxxx}\|_{L^2}\left(\|w_{nxx}\|_{L^\infty}\|w_n+U(t)V\|_{H^2(1-\psi_0(t))} + \|_{L^2}\|w_{nxx}\|_{H^2(1-\psi_0(t))}\right)$$
$$\leq \|w_n+U(t)V\|_{H^2(1-\psi_0(t))}\|w_{nxxx}\|_{L^2}^{3/2} + \|w_{nxx}\|_{L^2(1-\psi_0(t))}\|w_{nxxx}\|_{L^2}.$$

Therefore, for the whole term :

$$\left|\int w_{nxxx}w_{nxx}u_{nxx}u_n^2\right| \leq C\left(\frac{1}{t^{(5/3}} + \|w_{nxx}\|_{L^2(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2}$$
$$+ C\left(\frac{1}{t^{7/6}} + \|w_n+U(t)V\|_{H^2(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2}^{3/2}. \qquad (4.71)$$

- $\int w_{nxxx}w_{nxx}u_{nx}^2u_n.$

$$u_{nx}^2u_n = (w_n+U(t)V)_x^2(w_n+U(t)V)$$
$$+ (w_n+U(t)V)_x^2RA_7 + (w_n+U(t)V)_xR_xu_n + R_x^2u_n,$$

with $\|A_7\|_{L^\infty} \leq C$. As $\|(w_n+U(t)V)_x\|_{L^\infty} \leq C$ we get

$$\left|\int w_{nxxx}w_{nxx}(w_n+U(t)V)_x^2(w_n+U(t)V)\right|$$
$$\leq \|w_{nxxx}\|_{L^2}\|w_{nxx}\|_{L^2}\|(w_n+U(t)V)_x\|_{L^\infty}\|(w_n+U(t)V)_x(w_n+U(t)V)\|_{L^\infty}$$
$$\leq \frac{C}{t^{4/3}}\|w_{nxxx}\|_{L^2},$$

and for the remaining terms, we clearly have

$$\left|\int w_{nxxx}w_{nxx}(w_n+U(t)V)_x^2RA_7 + (w_n+U(t)V)_xR_xu_n + R_x^2u_n\right|$$
$$\leq \|w_{nxxx}\|_{L^2}\|w_{nxx}\|_{L^2(1-\psi_0(t))}.$$

So that

$$\left|\int w_{nxxx}w_{nxx}u_{nx}^2u_n\right| \leq C\left(\frac{1}{t^{4/3}} + \|w_{nxx}\|_{L^2(1-\psi_0(t))}\right)\|w_{nxxx}\|_{L^2}. \qquad (4.72)$$

- $\int w_{nxx}^2u_{nx}u_n^5 + \int (u_n^4 - R^4)_{xxx}w_{nxx}u_n^3.$

$$u_{nx}u_n^5 = (w_n+U(t)V)_x(w_n+U(t)V)^5 + (w_n+U(t)V)_xRA_9 + R_xu_n^5,$$

with $\|A_8\|_{L^\infty} \leq C$. So that we get directly

$$\left|\int w_{nxx}^2 u_{nx} u_n^5\right| \leq \frac{C}{t^{2/3+1+4/3}} + \|w_{nxx}\|_{L^2(1-\psi_0(t))}\|w_n + U(t)V\|_{H^1(1-\psi_0(t))} + C\|w_{nxx}\|^2_{L^2(1-\psi_0(t))}.$$

Now for the right term

$$\int (u_n^4 - R^4)_{xxx} w_{nxx} u_n^3 = -\int (u_n^4 - R^4)_{xx} w_{nxxx} u_n^3 - 3\int (u_n^4 - R^4)_{xx} w_{nxx} u_{nx} u_n^2.$$

As

$$(u_n^4 - R^4)_{xx} = 4(u_{nxx}u_n^3 - R_{xx}R^2) + 12(u_{nx}^2 u_n^2 - R_x^2 R^2),$$

we get that :

$$\begin{aligned}(u_n^4 - R^4)_{xx} u_n^3 = &\Big(4\big(w_{nxx}(w_n + U(t)V)^3 + U(t)V_{xx}(w_n + U(t)V)^3\\ &+ (w_n + U(t)V)_{xx} R A_8' + R_{xx}(w_n + U(t)V)A_8''\big)\\ &+ 12\big((w_n + U(t)V)_x^2 (w_n + U(t)V)^2 + (w_n + U(t)V)_x^2 R A_8'''\\ &+ 2(w_n + U(t)V)_x R_x u_n^3 + R_{xx}^2 (w_n + U(t)V) A_8''''\big)\Big)(A_8''''' R + (w_n + U(t)V)^3),\end{aligned}$$

where all the $A_8'^{\cdots\prime}$ are bounded in L^∞. Now when developing carefully, we get that :

$$\left|\int (u_n^4 - R^4)_{xxx} w_{nxx} u_n^3\right| \leq \frac{C}{t^{8/3}} + C\|w_n\|^2_{H^2(1-\psi_0(t))} + C\|U(t)V\|^2_{H^2(1-\psi_0(t))} \tag{4.73}$$

- $\displaystyle\int \left(R^4 - \sum_{j=1}^{n} R_j^4\right)_{xxx} w_{nxx} u_n^3$ and $\displaystyle\int \left(R^4 - \sum_{j=1}^{N} R_j^4\right)_{xxxx} w_{nxxx}$.

  We obviously have exponential decay :

$$\left|\int \left(R^4 - \sum_{j=1}^{n} R_j^4\right)_{xxx} w_{nxx} u_n^3\right| + \left|\int \left(R^4 - \sum_{j=1}^{N} R_j^4\right)_{xxxx} w_{nxxx}\right| \leq C e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|w_n\|_{H^3}. \tag{4.74}$$

And finally :

- $\displaystyle\int w_{nxx}^2 u_n^3$.

  As $u_n^3 = (w_n + U(t)V)^3 + RA_{10}$, with $\|A_{10}\|_{L^\infty} \leq C$, we have :

$$\left|\int w_{nxx}^2 u_n^3\right| \leq \frac{C}{t^{2/3+1}} + C\|w_{nxx}\|^2_{L^2(1-\psi_0(t))} \leq \frac{C}{t^{5/3}} + C\|w_{nxx}\|^2_{L^2(1-\psi_0(t))}. \tag{4.75}$$

Step 3. We can now conclude our estimate of (4.65). Let us sum all our estimates (4.66)-(4.74). Then let us integrate in time between t and S_n, and plug in (4.75). We get

$$\begin{aligned}&\|w_{nxxx}\|^2_{L^2}\\ &\leq \frac{C}{t^{5/3}} + C\|w_{nxx}\|^2_{L^2(1-\psi_0(t))} + C\int_t^{S_n} \frac{\|w_{nxxx}(\tau)\|^{3/2}_{L^2}}{t^{7/6}} d\tau\\ &\quad + C\int_t^{S_n}\left(\frac{1}{t^{4/3}} + \|w_n\|_{H^2(1-\psi_0(t))} + \|U(t)V\|_{H^3(1-\psi_0(t))}\right)\|w_{nxxx}(\tau)\|_{L^2} d\tau.\end{aligned}$$

Now, $\left(\frac{7}{6}-1\right)\cdot\frac{1}{1-\frac{3}{4}}=\frac{2}{3}$, so that from Lemma 4.4, we get

$$\|w_{nxxx}\|_{L^2}\leq\frac{C}{t^{1/3}},$$

as soon as $V\in H^{4,1}\cap H^{2,2}$ and

$$\|w_n\|_{H^2(1-\psi_0(t))}+\|U(t)V\|_{H^4(1-\psi_0(t))}\leq\frac{C}{t^{4/3}}.$$

This conclude the $\dot{H}^3$ estimate.

$\dot{H}^4$ estimate

Let us summarize what we obtained until now. We dispose of the global estimates

$$\|w_n(t)\|_{H^4}+M_0^t(w_n(t))\leq\varepsilon_0,\quad\text{and}\quad\|w_n(t)\|_{H^3}\leq\frac{C}{t^{1/3}},$$

along with the following decay on the right estimates (from Corollary 4.1) :

$$t\|w_n\|_{H^3(1-\psi_0(t))}+\|U(t)V\|_{H^5(1-\psi_0(t))}\leq\frac{C}{t^{4/3}}.$$

Step 1 : deriving the H^4 conservation law. Let us differentiate (4.11) four times :

$$w_{nxxxxt}+w_{nxxxxxxx}+\left(u_n^4-\sum_{j=1}^N R_j^4\right)_{xxxxx}=0.$$

We multiply it by w_{nxxxx}, and do an integration by parts, to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int w_{nxxxx}^{\,2}&=\int\left(u_n^4-\sum_{j=1}^N R_j^4\right)_{xxxx}w_{nxxxxx}\\&=\int\left(u_n^4-R^4\right)_{xxxx}w_{nxxxxx}+\int\left(R^4-\sum_{j=1}^N R_j^4\right)_{xxxxx}w_{nxxxx}.\end{aligned}$$

The second integral is harmless. Let us develop the first term

$$\begin{aligned}\left(u_n^4-R^4\right)_{xxxx}&=4(u_{nxxxx}u_n^3-R_{xxxx}R^3)+48(u_{nxxx}u_{nx}u_n^2-R_{xxx}R_xR^2)\\&\quad+36(u_{nxx}^{\,2}u_n^2-R_{xx}^2R^2)+144(u_{nxx}u_{nx}^{\,2}u_n-R_{xx}R_x^2R)+24(u_{nx}^{\,4}-R_x^4)\\&=4w_{nxxxx}u_n^3+4((U(t)V+R)_{xxx}u_n^3-R_{xxx}R^3)\\&\quad+48(u_{nxxx}u_{nx}u_n^2-R_{xxx}R_xR^2)+36(u_{nxx}^{\,2}u_n^2-R_{xx}^2R^2)\\&\quad+144(u_{nxx}u_{nx}^{\,2}u_n-R_{xx}R_x^2R)+24(u_{nx}^{\,4}u_n-R_x^4R).\end{aligned}$$

We try to get rid of the w_{nxxxx} terms, by integration by parts.

$$\int\left(u_n^4-R^4\right)_{xxxx}w_{nxxxxx}$$

$$
\begin{aligned}
&= -6\int w_{nxxxx}^2 u_{nx}u_n^2 - 4\int w_{nxxxx}((U(t)V+R)_{xxxxx}u_n^3 - R_{xxxxx}R^3)\\
&- 60\int w_{nxxxx}((U(t)V+R)_{xxxx}u_{nx}u_n^2 - R_{xxxx}R_xR^2)\\
&- 48\int w_{nxxxx}^2 u_{nx}u_n^2 - 120\int w_{nxxxx}(u_{nxxx}u_{nxx}u_n^2 - R_{xxx}R_{xx}R^2)\\
&- 240\int w_{nxxxx}(u_{nxxx}u_{nx}^2u_n - R_{xxx}R_x^2R)\\
&- 360\int w_{nxxxx}(u_{nxx}^2u_{nx}u_n - R_{xx}^2R_xR) - 240\int w_{nxxxx}(u_{nxx}u_{nx}^3 - R_{xx}R_x^3)
\end{aligned}
$$

We now want to get rid of the troublesome term $-54\int w_{nxxx}^2 u_{nx}u_n^2$. We thus introduce

$$
\begin{aligned}
\frac{d}{dt}\int w_{nxxx}^2u_n^3 &= 2\int w_{nxxxt}w_{nxxx}u_n^3 + 3\int w_{nxxx}^2 u_{nt}u_n^2\\
&= -2\int w_{nxxxxxx}w_{nxxx}u_n^3 - \int\left(u_n^4 - \sum_{j=1}^n R_j^4\right)_{xxxx} w_{nxxx}u_n^3\\
&\quad - 3\int w_{nxxx}^2u_{nxxx}u_n^2 - 12\int w_{nxxx}^2u_{nx}u_n^5.
\end{aligned}
$$

First :

$$
\begin{aligned}
&-\int\left(u_n^4 - \sum_{j=1}^n R_j^4\right)_{xxxx} w_{nxxx}u_n^3\\
&\qquad = -\int (u_n^4 - R^4)_{xxxx}w_{nxxx}u_n^3 - \int\left(R^4 - \sum_{j=1}^n R_j^4\right)_{xxxx} w_{nxxx}u_n^3\\
&\qquad = \int (u_n^4 - R^4)_{xxx}w_{nxxxx}u_n^3 + 3\int (u_n^4 - R^4)_{xxx}w_{nxxx}u_{nx}u_n^2\\
&\qquad\quad - \int\left(R^4 - \sum_{j=1}^n R_j^4\right)_{xxxx} w_{nxxx}u_n^3,
\end{aligned}
$$

where the third integral is immediately controlled. Now we rearrange the term with high derivatives (more than 3) through integrations by parts.

$$
\begin{aligned}
-2\int w_{nxxxxxx}w_{nxxx}u_n^3 &= 2\int w_{nxxxxx}w_{nxxxx}u_n^3 + 6\int w_{nxxxxx}w_{nxxx}u_{nx}u_n^2\\
&= -9\int w_{nxxxx}^2u_{nx}u_n^2 - 6\int w_{nxxxx}w_{nxxx}u_{nxx}u_n^2\\
&\quad - 12\int w_{nxxxx}w_{nxxx}u_{nx}^2u_n.
\end{aligned}
$$

So that we get :

$$
\begin{aligned}
&\frac{d}{dt}\int w_{nxxx}^2u_n^3\\
&= -9\int w_{nxxxx}^2u_{nx}u_n^2 - 6\int w_{nxxxx}w_{nxxx}u_{nxx}u_n^2
\end{aligned}
$$

$$
\begin{aligned}
&- 12 \int w_{nxxxx} w_{nxxx} u_{nx}^2 u_n - 3 \int w_{nxxx}^2 u_{nxxx} u_n^2 - 12 \int w_{nxxx}^2 u_{nx} u_n^5 \\
&+ \int (u_n^4 - R^4)_{xxx} w_{nxxxx} u_n^3 + 3 \int (u_n^4 - R^4)_{xxx} w_{nxxx} u_{nx} u_n^2 \\
&- \int \left(R^4 - \sum_{j=1}^{n} R_j^4 \right)_{xxxx} w_{nxxx} u_n^3
\end{aligned}
$$

And we obtain the following (and last) relation, at level $\dot{H}^4$:

$$
\begin{aligned}
&\frac{d}{dt} \left(\frac{1}{2} \int w_{nxxxx}^2 - 12 \int w_{nxxx}^2 u_n^3 \right) = \\
&- 4 \int w_{nxxxx} ((U(t)V + R)_{xxxxx} u_n^3 - R_{xxxxx} R^3) \\
&- 60 \int w_{nxxxx} ((U(t)V + R)_{xxxx} u_{nx} u_n^2 - R_{xxxx} R_x R^2) \\
&- 120 \int w_{nxxxx} (u_{nxxx} u_{nxx} u_n^2 - R_{xxx} R_{xx} R^2) \\
&- 240 \int w_{nxxxx} (u_{nxxx} u_{nx}^2 u_n - R_{xxx} R_x^2 R) \\
&- 360 \int w_{nxxxx} (u_{nxx}^2 u_{nx} u_n - R_{xx}^2 R_x R) - 240 \int w_{nxxxx} (u_{nxx} u_{nx}^3 - R_{xx} R_x^3 R) \\
&+ 72 \int w_{nxxxx} w_{nxxx} u_{nxx} u_n^2 + 144 \int w_{nxxxx} w_{nxxx} u_{nx}^2 u_n \\
&+ 36 \int w_{nxxx}^2 u_{nxxx} u_n^2 + 144 \int w_{nxxx}^2 u_{nx} u_n^5 - 12 \int (u_n^4 - R^4)_{xxx} w_{nxxxx} u_n^3 \\
&- 36 \int (u_n^4 - R^4)_{xxx} w_{nxxx} u_{nx} u_n^2 + 12 \int \left(R^4 - \sum_{j=1}^{n} R_j^4 \right)_{xxxx} w_{nxxx} u_n^3 \\
&+ \int \left(R^4 - \sum_{j=1}^{N} R_j^4 \right)_{xxxxx} w_{nxxxx}. \qquad (4.76)
\end{aligned}
$$

Step 2. Estimating terms in (4.76). There are 13 lines to consider, and as for the H^3 norm, we will do them one by one. We now note $B_i^{\prime\cdots\prime}$ in place of $A_i^{\prime\cdots\prime}$ in the previous lemma : all B_i are bounded in L^∞.

- $\int w_{nxxxx} ((U(t)V + R)_{xxxxx} u_n^3 - R_{xxxxx} R^3)$.

$$
\begin{aligned}
(U(t)V + R)_{xxxxx} u_n^3 - R_{xxxxx} R^3 = {} & U(t)V_{xxxxx} (w_n + U(t)V)^3 \\
& + U(t)V_{xxxxx} R B_1 + R_{xxxxx} (w_n + U(t)V) B_1'.
\end{aligned}
$$

So that as $V \in H^{4,1}$, we obtain :

$$
\left| \int w_{nxxxx} ((U(t)V + R)_{xxxxx} u_n^3 - R_{xxxxx} R^3 \right|
$$

$$\leq C\left(\frac{1}{t^{4/3}} + \|w_n\|_{H^1(1-\psi_0(t))} + \|U(t)V\|_{H^5(1-\psi_0(t))}\right)\|w_{nxxxx}\|_{L^2}. \quad (4.77)$$

- $\int w_{nxxxx}((U(t)V+R)_{xxxx}u_{nx}u_n^2 - R_{xxxx}R_xR^2)$.

$$\begin{aligned}&(U(t)V+R)_{xxxx}u_{nx}u_n^2 - R_{xxxx}R_xR^2\\ &= U(t)V_{xxxx}(w_n+U(t)V)_x(w_n+U(t)V)^2 + U(t)V_{xxxx}(w_n+U(t)V)_xRB_2\\ &\quad + U(t)V_{xxxx}R_xu_n^2 + R_{xxxx}(w_n+U(t)V)_xu_n^2\\ &\quad + R_{xxxx}R_x(w_n+U(t)V)B_2'.\end{aligned}$$

And as $V\in H^4$, we simply get :

$$\begin{aligned}&\left|\int w_{nxxxx}((U(t)V+R)_{xxxx}u_{nx}u_n^2 - R_{xxxx}R_xR^2)\right|\\ &\qquad\leq C\left(\frac{1}{t^{4/3}} + \|U(t)V\|_{H^4(1-\psi_0(t))} + \|w_n\|_{H^1(1-\psi_0(t))}\right)\|w_{nxxxx}\|_{L^2}. \quad (4.78)\end{aligned}$$

- $\int w_{nxxxx}(u_{nxxx}u_{nxx}u_n^2 - R_{xxx}R_{xx}R^2)$.

$$\begin{aligned}&u_{nxxx}u_{nxx}u_n^2 - R_{xxx}R_{xx}R^2\\ &= (w_n+U(t)V)_{xxx}(w_n+U(t)V)_{xx}(w_n+U(t)V)^2\\ &\quad + (w_n+U(t)V)_{xxx}(w_n+U(t)V)_{xx}RB_3 + (w_n+U(t)V)_{xxx}R_{xx}u_n^2\\ &\quad + R_{xxx}(w_n+U(t)V)_{xx}u_n^2 + R_{xxx}R_{xx}(w_n+U(t)V)B_3'.\end{aligned}$$

Then only considering the first term :

$$\begin{aligned}&\left|\int w_{nxxxx}(w_n+U(t)V)_{xxx}(w_n+U(t)V)_{xx}(w_n+U(t)V)^2\right|\\ &\quad\leq C\|w_{nxxxx}\|_{L^2}\Big(\|w_{nxxx}\|_{L^2}\|w_{nxx}\|_{L^\infty}\|w_n+U(t)V\|_{L^\infty}\\ &\qquad + \|w_{nxxx}\|_{L^2}\|U(t)V_{xx}(w_n+U(t)V)\|_{L^\infty}\\ &\qquad + \|U(t)V_{xxx}(w_n+U(t)V)\|_{L^\infty}\|w_n+U(t)V\|_{H^2}\Big)\|w_n+U(t)V\|_{L^\infty}^2\\ &\quad\leq\left(\frac{C}{t^{4/3}} + \frac{C}{t^{5/3}} + \frac{C}{t^{4/3}}\right)\|w_{nxxxx}\|_{L^2}.\end{aligned}$$

(where we used $V_{xx}\in H^{1,1}$). So that for this term :

$$\begin{aligned}&\left|\int w_{nxxxx}(u_{nxxx}u_{nxx}u_n^2 - R_{xxx}R_{xx}R^2)\right|\\ &\qquad\leq C\left(\frac{1}{t^{4/3}} + \|w_n+U(t)V\|_{H^3(1-\psi_0(t))}\right)\|w_{nxxxx}\|_{L^2}. \quad (4.79)\end{aligned}$$

- $\int w_{nxxxx}(u_{nxxx}u_{nx}^2u_n - R_{xxx}R_x^2R)$.

$$\begin{aligned}
&u_{nxxx}u_{nx}^2u_n - R_{xxx}R_x^2R \\
&= (w_n + U(t)V)_{xxx}(w_n + U(t)V)_x^2(w_n + U(t)V) \\
&\quad + (w_n + U(t)V)_{xxx}(w_n + U(t)V)_x^2 RB_4 + (w_n + U(t)V)_{xxx}R_xB_4'u_n \\
&\quad + R_{xxx}R_x^2(w_n + U(t)V)B_4'' + R_{xxx}R_x(w + U(t)V)_xB_4'''u_n.
\end{aligned}$$

Now, we have :

$$\|w_{nx} + U(t)V_x\|_{L^\infty} \leq \|w_n\|_{H^2} + \|U(t)V_x\|_{L^\infty} \leq \frac{C}{t^{1/3}},$$

as $V_x \in L^1$. So that

$$\|(w_n + U(t)V)_{xxx}(w_n + U(t)V)_x^2(w_n + U(t)V)\|_{L^2} \leq \frac{C}{t^{1/3}} \cdot \frac{C}{t} \leq \frac{C}{t^{4/3}}.$$

And we get

$$\left|\int w_{nxxxx}(u_{nxxx}u_{nx}^2u_n - R_{xxx}R_x^2R)\right| \leq C\left(\frac{1}{t^{4/3}} + \|w_n + U(t)V\|_{H^3(1-\psi_0(t))}\right)\|w_{nxxxx}\|_{L^2}. \tag{4.80}$$

- $\int w_{nxxxx}(u_{nxx}^2u_{nx}u_n - R_{xx}^2R_xR)$.

$$\begin{aligned}
&u_{nxx}^2u_{nx}u_n - R_{xx}^2R_xR \\
&\qquad = (w_n + U(t)V)_{xx}^2(w_n + U(t)V)_x(w_n + U(t)V) \\
&\qquad\quad + (w_n + U(t)V)_{xx}^2(w_n + U(t)V)_xRB_5 + (w_n + U(t)V)_{xx}^2R_xu_n \\
&\qquad\quad + (w_n + U(t)V)_{xx}R_xB_5'u_{nx}u_n + R_{xx}^2(w_n + U(t)V)_xu_n \\
&\qquad\quad + R_{xx}^2R_x(w_n + U(t)V)B_5''.
\end{aligned}$$

As previously as $V_{xx} \in H^{1,1}$:

$$\|(w_n + U(t)V)_{xx}\|_{L^\infty} \leq \|w_n\|_{H^3} + \|U(t)V_{xx}\|_{L^\infty} \leq \frac{C}{t^{1/3}}.$$

So that

$$\|(w_n + U(t)V)_{xx}^2(w_n + U(t)V)_x(w_n + U(t)V)\|_{L^2} \leq C\|(w_n + U(t)V)_{xx}\|_{L^2} \cdot \frac{1}{t^{\frac{1}{3}+1}} \leq \frac{C}{t^{4/3}}.$$

And we get

$$\left|\int w_{nxxxx}(u_{nxx}^2u_{nx}u_n - R_{xx}^2R_xR)\right| \leq C\left(\frac{1}{t^{4/3}} + \|w_n + U(t)V\|_{H^3(1-\psi_0(t))}\right)\|w_{nxxxx}\|_{L^2}. \tag{4.81}$$

- $\int w_{nxxxx}(u_{nxx}u_{nx}^3 - R_{xx}R_x^3)$.

$$\begin{aligned} u_{nxx}u_{nx}^3 - R_{xx}R_x^3 =& w_{nxx}(w_n + U(t)V)_x^3 + U(t)V_{xx}w_{nx}B_6 + U(t)V_{xx}U(t)V_x^3 \\ &+ (w_n + U(t)V)_{xx}R_xB_6' + R_{xx}(w_n + U(t)V)_xB_6'', \end{aligned}$$

where $\|B_6\|_{L^\infty} \leq Ct^{-2/3}$ (it is a homogeneous polynomial of degree 2 in w_{nx} and $U(t)V_x$), and B_6', B_6'' are bounded in L^∞. Now (the L^2 norm goes to a w_n-type term when possible, and $V_x \in H^{1,1}$) :

$$\begin{aligned} \|w_{nxx}(w_n + U(t)V)_x^3\|_{L^2} &\leq \frac{C}{t^{1/3}} \cdot \frac{C}{t}, \\ \|U(t)V_{xx}w_{nx}B_6\|_{L^2} &\leq \frac{C}{t^{1/3}} \cdot \frac{C}{t^{1/3}} \cdot \frac{C}{t^{2/3}}, \\ \|U(t)V_{xx}U(t)V_x^3\|_{L^2} &\leq \frac{C}{t} \cdot \frac{C}{t^{1/3}}. \end{aligned}$$

So that the "linear term" is bounded by $Ct^{-4/3}$, and we have

$$\left|\int w_{nxxxx}(u_{nxx}u_{nx}^3 - R_{xx}R_x^3)\right| \leq C\left(\frac{1}{t^{4/3}} + \|w_n + U(t)V\|_{H^2(1-\psi_0(t))}\right)\|w_{nxxxx}\|_{L^2}. \quad (4.82)$$

- $\int w_{nxxxx}w_{nxxx}u_{nxx}u_n^2$.

$$u_{nxx}u_n^2 = (w_n + U(t)V)_{xx}(w_n + U(t)V)^2 + (w_n + U(t)V)_{xx}RB_8 + R_{xx}u_n^2.$$

No as $\|w_{nxxx}\|_{L^2} \leq Ct^{-1/3}$ and :

$$\|(w_n + U(t)V)_{xx}(w_n + U(t)V)^2\|_{L^\infty} \leq \frac{C}{t},$$

we get

$$\left|\int w_{nxxxx}w_{nxxx}u_{nxx}u_n^2\right| \leq C\left(\frac{1}{t^{5/3}} + \|w_n\|_{H^3(1-\psi_0(t))} + \|U(t)V\|_{H^2(1-\psi_0(t))}\right)\|w_{nxxxx}\|_{L^2}. \quad (4.83)$$

- $\int w_{nxxxx}w_{nxxx}u_{nx}^2u_n$.
  It is almost like the previous one.

$$u_{nx}^2u_n = (w_n + U(t)V)_x^2(w_n + U(t)V) + (w_n + U(t)V)_x^2RB_9 + RB_9'u_n.$$

Now along with $\|w_{nxxx}\|_{L^2} \leq Ct^{-1/3}$, we have :

$$\|(w_n + U(t)V)_x^2(w_n + U(t)V)\|_{L^\infty} \leq \frac{C}{t^{1+1/3}} = \frac{C}{t^{4/3}}.$$

So that

$$\left|\int w_{nxxxx}w_{nxxx}u_{nx}^2u_n\right| \leq C\left(\frac{1}{t^{5/3}} + \|w_n\|_{H^3(1-\psi_0(t))}\right)\|w_{nxxxx}\|_{L^2}. \quad (4.84)$$

- $\int w_{nxxx}^2u_{nxxx}u_n^2$.

$$u_{nxxx}u_n^2 = w_{nxxx}(w_n + U(t)V)^2 + U(t)V_{xxx}(w_n + U(t)V)^2 + (w_n + U(t)V)_{xxx}RB_{10} + R_{xxx}u_n^2.$$

Now as $V \in H^{3,1}$, we have

$$\|U(t)V_{xxx}(w_n + U(t)V)^2\|_{L^\infty} \leq \frac{C}{t^{4/3}}.$$

Then, of course

$$\left| \int w_{nxxx}^2((w_n + U(t)V)_{xxx}RB_{10} + R_{xxx}u_n^2) \right| \leq \|w_n\|^2_{H^3(1-\psi_0(t))}.$$

And for the first term, we have to be a bit more foxy :

$$\int w_{nxxx}^3(w_n + U(t)V)^2 = -3 \int w_{nxxxx}w_{nxxx}^2(w_n + U(t)V)_x(w_n + U(t)V).$$

This last term is bounded by

$$\|w_{nxxxx}\|_{L^2}\|w_{nxxx}\|_{L^2}\|w_{nxxx}\|_{L^\infty}\|(w_n + U(t)V)_x(w_n + U(t)V)\|_{L^\infty} \leq \frac{C}{t^{4/3}}\|w_{nxxxx}\|_{L^2}.$$

And we get

$$\left| \int w_{xxx}^2 u_{nxxx}u_n^2 \right| \leq \frac{C}{t^{5/3}} + \frac{C}{t^{4/3}}\|w_{nxxxx}\|_{L^2} + \|w_n\|^2_{H^3(1-\psi_0(t))}. \tag{4.85}$$

- $\int w_{nxxx}^2 u_{nx}u_n^5$

$$u_{nx}u_n^5 = (w_n + U(t)V)_x(w_n + U(t)V)^5 + (w_n + U(t)V)_xRB_{11} + R_xu_n^5.$$

We can use directly the usual L^∞ bound for the first term and get a $Ct^{-7/3}$ decay, so that

$$\left| \int w_{nxxx}^2 u_{nx}u_n^5 \right| \leq \frac{C}{t^3} + \frac{C\|w_n\|_{H^1(1-\psi_0(t))}}{t^{2/3}}. \tag{4.86}$$

- $\int (u_n^4 - R^4)_{xxx}w_{nxxxx}u_n^3.$

  The only trouble with this 8-power integral is the expression of the differentiated term.

$$\begin{aligned}
&(u_n^4 - R^4)_{xxx} \\
&= 4(u_{nxxx}u_n^3 - R_{xxx}R^3) + 36(u_{nxx}u_{nx}^2R^2 - R_{xx}R_x^2R^2) + 24(u_{nx}^3u_n - R_x^3R) \\
&= 4(w_{nxxx}(w_n + U(t)V)^3 + U(t)V_{xxx}(w_n + U(t)V)^3 \\
&\quad + (w_n + U(t)V)_{xxx}RB_{12} + R_{xxx}(w_n + U(t)V)B'_{12}) \\
&\quad + 36((w_n + U(t)V)_{xx}(w_n + U(t)V)_x^2(w_n + U(t)V)^2 \\
&\quad + (w_n + U(t)V)_{xx}(w_n + U(t)V)_x^2RB''_{12} + (w_n + U(t)V)_{xx}RB'''_{12}u_n^2
\end{aligned}$$

$$+ R_{xx}R_x^2(w_n+U(t)V)B_{12}'''' + R_{xx}(w_n+U(t)V)_x B_{12}''''' u_n^2)$$
$$+ 24((w_n+U(t)V)_x^3(w_n+U(t)V) + (w_n+U(t)V)_x^3 RB_{12}''''''$$
$$+ (w_n+U(t)V)_x RB_{12}''''''' u_n + R_x^3(w_n+U(t)V)B_{12}'''''''). \tag{4.87}$$

Now along with $u_n^3 = (w_n+U(t)V)^3 + RB_{12}''''''''$, we develop the product $(u_n^4 - R^4)_{xxx}u_n^3$. Looking only on terms without R, we have the L^2 bound on these terms :

$$\left(\frac{C}{t^{\frac{1}{3}+1}} + \frac{C}{t^{1+1/3}} + \frac{C}{t^2} + \frac{C}{t^{\frac{1}{3}+1}}\right)\cdot\frac{C}{t^1} \le \frac{C}{t^{7/3}}.$$

On the other side, for any of the terms containing R, we have the following on-the-right bound

$$C\|w_n+U(t)V\|_{H^3(1-\psi_0(t))}.$$

So that finally

$$\left|\int (u_n^4-R^4)_{xxx} w_{nxxxx} u_n^3\right| \le C\left(\frac{1}{t^{7/3}} + \|w_n+U(t)V\|_{H^3(1-\psi_0(t))}\right)\|w_{nxxxx}\|_{L^2}. \tag{4.88}$$

- $\int (u_n^4-R^4)_{xxx} w_{nxxx} u_{nx} u_n^2$.

  We reuse the development (4.87), along with

$$u_{nx}u_n^2 = (w_n+U(t)V)_x(w_n+U(t)V)^2 + (w_n+U(t)V)_x RB_{14} + R_x u_n^2,$$

to have L^2 bounds on the product $(u_n^4-R^4)_{xxx}u_{nx}u_n^2$. For the terms with no R, we get

$$\left(\frac{C}{t^{\frac{1}{3}+1}} + \frac{C}{t^{1+\frac{1}{3}}} + \frac{C}{t^2} + \frac{C}{t^{\frac{1}{3}+1}}\right)\cdot\frac{C}{t^{\frac{4}{3}}} \le \frac{C}{t^{8/3}}.$$

And as for the previous integral, for any of the terms containing R, we have the on-the-right bound :

$$C\|w_n+U(t)V\|_{H^3(1-\psi_0(t))}.$$

Then $\|w_{nxxx}\|_{L^2} \le Ct^{-1/3}$ gives the estimate :

$$\left|\int (u_n^4-R^4)_{xxx} w_{nxxx} u_{nx} u_n^2\right| \le C\left(\frac{1}{t^3} + \frac{\|w_n+U(t)V\|_{H^3(1-\psi_0(t))}}{t^{1/3}}\right). \tag{4.89}$$

- $\int \left(R^4 - \sum_{j=1}^n R_j^4\right)_{xxxx} w_{nxxx}u_n^3$ and $\int \left(R^4 - \sum_{j=1}^N R_j^4\right)_{xxxxx} w_{nxxxx}$.

  We obviously have exponential decay :

$$\left|\int \left(R^4 - \sum_{j=1}^n R_j^4\right)_{xxx} w_{nxx}u_n^3\right| + \left|\int \left(R^4 - \sum_{j=1}^N R_j^4\right)_{xxxx} w_{nxxx}\right| \le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|w_n\|_{H^4}. \tag{4.90}$$

And finally :

- $\int w_{n\,xxx}^{\,2} u_n^3$.

As $u_n^3 = (w_n + U(t)V)^3 + RB_{15}$, we have

$$\left| \int w_{n\,xxx}^{\;2} u_n^3 \right| \leq \frac{C}{t^{\frac{2}{3}+1}} + C\|w_{nxxx}\|^2_{L^2(1-\psi_0(t))} \leq \frac{C}{t^{5/3}} + C\|w_{nxxx}\|^2_{L^2(1-\psi_0(t))}. \tag{4.91}$$

Step 3. Let us sum all our estimates (4.77)-(4.90) (aside from 4.87). Then we integrate in time between t and S_n, and plug in (4.91). We get

$$\begin{aligned}\|w_{nxxx}\|^2_{L^2} \leq \frac{C}{t^{2/3}} + C\int_t^{S_n} \|w_n(\tau\|^2_{H^3(1-\psi_0(t))} d\tau + C\|w_{nxxx}\|^2_{L^2(1-\psi_0(t))} \\ + C\int_t^{S_n} \left(\frac{1}{t^{4/3}} + \|w_n\|_{H^3(1-\psi_0(t))} + \|U(t)V\|_{H^5(1-\psi_0(t))}\right) \|w_{nxxxx}(\tau)\|_{L^2} d\tau\end{aligned}$$

(Notice that we don't have an exponent greater than 1 on $\|w_{nxxxx}(\tau)\|_{L^2}$). So that we obtain

$$\|w_{nxxx}\|_{L^2} \leq \frac{C}{t^{1/3}}.$$

as soon as $V \in H^{5,1} \cap H^{2,2}$ and :

$$\|w_n\|_{H^3(1-\psi_0(t))} + \|U(t)V\|_{H^5(1-\psi_0(t))} \leq \frac{C}{t^{4/3}}.$$

This is follows from Corollary 4.1, and completes the proof of Proposition 4.3. ☐

Chapitre 5

Instability of non-constant harmonic maps for the $1+2$-dimensional equivariant wave map system[1]

5.1 Introduction

5.1.1 Recall of known results

Let us introduce a function $g \in C^1$, such that $g(0) = 0$, and $f = g \cdot g'$. We consider the following initial value problem on function $u : \mathbb{R}_t \times \mathbb{R}_r^+ \to \mathbb{R}$:

$$\left\{ \begin{array}{rcl} u_{tt} - \Delta u & = & -\dfrac{f(u)}{r^2}, \\ (u, u_t)|_{t=0} & = & (u_0, u_1). \end{array} \right. \tag{5.1}$$

(We denote $\Delta u = u_{rr} + u_r/r = 1/r(ru_r)_r$ the radial Laplacian in $\mathbb{R}^2$). This problem has the following geometric interpretation. Let N be a surface of revolution with polar coordinates $(\rho, \theta) \in [0, \infty) \times \mathbb{S}^1$. Let ds^2 be the metric on N :

$$ds^2 = d\rho^2 + g^2(\rho) d\theta^2. \tag{5.2}$$

A wave map is a function $U : \mathbb{R}^{1+2} \to N$ satisfying the system :

$$\left\{ \begin{array}{l} \Box U^\alpha + \Gamma^\alpha_{\beta\gamma}(U) \partial_a U^\beta \partial^a U^\gamma = 0, \\ (U, U_t)|_{t=0} = (U_0, U_1). \end{array} \right. \tag{5.3}$$

($\Gamma^\alpha_{\beta\gamma}$ are the Christoffel symbols for (N, ds^2), $\alpha = \rho$ or θ). Denote (r, ϕ) the usual polar coordinates on $\mathbb{R}^2$. We are concerned with the corotationnal equivariant case, that is, we impose that U has the form

$$\rho = u(t, r), \quad \theta = \phi.$$

The wave map system (5.3) then simplifies to a single nonlinear scalar equation for u : $\mathbb{R}_t \times \mathbb{R}_r^+ \to \mathbb{R}$, which is (5.1) (see the book by Shatah and Struwe [44] for further details). Of course, any result on u has a reformulation for U.

[1]Ce chapitre a fait l'objet une publication dans à *International Mathematics Research Notices*, **2005**, no. 57, 3525–3549.

At least formally, one has conservation of one quantity, that is energy :

$$E(u) = \int \left(u_t^2 + u_r^2 + \frac{g^2(u)}{r^2} \right) rdr = E(u_0, u_1). \tag{5.4}$$

(In fact $E(u) = \|U_t\|_{L^2}^2 + \|\nabla U\|_{L^2}^2$). Let us introduce :

$$H = \left\{ (u,v) \middle| \; \|(u,v)\|_H^2 \stackrel{\text{def}}{=} E(u,v) = \int \left(v^2 + u_r^2 + \frac{g(u)^2}{r^2} \right) rdr < \infty \right\}. \tag{5.5}$$

H appears as an energy space, in which it is natural to study the solutions to (5.1). In [46], Shatah and Tahvildar-Zadeh proved local in time existence and uniqueness of solutions to (5.1) arising from initial data in the energy space :

Local existence in H [46, Theorem 1.1]. *Let $(u_0, u_1) \in H$. Then there exist $T > 0$ and a unique solution u to Problem (5.1) such that :*

$$(u, u_t) \in L^\infty([0,T), H), \qquad u \in L^{10/3}([0,T), \dot{B}^{1/2}_{10/3,10/3}(\mathbb{R}^+, rdr)).$$

Let us define the usual notation :

$$E(u,a,b) \stackrel{\text{def}}{=} \int_a^b \left(u_t^2 + u_r^2 + \frac{g^2(u)}{r^2} \right) rdr.$$

One way to express the finite speed of propagation is the fact that the energy is decreasing on light cones :

$$\forall R \geq 0, \; \forall |\tau| \leq R, \quad E(u(t), 0, R - |\tau|) \leq E(u(t+\tau), 0, R). \tag{5.6}$$

One should also notice that Problem (5.1) has a natural scaling :

$$u(t,r) \text{ is a wave map} \quad \Longleftrightarrow \quad u_\lambda(t,r) \stackrel{\text{def}}{=} u(\lambda t, \lambda r) \text{ is a wave map.}$$

A straightforward computation gives :

$$E(u_\lambda) = \int \left(\lambda^2 u_t^2 + \lambda^2 u_r^2 + \lambda^2 \frac{g^2(u)}{(\lambda r)^2} \right) rdr = E(u).$$

The energy remains unaffected by scaling : Problem (5.1) is thus said to be (scaling-)critical for the energy.

The main remaining open question for this problem is global well-posedness.

The result by Shatah and Tavildar-Zadeh [46] solves in particular the case of small energy data : there exists a constant $\varepsilon_0 > 0$ such that if $E(u) = E(u_0, u_1) < \varepsilon_0$, then u is global in time. Another direct consequence of the proof of Theorem 1.1 of [46] is the following condition for blow-up : u blows-up at time T only if

$$\liminf_{t \uparrow T} E(u(t), 0, T-t) \geq \varepsilon_0. \tag{5.7}$$

Recall that due to radial symmetry, concentration of energy can only happen at point $r = 0$.

Under some assumptions on N, the energy can not concentrate, and this is enough to ensure global existence in time : in [45], Shatah and Tavildar-Zadeh proved non-concentration under the assumption $g' \geq 0$ (geodesical convexity). This condition was later laxed by

Grillakis to $g(\rho) + g'(\rho)\rho > 0$, and finally by Struwe in [50], to $g > 0$ for $\rho > 0$. On the other hand, in [1], Bizoń and al. gave strong evidence of blow-up for system (5.1), in the case $N = \mathbb{S}^2$, and $g = \sin$.

In [50], Struwe proved further that if u does blow up (say at time $t = 0$), then for a subsequence t_n, there exists a scaling parameter $\lambda(t_n)$ such that $\lambda(t_n)|t_n| \to \infty$ and :

$$u(t_n + t/\lambda(t_n), r/\lambda(t_n)) \to Q(r) \quad \text{in} \quad H^1_{\text{loc}}(\mathbb{R}^{1+2}), \tag{5.8}$$

where Q is a non-constant harmonic map, i.e. a stationary solution to (5.1) :

$$\Delta Q = \frac{f(Q)}{r^2}. \tag{5.9}$$

This proves in particular that if there is no harmonic map, then there is no blow up : this is the case when $g > 0$ for $\rho > 0$. Furthermore, [50, Theorem 1] has two corollaries :

1. Let Q be a non-constant harmonic map with least energy. Suppose $E(u) \leq E(Q)$: then u is global in time.
2. Blow-up (at T) is characterized by $\liminf_{t\uparrow T} E(u(t), 0, T-t) \geq E(Q)$. In particular, if the initial data is such that $E((u_0, u_1), 0, R) \leq E(Q)$, then the corresponding wave map $u(t)$ is defined at least up to time $t = R$ (recall that the energy is decreasing on cones).

These results can be reformulated by saying that one can choose $\varepsilon_0 = E(Q)$.

5.1.2 Statement of the results

For a single function u, not depending on time, we shall often note $\|u\|_H$ instead of $\|(u, 0)\|_H$, as well as $E(u, a, b)$ for $E((u, 0), a, b)$.

From now on, Q will denote a non-constant harmonic map, i.e. Q satisfies (5.9). Q is important for the qualitative study of (5.1). Indeed, notice that our first criterion (5.7) for blow-up states the impossibility to apply the local well-posedness result. Due to [50], it is in fact equivalent to the formation of a singularity with the well-defined blow-up profile Q, which is a descriptive result.

As we shall see, the existence of Q is equivalent to the vanishing of g at some point. The problematic is now : if Q exists, does blow-up occur ? In particular, for initial data in a neighborhood of Q, does one has blow-up ?

Our goal in this paper is to prove the instability of Q in the energy space. This result should be related to previous works on instability of stationary states for other equations. In particular, we refer to [31] for the critical non-linear Schrödinger equation :

$$\begin{cases} iu_t + \Delta u + |u|^{4/N} u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

and [26] for the critical generalized Korteweg-de Vries equation :

$$\begin{cases} u_t + (u_{xx} + u^5)_x = 0, & (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases}$$

In this context, proving instability for Q should be viewed as a first step toward understanding the blow-up mechanism.

More precisely we can build wave maps, which are in an arbitrary neighborhood of Q at time 0, and which change profile to a certain $Q(\lambda\cdot)$, $\lambda \neq 1$ (in fact $\lambda > 1$) :

Theorem 5.1 (Geometric instability of Q in H)**.** *Suppose g vanishes at some point $C^* > 0$ (isolated zero). Then there exists a non-constant finite energy harmonic map Q for Problem (5.1). Moreover, for any $\lambda_0 > 1$, there exist a sequence of initial data (u_n^0, u_n^1) such that*

$$\|(u_n^0, u_n^1) - (Q, 0)\|_H \to 0 \quad as \quad n \to \infty, \tag{5.10}$$

and if we denote the arising wave maps u_n (solutions to (5.1)), u_n is defined at least up to some time t_n such that

$$\|(u_n, u_{nt})(t_n) - (Q(\lambda_0 \cdot), 0)\|_H \to 0 \quad as \quad n \to \infty. \tag{5.11}$$

We shall call this change of profile (from Q to $Q(\lambda_0\cdot)$) *geometric instability*. Notice, as proven in Corollary 5.4, that one must have $t_n \to \infty$. As a direct consequence, we have :

Corollary 5.1. *Let Q be a finite energy harmonic map for Problem (5.1). Then Q is unstable in H. For some constant $C(\lambda_0) > 0$ independent of n :*

$$\sup_{t \in [0, t_n]} \|(u_n, u_{nt}) - (Q, 0)\|_H \geq C(\lambda_0).$$

As Problem (5.1) has no symmetry besides scaling, this result also proves orbital instability.

Of course, as mentioned earlier, this result can be rewritten as geometrical instability of harmonic maps for the wave map system (5.3).

Theorem 5.1 deals in particular with the following two special cases.

Corollary 5.2 (Wave maps to the 2-sphere)**.** *It corresponds to the case $g = \sin$: $C^* = \pi$. The equation takes the form (cf. [1] for more details) :*

$$u_{tt} - \Delta u = -\frac{\sin 2u}{2r^2}.$$

And the harmonic solution is explicit : $Q(r) = 2\arctan(r)$. Then Q is geometrically unstable.

Q can be seen as the minimal (in the sense of energy) connection between the north and south poles. Theorem 5.1 is here in agreement with the numerical investigation of [1].

Corollary 5.3 (Critical Yang-Mills equation in dimension 4)**.** *It corresponds to $g(\rho) = (1 - \rho^2)$: $C^* = 1$, which gives the equation:*

$$u_{tt} - \Delta u = \frac{2u(1 - u^2)}{r^2}.$$

Then the harmonic map $Q(r) = \tanh(\ln r) = \frac{r^2-1}{r^2+1}$ is geometrically unstable.

Here, one should think of a slight modification of our setting, as $\tilde{u} = u + 1$ is the wave map - with g replaced by $\tilde{g}(\rho) = g(\rho - 1) = \rho(2 - \rho)$: the main difference will be that $\rho \in [-1, \infty)$.

The proof is organized as follows : in Section 2.1 we characterize harmonic maps and their properties. In Section 2.2, we study special, infinite energy wave maps, which are in fact related to harmonic maps : the self similar solutions. Then, in Section 2.3, we regularize these solutions to obtain wave maps with initial data in a neighborhood of Q. Finally in Section 2.4, using finite speed of propagation, we exhibit a family of initial data satisfying Theorem 5.1.

5.2 Proofs

5.2.1 On stationary solutions to (5.1)

Recall that the existence of a non constant harmonic map implies that g vanishes at some point (besides 0) : it is a consequence of [50]. In the following, we suppose that there exists a least positive real number C^* such that $g(C^*) = 0$. Without loss of generality, we can suppose that $g(\rho) > 0$ for $\rho \in (0, C^*)$. Finally, as to avoid degeneracy of the energy, we assume that g has only isolated zeros. We shall denote $G(\rho) = \int_0^\rho |g(\rho')|d\rho'$: G is increasing.

Proposition 5.1. *For any g such that $g(C^*) = 0$, there exists a non constant finite energy harmonic map Q, i.e. a solution to (5.9). Furthermore :*

1. *Regularity for Q. Any harmonic map Q is of class C^2 and satisfies one of the equations :*
$$rQ_r = g(Q), \quad \text{or} \quad rQ_r = -g(Q).$$
As a consequence, Q is monotone, and joins 2 consecutive zeros of g.

2. *Variational characterization of Q. Suppose that Q joins 0 to C^*. Then Q is of minimal energy for this property : for a function v, such that $v(0) = 0$, and $v(r) \to C^*$ as $r \to \infty$, then*
$$E(v) \leq E(Q) \Longrightarrow v(r) = Q(\lambda r) \text{ for some } \lambda > 0.$$

We will use many times the following simple inequality :

Lemma 5.1 (Pointwise inequality)**.** *For a finite energy function v :*

$$E(v, \alpha, \beta) = \int_\alpha^\beta \left(v_r^2 + \frac{g^2(v)}{r^2} \right) rdr \geq 2 \int_\alpha^\beta |g|(v) v_r dr \geq 2|G(v(\beta)) - G(v(\alpha))|. \tag{5.12}$$

Proof. First, let us prove the existence of a harmonic map. Let Q be a maximal (in the sense of Cauchy-Lipschitz) solution to

$$rQ_r = g(Q), \quad Q(1) = C^*/2. \tag{5.13}$$

Suppose Q is defined on (a, b). Then :

$$\int_a^b \left(Q_r^2 + \frac{g^2(Q)}{r^2} \right) rdr = 2 \int_a^b Q_r g(Q) dr = 2(G(Q(b)) - G(Q(a))).$$

Now, as $u = 0$ and $u = C^*$ are solutions, by uniqueness, we always have $Q(r) \in (0, C^*)$ for $r > 0$. This proves both that Q defined for $r \in [0, \infty)$ and that Q is of finite energy $2G(C^*)$. If we differentiate (5.13), we obtain :

$$\Delta Q = \frac{1}{r}(rQ_r)_r = \frac{f(Q)}{r^2}.$$

As $Q_r(1) \neq 0$, Q is not constant : it is the desired harmonic map.

Let us now prove the properties for any harmonic map. We denote $r = e^x$, so that $r\partial/\partial r = \partial/\partial x$, and thus equation 5.9 writes :

$$Q_{xx} = f(Q). \tag{5.14}$$

The Theorem of Cauchy-Lipschitz allows us to solve (5.14) ; this ensures that Q is C^2 where it is defined. Multiply (5.14) by Q_x and integrate between a and b :

$$[Q_x^2]_a^b = [g^2(Q(x))]_a^b. \tag{5.15}$$

On another side, for any finite energy function $v(r)$, in view of (5.12), $G(v)$ satisfies the Cauchy criterion as $\alpha, \beta \to 0$, and as $\alpha, \beta \to \infty$. Thus $G(v)$ admits limits $G(v)(0)$ and $G(v)(\infty)$, at $r = 0$ and $r \to \infty$ respectively. As G is increasing and continuous, it is a homeomorphism, and in fact v admits limits in 0 and ∞, $v(0)$ and $v(\infty)$. Of course, as v is of finite energy, we must have :

$$g(v(0)) = g(v(\infty)) = 0.$$

Therefore, Q admits limits at $r = 0$ and at ∞ (or in the x variable, at $\pm\infty$), that are zeros of g. In (5.15), fix b, and let $a \to -\infty$. The left hand side has a limit, so that Q_x has a limit ℓ at $-\infty$. If $\ell \neq 0$, then Q has no limit at $-\infty$, a contradiction. Hence $Q_x \to 0$, and finally we get

$$Q_x^2(b) = g^2(Q(b)).$$

We already know that Q joins two zeros of g, say α and β : if they are not consecutive, then for a certain c, $g(Q(c)) = 0$, so $Q_x(c) = 0$. By uniqueness, $Q = Q(c) = \text{constant}$: a contradiction, so a and b are two consecutive zeros. The same argument shows that for all $x \in \mathbb{R}$, $Q(x) \in (a, b)$, and that $g(Q)$ has a constant sign : as a consequence, Q_x does not vanish, hence Q is monotone. We then deduce (for sign reasons) that either :

$$\forall x, \quad Q_x = g(Q) \qquad \text{or} \qquad \forall x, \quad Q_x = -g(Q).$$

Finally, let us prove the the minimizing property. Up to rescaling of v, we can suppose that $v(1) = Q(1)$. Now, using (5.12) :

$$E(Q, 0, 1) = 2|G(Q(1)) - G(Q(0))| = 2|G(v(1)) - G(v(0)| \leq E(v, 0, 1).$$

In the same way, $E(Q, 1, \infty) \leq E(v, 1, \infty)$, and we obtain $E(v) \geq E(Q)$. Thus $E(v) = E(Q)$ (note that this already shows that Q is minimizing). More precisely, using 5.12, we have for all r :

$$\begin{aligned} E(v) &= E(v, 0, r) + E(v, r, \infty) \\ &\geq 2\left(|G(v(r)) - G(0)| + |G(C^*) - G(v(r))|\right) \geq E(Q) = E(v). \end{aligned}$$

So there is in fact equality everywhere. As a consequence, v is non-decreasing (so as not to lose any energy), and takes its values in $(0, C^*)$ for $r > 0$. The energy equality also gives :

$$v_r^2 = g^2(v)/r^2.$$

As before, using the fact that v is non-decreasing, we get : $v_r = g(v)/r$. As $v(1) = Q(1)$, $v = Q$. □

From now on, Q will denote a harmonic map joining 0 to C^*. As $g > 0$ on $(0, C^*)$:

$$rQ_r = g(Q). \tag{5.16}$$

Proposition 5.2 (Decomposition). *There exists $\alpha_0 > 0$ and an increasing function $\delta : [0,\alpha_0] \to \mathbb{R}^+$, with $\delta(\alpha) \to 0$ as $\alpha \to 0$, such that the following is true.*

Suppose v is a function of finite energy, with $v(0) = 0$ and $v(r) \to C^*$ as $r \to \infty$ and such that :

$$E(v) = E(Q) + \alpha < E(Q) + \alpha_0.$$

Then there exist $\lambda \in \mathbb{R}^+_*$, $\epsilon \in H$, such that :

$$v(r) = Q(\lambda r) + \epsilon(r), \quad \|\epsilon\|_H \le \delta(\alpha).$$

Proof. Follows from the variational characterization of Q : cf. Appendix A. □

Let u be a wave map with energy lower than $E(Q) + \alpha_0$. Such a decomposition exists for all $u(t)$: this gives two functions $\lambda(t)$ and $\epsilon(t,r)$ (the proof of Proposition 5.2 shows that they are continuous functions of t) such that

$$u(t,r) = Q(\lambda(t)r) + \epsilon(t,r).$$

Now blow up occurs only when there is concentration of at least $E(Q)$ energy. As $\epsilon(t,r)$ has a small energy, it cannot lead to blow up : only the $Q(\lambda(t)r)$ can do it, which it equivalent to $\lambda(t) \to \infty$. On the other side, if $\lambda(t) \to \infty$, then the initial blow-up criterion is fulfilled, and the result of [50] applies.

Finally u blows up at time T if and only if $\lambda(t) \to \infty$ as $t \uparrow T$. In this setting, Theorem 5.1 gives the existence of a wave map whose scaling parameter $\lambda(t)$ goes from $\lambda(0) = 1$ to $\lambda(t_n) = \lambda_0 > 0$: this can be seen as a first step toward existence of blow up.

Another consequence of Proposition 5.2 is the following :

Corollary 5.4. *Let $T > 0$ and $\varepsilon > 0$. Then there exists $\eta > 0$ such that for all initial data (u_0, u_1) satisfying $\|(u_0,u_1) - (Q,0)\|_H \le \eta$, the arising wave map u is defined at least up to time T and :*

$$\sup_{t\in[0,T]} \|(u,u_t) - (Q,0)\|_H \le \varepsilon.$$

In particular, this shows that one can not expect instability of Q for bounded times, and in Theorem 5.1, $t_n \to \infty$. The proof is postponed to Appendix A.

5.2.2 Self-similar solutions on light cones

The proof of Theorem 5.1 relies on the study of self-similar solutions. A self similar solution is a solution u to (5.1), defined for $t < 0$, with the ansatz :

$$u(t,r) = v\left(\frac{r}{|t|}\right).$$

It of course blows up at time $t = 0$. Let us first exhibit self similar wave maps.

Corollary 5.5. *Define, for $\xi < 1$:*

$$P_\alpha(\xi) = Q\left(\frac{2\alpha\xi}{1+\sqrt{1-\xi^2}}\right).$$

Then P_α generates a self-similar solution (on the light cone $r < |t|$).

Proof. It is mainly computations with change of variables. We plug the ansatz $u(t,r) = v(-r/t)$ (and $-r/t = \xi$) in (5.1), and write the equation for $v(\xi)$:

$$u_t = \frac{r}{t^2}v'(\xi), \quad u_{tt} = \frac{r^2}{t^4}v''(\xi) - \frac{2r}{t^3}v'(\xi), \quad u_r = -\frac{1}{t}v'(\xi), \quad u_{rr} = \frac{1}{t^2}v''(\xi).$$

So v satisfies :

$$\left(\frac{r^2}{t^4} - \frac{1}{t^2}\right)v''(\xi) + \left(-\frac{2r}{t^3} + \frac{1}{rt}\right)v'(\xi) = -\frac{f(v(\xi))}{r^2}.$$

Now we multiply by t^2, replace $-r/t$ by ξ, and again multiply by ξ^2 : we get the equation for v in the ξ variable :

$$\xi^2(\xi^2-1)\frac{d^2v}{d\xi^2} + \left(2\xi^3 - \xi\right)\frac{dv}{d\xi} = -f(v). \tag{5.17}$$

We would like to make the first order term vanish. For this, let us now do the change of variable $\chi = \ln\left(\xi/(1+\sqrt{1-\xi^2})\right)$ i.e. $\xi = 1/\cosh\chi$ (diffeomorphism for $\xi \in (0,1)$, that is $r < |t|$), then :

$$\frac{dv}{d\xi} = \frac{1}{\xi\sqrt{1-\xi^2}}\frac{dv}{d\chi}, \quad \frac{d^2v}{d\xi^2} = \frac{2\xi^2-1}{\xi^2(1-\xi^2)^{3/2}}\frac{dv}{d\chi} + \frac{1}{\xi^2(1-\xi^2)}\frac{d^2v}{d\chi^2}.$$

We plug this last relation in (5.17), the equation simplifies to

$$\frac{d^2v}{d\chi^2} = f(v).$$

But this is simply equation (5.14), whose solutions are

$$Q(2\alpha\exp\chi) = Q\left(\frac{2\alpha\xi}{1+\sqrt{1-\xi^2}}\right),$$

according to Proposition 5.1 (recall $x = \exp r$). □

This computation motivates the definition, for $b > 0$:

$$S(b;r) \stackrel{\text{def}}{=} P_{1/b}(br) = Q\left(\frac{2r}{1+\sqrt{1-b^2r^2}}\right) \quad \text{for } r \leq \frac{1}{b}. \tag{5.18}$$

Indeed if we define

$$\mathcal{S}(b;t,r) \stackrel{\text{def}}{=} P_{1/b}\left(\frac{r}{|t|}\right) = S\left(b;\frac{r}{b|t|}\right), \tag{5.19}$$

$\mathcal{S}(b;t,r)$ satisfies system (5.1) with initial data (at time $t = -1/b$) :

$$\left\{\begin{array}{l} \mathcal{S}(b;-\frac{1}{b},r) = S(b;r), \\ \mathcal{S}_t(b;-\frac{1}{b},r) = brS_r(b;r). \end{array}\right. \tag{5.20}$$

$\mathcal{S}(b;t,r)$ is defined in the interior of the cone $\{(t,r)|r \leq b|t|\}$; it blows up at time $t = 0$, and its life-span is $1/b$. Observe that from the proof of Corollary 5.5, it follows that $S(b;r)$ satisfies the equation :

$$(b^2r^2-1)\Delta S(b;r) + b^2rS_r(b;r) = -\frac{f(S(b;r))}{r^2}.$$

One important thing to notice is the following :

Claim.

$$\begin{cases} \forall A \geq 0, & \lim_{b\to 0} \|(S(b;r), brS_r(b;r)) - (Q,0)\|_{H([0,A])} \to 0, \\ \text{but } \forall b > 0, & \|S(b,r)\|_H = +\infty. \end{cases}$$

This is due to the specific form of P_α. If the convergence were in $H(\mathbb{R}^+)$, we would obtain a family of blowing up wave maps, whose initial data would converge to Q (in H). But these blowing up wave maps are always of infinite energy. Indeed, if we compute at point $\xi = 1$ (c.f. (5.45)) :

$$S_r\left(b; \frac{1}{b} - \varepsilon\right) \sim Q'\left(\frac{1}{b}\right) \cdot \sqrt{\frac{b}{\varepsilon}},$$

(as $Q'(1/b) \neq 0$), and $S_r^2(b) \sim C\frac{b}{\varepsilon}$ gives a logarithmic divergence when integrating in rdr, and thus an infinite energy. However, if we were to forget this infinite energy tail, $\mathcal{S}(b;t,r) = S(b;r/(b|t|)$ has the "profile" $Q(\lambda(t)r)$ with $\lambda(t) = 1/(b|t|)$. We recover the fact that $\lambda(t) \to \infty$ as $t \uparrow 0$, that is, blow-up. Furthermore $1/|t|$ is the self-similar blow-up rate. Notice that this rate can never be the blow-up rate of a finite energy wave map (cf. (5.8)).

In higher dimension, self-similar solutions are some examples of blowing up wave maps with smooth initial conditions (see [44, ch. 7]). In dimension 2, no blowing up wave map of finite energy is known.

The next step is to regularize $S(b;r)$ to obtain initial conditions in H.

5.2.3 Construction of approximation of self-similar profiles in the energy space

For simplicity in writing expressions in throughout this section, let us first introduce the following notations.

Let $c_0 < 1$. **Notation :** We define :

- $C(c_0)$, a constant that may change form line to line, but which does only depend on $c_0 < 1$ (as $c_0 \uparrow 1$, $C(c_0$ may tend to $+\infty$).
- the interval $I(b,r) = [r, 2r/(1+\sqrt{1-b^2r^2})]$,
- the functions $h(r) = \sup_{\rho\in[Q(r),C^*]} |g(\rho)| = \sup_{\theta\in[r,\infty)} |g(Q)(\theta)|$, and also $h_2(r) = \frac{1}{r^2}\int_0^r h^2(s)sds$.

Observe that if g is decreasing in a neighborhood of C^*, (which is always the case if $g'(C^*) < 0$), $h(r) = g(Q)(r)$ in this neighborhood. In any case, h decreases to 0 as $r \to \infty$. In particular, $h_2(r) \to 0$ as $r \to \infty$. This gives the existence of a constant $A \geq 10$ such that for $\alpha \geq A$:

$$h(\alpha) \leq 0.01, \quad \text{and} \quad h_2(\alpha) \leq 0.01.$$

This section is devoted to the proof of the following proposition :

Proposition 5.3. *Let $\alpha, b > 0$ such that $b\alpha \leq c_0 < 1$, and $\alpha \geq A$. There exist C^2 functions $S^0(\alpha,b;r)$ and $S^1(\alpha,b;r)$, defined for $r \in \mathbb{R}^+$, and satisfying :*

$$\begin{cases} S^0(\alpha,b;r) &=& S(b;r) & \text{if} \quad 0 \leq r \leq \alpha, \\ S^0(\alpha,b;r) &=& Q(r) & \text{if} \quad r \geq \alpha(1+h(\alpha)), \end{cases} \tag{5.21}$$

$$\left\{\begin{array}{lllll} S^1(\alpha,b;r) & = & brS_r(b;r) & if & 0\leq r\leq \alpha, \\ S^1(\alpha,b;r) & = & 0 & if & r\geq \alpha(1+h(\alpha)). \end{array}\right. \tag{5.22}$$

With the following estimate :

$$\|(S^0(\alpha,b;r),S^1(\alpha,b;r))-(Q(r),0)\|_H^2\leq C(c_0)(b^2\alpha^2h_2(\alpha)+h(\alpha)). \tag{5.23}$$

***Remark* 5.1.** *1. These modified profiles are simply truncated self-similar wave maps (at point α), that were smoothly reconnected to Q (at point $\alpha(1+h(\alpha))$).*

2. As we shall see through the proof, the contribution before truncation is in $h_2(\alpha)$, and the contribution for joining is in $h(\alpha)$, which is often worse (one should think as $h_2\sim h^2$). So the global contribution is in $h(\alpha)$.

Proof. The proof goes as follows. As the values of $(S^0(\alpha,b;r),S^1(\alpha,b;r))$ are given on $[0,\alpha]$, we only have to compute the desired estimate on this interval : this is done in lemma 5.2 (pointwise estimate) and 5.3 (H estimate). On $[\alpha,\alpha(1+h(\alpha))]$, we need both the construction of a smooth reconnection, and the estimates that goes along with it : this is lemma 5.4. The part $[\alpha(1+h(\alpha)),\infty)$ does not add any contribution. The proofs are mainly computational, and are postponed to Appendix B.

Lemma 5.2. *If $br\leq c_0$, then :*

$$|S(b;r)-Q(r)|\leq C(c_0)b^2r^2h^2(r), \tag{5.24}$$

$$|S_r(b;r)-Q_r(r)|\leq C(c_0)b^2rh(r). \tag{5.25}$$

Lemma 5.3. *For $b\alpha\leq c_0$ and $\alpha\geq A$, then :*

$$\|S(b;r)-Q(r)\|^2_{H([0,\alpha])}\leq C(c_0)b^4\alpha^4h_2(\alpha), \tag{5.26}$$

$$\|brS_r(b;r)\|^2_{L^2([0,\alpha])}\leq C(c_0)b^2\alpha^2h_2(\alpha), \tag{5.27}$$

$$\|Q\|_{H([\alpha;\infty))}\leq 2|C^*-Q(\alpha)|h(\alpha). \tag{5.28}$$

Lemma 5.4 (Joining lemma)**.** *Let $v:I=[0,a]\cup[b,\infty)\to\mathbb{R}$. Then there exists $\tilde v:\mathbb{R}_+\to\mathbb{R}$ extending v, as smooth as v, such that :*

$$\|\tilde v\|_H^2\leq \|v\|^2_{H(I)}+(v(b)-v(a))^2\frac{b+a}{b-a}+2\max(v(a),v(b))^2\ln\frac{b}{a}. \tag{5.29}$$

To construct the $S^i(\alpha,b;r)$, we only have choice on the interval $r\in[\alpha,\alpha(1+h(\alpha))]$. For $S^0(\alpha,b;r)$ we use the joining Lemma 5.4 to obtain a smooth reconnection. For $S^1(\alpha,b;r)$, the density of smooth functions in L^2 allows us to have a smooth reconnection such that :

$$\|S^1(\alpha,b;r)\|^2_{L^2([\alpha,\alpha(1+h(\alpha))])}\leq h(\alpha).$$

Let us now turn to estimate (5.23). First, by estimate (5.27) we have :

$$\|S^1(\alpha,b;r)\|^2_{L^2}\leq C(c_0)b^2\alpha^2h_2(\alpha)+h(\alpha). \tag{5.30}$$

Let us now focus on $S^0(\alpha,b;r)-Q(r)$. For $r\geq\alpha(1+h(\alpha))$, there is no contribution. On $[0,\alpha]$, we use lemma 5.3 : it gives the bound

$$C(c_0)b^4\alpha^4h_2(\alpha)\leq C(c_0)b^2\alpha^2h_2(\alpha). \tag{5.31}$$

On $[\alpha, \alpha(1+h(\alpha))]$, in order to use estimate (5.29), we compute (with Rolle's theorem, using $h(\alpha) \leq 0.01)$) :

$$\begin{aligned}|S(b;\alpha) - Q(\alpha(1+h(\alpha))| &= \left| Q\left(\frac{2\alpha}{1+\sqrt{1-b^2\alpha^2}}\right) - Q\left(\alpha(1+h(\alpha))\right)\right| \\ &\leq \alpha \left|\frac{2}{1+\sqrt{1-b^2\alpha^2}} - (1+h(\alpha))\right| \sup_{\vartheta\geq\alpha} |Q_r(\vartheta)| \\ &\leq \alpha(1+h(\alpha))h(\alpha)/\alpha \leq h(\alpha).\end{aligned}$$

So the second term on the right hand side of (5.29) can be estimated by ($\alpha \geq 10$):

$$4h(\alpha)^2 \frac{\alpha(2+h(\alpha))}{\alpha h(\alpha)} \leq 4h(\alpha).$$

And for the third term of (5.29), it remains to notice that $S(b;\alpha)^2, Q(\alpha(1+h(\alpha))^2 \leq C^{*2}$ and the well known : $\ln(1+h(\alpha)) \leq h(\alpha)$. Let's also allow $h(\alpha)$ for the time-derivative estimate on this interval (using the density of regular function in L^2). So the contribution of $\|S^0(\alpha,b;r) - Q(r)\|_H$ on $[\alpha, \alpha(1+h(\alpha))]$ is bounded by :

$$(2C^{*2}+4)h(\alpha). \tag{5.32}$$

Summing up the contributions (5.31) and (5.32), we get :

$$\|S^0(\alpha,b;r) - Q(r)\|_H \leq C(c_0)b^2\alpha^2 h_2(\alpha) + Ch(\alpha). \tag{5.33}$$

(5.33) and (5.30) give the estimate (5.23). □

5.2.4 Finite speed of propagation and conclusion

Before proving Theorem 5.1, let us recall a consequence of the finite speed of propagation for system (5.1).

Proposition 5.4. *Let (u_0,u_1) and (v_0,v_1) be a couple of initial data (at time $t=0$) with finite energy on the interval $[0,R)$. Let u,v be the respective solutions to (5.1) arising from them. Suppose that :*

$$\forall r \in [0,R), \quad (u_0,u_1)(r) = (v_0,v_1)(r).$$

Suppose that u is defined up to time $T>0$, and let $T_0 = \min\{R,T\}$. Then v does not blow up before time T_0, and for $t \in [0,T_0]$ we have :

$$\forall r \in [0, R-|t|), u(t,r) = v(t,r).$$

***Remark* 5.2.** *Of course, if (u_0,u_1) and (v_0,v_1) coincide on (a,b), with $a>0$, then at time $t \in [(a-b)/2, (b-a)/2]$, $u(t)$ and $v(t)$ coincide on the interval $r \in (a+|t|, b-|t|)$.*

Proof. It relies on the proof of Theorem 1.1 of [46] : more precisely, this Theorem is a direct consequence of the following claim, which is what is proved indeed in [46]

Claim : There exists $\varepsilon_0 > 0$ such that the following is true. Let (u_0,u_1), initial data at time t_0, have energy less than ε_0 on $B(R_0,C) = (R_0-C, R_0+C)$. Then exists a unique solution u to (5.1) on the full cone of dependence $K(R_0,C) = \{(t,r) || r-R_0| < C - |t-t_0|\}$.

Proposition 5.4 is also a direct consequence of this claim. Let $T_1 \geq 0$ be the greatest time for which u and v coincide on the truncated cone

$$\{(t,r)|t \in [0,T_1] \text{ and } 0 \leq r < R-t\}.$$

First, $T_1 > 0$: indeed, we divide $[0,R]$ into finitely many overlapping intervals (a_n, b_n) such that on every interval, the energy of (u_0, u_1) is less than ε_0. Thanks to the claim, on every cone of dependence associated to (a_n, b_n), v (exist and) coincides with u. As the (a_n, b_n) overlap and are in finite number, there exists $\delta > 0$ such that u and v coincide on the truncated cone

$$\{(t,r)|t \in [0,\delta] \text{ and } 0 \leq r < R-t\}.$$

Thus, $T_1 \geq \delta > 0$. Now, if $T_1 < T_0$: $u(T_1) = v(T_1)$, $u_t(T_1) = v_t(T_1)$ on $[0, R-T_1)$, we can repeat the same argument at time T_1 to obtain a greater time for the truncated cone on which u and v coincide, and this contradicts the maximality of T_1. Hence $T_1 = T_0$. □

Proof of Theorem 5.1. The idea is the following. We know the evolution of $\mathcal{S}(b)$: given initial data $S^i(\alpha, b; r)$, we deduce from Proposition 5.4 what happens at time t on the space interval $r \in [0, \alpha - t]$.

The problem is now to choose $\alpha = \alpha(b)$ large enough so that our control takes place for large enough times (so that the scaling parameter λ changes), but not too large so that the initial data is close to $(Q, 0)$ in H : there we need the estimates of Proposition 5.3.

Choose a fixed $c_0 < 1$, and set :

$$\lambda_0 = \frac{1}{1-c_0^2} > 1 \quad \text{and for } b > 0, \quad \alpha = \alpha(b) = \frac{c_0}{b}.$$

Define $R(b,t;r)$ as the wave map arising from the regularized initial conditions with $\alpha(b) = 2c_0/b$:

$$\begin{cases} R(b;0,r) = S^0(c_0/b, b; r), \\ R_t(b,0,r) = S^1(c_0/b, b; r). \end{cases} \tag{5.34}$$

First, $(R(b;0,r), R_t(b;0,r))$ coincide with $(\mathcal{S}(b,-1/b,r), \mathcal{S}_t(b,-1/b,r))$ on the interval $[0, c_0/b]$, and is of finite total energy. Thus, Proposition 5.4 ensures that $R(b;t,r)$ is defined at least up to time $T = c_0/b$ (as $\mathcal{S}(b;t,r)$ is defined on an interval of length $1/b \geq 2c_0/b$), and :

$$\forall t \in \left[0; \frac{c_0}{b}\right], \forall r \leq \frac{c_0}{b} - t, \quad R(b;t,r) = \mathcal{S}\left(b; -\frac{1}{b} + t, r\right) = S\left(b; \frac{r}{1-bt}\right).$$

In particular, for $t(b) = c_0(1-c_0)/b$:

$$\forall r \leq \frac{c_0^2}{b}, \quad R(b; t(b), r) = S\left(b; \lambda_0 r\right). \tag{5.35}$$

Now, we use the estimate (5.23) (for $t = 0$) and we obtain :

$$\|(R(b;0,r), R_t(b;0,r) - (Q(r),0)\|_H^2 \leq C(h(c_0/b) + h_2(c_0/b)) \to 0 \quad \text{as} \quad b \to 0.$$

This is the first condition (5.10). It remains to estimate what happens at time $t(b) = c_0(1-c_0)/b$, i.e. to bound :

$$\|(R(b;t(b),r),R_t(b;t(b),r)) - (Q(\lambda_0 r),0)\|_H.$$

Consider separately the contributions on $[0, c_0^2/b]$ and on $[c_0^2/b, \infty)$. On the first interval, estimates of Lemma 5.3 give the bound

$$\|(R(b;t(b),r),R_t(b;t(b),r) - (Q(\lambda_0 r),0)\|^2_{H([0,c_0^2/b])} \leq Ch_2(c_0^2/b). \tag{5.36}$$

On the second interval, we work out separately

$$\|(R(b;t(b),r),R_t(b;t(b),r)\|_{H([c_0^2/b,\infty))} \quad \text{and} \quad \|Q(\lambda_0 r)\|_{H([c_0^2/b,\infty))}. \tag{5.37}$$

Let us focus on the first term of (5.37). In view of the initial conditions (5.34),

$$E(R(b)) \leq E(Q) + C(h_2(c_0/b) + h(c_0/b)).$$

On the other side, we use our control on the interval $[0, c_0/b]$:

$$\|(S(b;\lambda_0 r), b\lambda_0 r S(b;\lambda_0 r))\|_{H([0,c_0^2/b])} \geq \|Q\|_{H([0,c_0^2/b])} \geq E(Q) - 2(C^* - Q(c_0^2/b))h(c_0^2/b).$$

(First inequality because $S(b;\lambda_0 r) \geq Q(r)$ and the pointwise inequality, second inequality due to (5.28)). Hence we have (as h is decreasing) :

$$\begin{aligned}
\|(R(b;t(b),r),\, & R_t(b;t(b),r))\|_{H([c_0^2/b,\infty))} \\
&= E(R(b)) - \|(S(b;\lambda_0 r), b\lambda_0 r S(b;\lambda_0 r))\|_{H[0,c_0^2/b]} \\
&\leq E(Q) + C(h(c_0^2/b) + h_2(c_0^2/b)) - (E(Q) - Ch(c_0^2/b)) \\
&\leq C(h(c_0^2/b) + h_2(c_0^2/b)).
\end{aligned} \tag{5.38}$$

For the second term of (5.37), using (5.28), we obtain :

$$\|Q(\lambda_0 r)\|_{H([c_0^2/b,\infty))} \leq 2(C^* - Q(\lambda_0 c_0^2/b))h(c_0^2/b) = o(h(c_0^2/b)). \tag{5.39}$$

Finally adding up (5.36), (5.38) and (5.39), we have :

$$\begin{aligned}
&\|(R(b;t(b),r),R_t(b;t(b),r)) - (Q(\lambda_0 r),0)\|_H \\
&\qquad\qquad \leq C(h(c_0^2/b) + h_2(c_0/b) + h_2(c_0^2/b)) \to 0 \quad \text{as} \quad b \to 0.
\end{aligned}$$

This exactly the second condition (5.11). To conclude, choose a sequence (b_n) decreasing to 0, and define the sequence :

$$u_n(t,r) = R(b_n;t,r), \quad t_n = (1-c_0)c_0/b_n.$$

The previous computations show that the initial data $(u_n(0), u_{nt}(t)) \to (Q,0)$ in H and that $(u_n(t_n), u_{nt}(t_n)) \to (Q(\lambda_0 \cdot), 0)$ in H as $n \to \infty$. This is true for any $c_0 < 1$, and thus for any $\lambda_0 = 1/(1-c_0^2) > 1$. The instability in H is then straightforward. □

***Remark* 5.3.** *1. As $\|(R(b;0,r),R_t(b,0,r)-(Q,0)\|_H \to 0$, $E(R(b)) \to E(Q)$: for b small enough, $R(b)$ admits a decomposition and a scaling factor $\lambda_b(t)$ for all t up to an possible blow-up time. Theorem 5.1 simply says that $\lambda_b(c_0/b) \to \lambda_0$.*

2. The proof of Theorem 5.1 also gives the time to leave a neighborhood of Q. Suppose we start in a δ-neighborhood of Q : then the corresponding b is such that $h(c_0/b)+h_2(c_0/b) \sim \delta$. We have to wait time $c_0/2b$ to leave the neighborhood, that is approximately $h^{-1}(\delta)$. Let us give an example.

In the case of Corollary 5.2 ($g=\sin$), studied in [1], we have : $C^*=\pi$, and $g'(\pi)=-1$, so $h(r)\sim 1/r$, $h_2(r)\sim \ln r/r^2$, so $\|\tilde{Q}_b-Q\|_H \sim 1/\sqrt{b}$. Thus the time to exit a δ-neighborhood is $O(1/\delta^2)$.

In the case of Corollary 5.3 ($g(\rho)=(1-\rho^2)$), we have : $C^*=1$, $g'(1)=-2$, so $h(r)\sim 1/r^2$, $h_2(r)\sim 1/r^2$, so $\|\tilde{Q}_b-Q\|_H\sim 1/b$, and the time to exit a δ-neighborhood is $O(1/\delta)$.

Appendix A : Decomposition of a wave map with energy close to $E(Q)$

Proposition 5.2 (Decomposition)**.** *There exist $\alpha_0>0$ and an increasing function $\delta : [0,\alpha_0]\to\mathbb{R}^+$, with $\delta(\alpha)\to 0$ as $\alpha\to 0$, such that the following is true.*

Suppose v is a function of finite energy, with $v(0)=0$ and $v(r)\to C^*$ as $r\to\infty$ and such that :

$$E(v)=E(Q)+\alpha<E(Q)+\alpha_0$$

Then there exist $\lambda\in\mathbb{R}_*^+$ and $\epsilon\in H$ such that :

$$v(r)=Q(\lambda r)+\epsilon(r), \qquad \textit{and} \qquad \|\epsilon\|_H\le\delta(\alpha).$$

Proof. Recall that $G(s)=\int_0^s|g|$, $E(Q)=2G(C^*)$, and that G is increasing. Denote D^* the unique point such that $G(Q(D^*))=G(C^*)/2$ (or equivalently $E(Q,0,D^*)=E(Q)/2$.

We proceed by contradiction. Suppose that there exist $\delta_0>0$ and a sequence of finite energy functions v_n, such that

$$E(v_n)\le E(Q)+\frac{1}{n}, \quad v_n(0)=0, \quad v_n(r)\to C^* \quad \text{as} \quad r\to\infty,$$

and :

$$\forall n\in\mathbb{N},\ \ \forall\lambda>0,\quad \|v_n-Q(\lambda\cdot)\|_H\ge\delta_0. \tag{5.40}$$

Set $w_n(r)=v_n(r/\lambda_n)$, where λ_n is such that $w_n(D^*)=Q(D^*)$ (this is possible because v_n is continuous). Using scaling invariance, we have :

$$E(w_n)=E(v_n)\le E(Q)+\frac{1}{n}.$$

(w_{nr}) is $L^2(rdr)$-bounded, and so, on $(0,\infty)$, (w_n) is locally $C^{1/2}$-continuous, and so locally equicontinuous. Furthermore :

$$|G(w_n(a))-G(w_n(b))|\le\int_a^b|g(w_n(\rho)w_{nr}(\rho)|d\rho\le E(w_n,a,b).$$

Apply with $a\to 0$, $b=r$, and then $a=r$, $b\to\infty$. As g is not uniformly 0 outside $[0,C^*]$, we deduce that for N large enough, the $(w_k)_{k\ge N}$ are uniformly bounded in $C^0([0,\infty))$ by a constant K. So one can apply Ascoli's theorem : for any compact set X of $(0,\infty)$, $(w_n|_X)_n$ has a compact closure in $C^0(X)$. Let us construct a diagonal extraction. Let $m \in \mathbb{N}$, and suppose we already constructed an extraction $\phi_1 \circ \cdots \circ \phi_m : \mathbb{N} \to \mathbb{N}$ to obtain a converging subsequence

$$w_{\phi_1 \circ \cdots \circ \phi_m(n)}|_{[1/m,m]} \to w \text{ in } C^0([1/m,m]) \text{ as } n \to \infty.$$

We can then construct ϕ_{m+1} so that the convergence of $w_{\phi_1 \circ \cdots \circ \phi_m \circ \phi_{n+1}(n)}$ takes place in $C^0([1/(m+1), m+1])$ (by applying Ascoli's theorem on $[1/(m+1), m+1]$). Now, define :

$$\varphi(n) = \phi_1 \circ \cdots \circ \phi_n(n).$$

For any m, and $n \geq m$, $w_{\varphi(n)}$ is a subsequence of $w_{\phi_1 \circ \cdots \circ \phi_m(n)}$, and hence converges in the space $C^0([1/m,m])$. Let us denote again $w \in C^0(R_*^+)$ the common limit. For convenience, we can consider the subsequence as the initial sequence, and thus drop the φ. We obtained :

$$\forall m \in \mathbb{N}, \quad w_n \to w \quad C^0([1/m,m]).$$

We can also suppose that $w_{nx} \rightharpoonup w_x$ weakly in $L^2(rdr)$. In particular, w is continuous, $w(D^*) = Q(D^*)$. By weak limit :

$$\int w_r^2 rdr \leq \liminf w_{nr}^2 rdr.$$

And by Fatou lemma (of course, there is a.e. convergence) :

$$\int g^2(w) dr/r \leq \liminf \int g^2(w_n) dr/r.$$

So that $E(w) \leq E(Q)$. Moreover $w(D^*) = Q(D^*)$. As w is of finite energy, w admits limits in $r = 0$ and $r \to \infty$, where g vanishes : using the pointwise inequality between 0, D^* and ∞, we deduce that the only possibility for these limits are 0 and C^*. Let us now prove that :

$$\forall r \geq D^*, \quad w(r) \geq Q(D^*), \qquad \text{and} \qquad \forall r \leq D^*, \quad w(r) \leq Q(D^*). \tag{5.41}$$

We prove only one of these inequalities, the second one can be deduced in the same way. We again proceed by contradiction. Let $r < D^*$ such that $w(r) > Q(D^*)$: denote $\varepsilon = G(w(r)) - G(C^*)/2 > 0$. Due to uniform convergence on compact sets, there exists N such that for $n \geq N$, $G(w_n(r)) - G(C^*)/2 \geq \varepsilon$. Then :

$$\begin{aligned} E(Q) + \frac{1}{n} \geq E(w_n) &\geq E(w_n, 0, r) + E(w_n, r, D^*) + E(w_n, D^*, \infty) \\ &\geq 2\big[G(w_n(r)) + |G(w_n(r)) - G(C^*)/2| + G(C^*)/2\big] \\ &\geq 2(G(C^*) + 2\varepsilon) \geq E(Q) + 4\varepsilon. \end{aligned}$$

This is impossible if $n \geq 1/(4\varepsilon)$, and proves (5.41). Thus, we conclude that $w(0) = 0$ and $w \to C^*$ as $r \to \infty$. Together with $E(w) \leq E(Q)$ and $w(D^*) = Q(D^*)$, the variational characterization of Q allows to conclude that $w = Q$.

Let us now prove that $\|w_n - Q\|_H = E(w_n - Q) \to 0$. First, $E(w_n) \to E(Q)$, so that $\lim \|w_{nr}\|_{L^2(rdr)} = \|Q_r\|_{L^2(rdr)}$, and the weak convergence $w_{nr} \rightharpoonup Q_r$ is in fact strong-$L^2(rdr)$:

$$\|w_{nr} - Q_r\|_{L^2(rdr)} \to 0. \tag{5.42}$$

Let us now consider $\int g^2(w_n - Q)dr/r$. Let $\varepsilon > 0$. Define $c, d > 0$ such that $E(Q, 0, c) \leq \varepsilon/16$ and $E(Q, d, \infty) \leq \varepsilon/16$. The convergence of w_n to Q in $C^0([c, d])$ gives the existence of $N \geq 4/\varepsilon$ such that

$$\forall n \geq N, \quad \int_c^d g^2(w_n - Q)\frac{dr}{r} \leq \varepsilon/2.$$

(because g is continuous at 0). Again by convergence in $C^0([c, d])$, we can choose N such that :

$$\forall n \geq N, \quad |G(w_n(c)) - G(Q(c)| + |G(w_n(d)) - G(Q(d))| \leq \varepsilon/16.$$

Hence (pointwise inequality), $E(w_n, c, d) \geq E(Q, c, d) - \varepsilon/8 \geq E(Q) - \varepsilon/4$. In view of $N \geq 4/\varepsilon$, we obtain

$$E(w_n, 0, c) + E(w_n, d, \infty) \leq \varepsilon/2.$$

So that for $n \geq N$:

$$\int g^2(w_n - Q)\frac{dr}{r} \leq \int_0^c + \int_c^d + \int_d^\infty \leq E(w_n, 0, c) + \varepsilon/2 + E(w_n, d, \infty) \leq \varepsilon.$$

This together with (5.42) proves that $\|w_n - Q\|_H = \|v_n - Q(\lambda_n \cdot)\|_H \to 0$ as $n \to \infty$: a contradiction with (5.40). □

Corollary 5.4. *Let $T > 0$ and $\varepsilon > 0$. Then there exists $\eta > 0$ such that for all initial data (u_0, u_1) satisfying $\|(u_0, u_1) - (Q, 0)\|_H \leq \eta$, the arising wave map u is defined at least up to time T and :*

$$\sup_{t \in [0,T]} \|(u, u_t) - (Q, 0)\|_H \leq \varepsilon.$$

Proof. We will choose $\eta > 0$ small enough later.

First, let us notice that we can use the decomposition. Observe indeed that the energy $E(u_0, u_1) \leq E(Q) + C\eta^2 \leq E(Q) + \alpha_0$, if η is so small that $C\eta^2 \leq \alpha_0$. Hence, u_0 has limits at $r = 0$ and as $r \to \infty$, that are zeros for g. As 0 and C^* are isolated zeros of g, we necessarily have $u_0(0) = 0$, $u_0(r) \to C^*$ as $r \to \infty$, if we choose $\eta > 0$ small enough.

The local existence Theorem gives a maximal time $T^* > 0$ of existence : we can consider $u(t)$ the arising wave map. By conservation of energy $E(u) \leq E(Q) + C\eta^2$, we get :

$$\int u_t^2 r dr \leq C\eta^2. \tag{5.43}$$

In particular, as $u(t, 0)$ and $\lim_{r\to\infty} u(t, r)$ are always zeros for g, we obtain that for all $t < T^*$, $u(t, 0) = u_0(0) = 0$ and $\lim_r u(t, r) = \lim_r u_0(r) = C^*$ (see [50, Lemma 1]). Therefore, we can apply Proposition 5.2 : for all $t < T^*$, there exists $\lambda(t)$, $\epsilon(t, r)$ such that

$$u(t, r) = Q(\lambda(t)r) + \epsilon(t, r), \quad \text{and} \quad \|\epsilon(t)\|_H \leq \delta(C\eta^2).$$

We can choose $\lambda(0) = 1$. Now, as noticed earlier, blow-up for u is characterized by $\lambda(t) \to \infty$. From now on, we will consider $T' \leq T^*$ maximal such that :

$$\forall t < T', \quad \lambda(t) \leq 2 + \varepsilon.$$

Then (if $\delta(C\eta^2)$ is small enough) :

$$\|u(t) - Q\|_H \leq C|\lambda(t) - 1|. \tag{5.44}$$

Now fix $a \in (0, 1/2)$, such that for $r \in [1-a, 1+a]$, $Q_r(r) \geq Q_r(1)/2$. Using (5.43), we have :

$$\int_0^t \int_{1-a}^{1+a} u_t dr dt \leq \int_0^t \sqrt{\int u_t^2 r dr} \sqrt{\int_{1-a}^{1+a} \frac{dr}{r}} dt \leq C\eta t.$$

On the other side :

$$\begin{aligned}\int_0^t \int_{1-a}^{1+a} u_t dr dt &= \int_{1-a}^{1+a} (u(t,r) - u(0,r)) dr \\ &= \int_{1-a}^{1+a} \left(Q(\lambda(t)r) - Q(r) + \epsilon(t,r) - \epsilon(0,r)\right) dr.\end{aligned}$$

Now, as $\|\epsilon(t)\|_{L^\infty} \leq C(\|\epsilon(t)\|_H) \leq c(\eta)$ (for some function c such that $c \to 0$ at 0) :

$$\int_{1-a}^{1+a} (\epsilon(t,r) - \epsilon(0,r)) dr \leq 2c(\eta).$$

And :

$$\left| \int_{1-a}^{1+a} Q(\lambda(t)r) - Q(r) dr \right| \geq \frac{1}{2} |\lambda(t) - 1| \int_{1-a}^{1+a} Q_r(1) r dr \geq C|\lambda(t) - 1|.$$

Combining our two expressions for the integral, we deduce :

$$|\lambda(t) - 1| \leq C(\eta t + c(\eta)).$$

And finally, using (5.44) and $\|u_t\|_L^2 \leq \eta$:

$$\|(u(t), u_t(t) - (Q, 0)\|_H \leq C(\eta t + \eta + c(\eta)).$$

It is enough to choose $\eta \leq \alpha_0/C$ so that $C(\eta T + \eta + c(\eta)) < \varepsilon$. Indeed, the previous computations are then valid up to T'. Now if $T' < T$, we have that for all $t \in [0, T']$, $\lambda(t) \leq 1 + \varepsilon < 2 + \varepsilon$, therefore the solution can be continued (blow-up hasn't occurred) and the continuity of $\lambda(t)$ contradicts the maximality of T'. □

***Remark* 5.4.** *The proof gives a more accurate result : the time to leave a η neighborhood is at least $O(1/\eta)$. This is coherent with the computations for the sequence u_n of Theorem 5.1.*

Appendix B

Here, we prove the computational lemmas needed for Proposition 5.3.

Proof of Lemma 5.2. Let us note $\vartheta = \frac{2r}{1+\sqrt{1-b^2r^2}}$. We compute explicitly :

$$S_b(b; r) = Q_r(\vartheta) \frac{2br^3}{\left(1 + \sqrt{1 - b^2r^2}\right)^2 \sqrt{1 - b^2r^2}}.$$

So if we plug in $br \leq 0.01$ and $Q_r = g(Q)/r$:

$$|S_b(b; r)| \leq C(c_0) Q_r(\vartheta) br^3 \leq C(c_0) g(Q)(\vartheta) br^2.$$

Now

$$|S(b;r)-Q(r)| \leq \int_0^b |S_b(b;r)|db \leq C(c_0) \sup_{\theta\in I(r,b)} |(g(Q))(\theta)|b^2r^2.$$

For the second estimate, we again compute explicitly :

$$S_r(b;r) = Q_r(\vartheta)\left(\frac{2}{1+\sqrt{1-b^2r^2}} + \frac{2r}{(1+\sqrt{1-b^2r^2})^2}\frac{b^2r}{\sqrt{1-b^2r^2}}\right). \tag{5.45}$$

Now, we can compute :

$$S_{r,b}(b;r) = Q_{rr}(\vartheta)\frac{4r^3b}{\left(1+\sqrt{1-b^2r^2}\right)^3(1-b^2r^2)} - Q_r(\vartheta)\frac{2br^2\left(2b^2r^2-3-3\sqrt{1-b^2r^2}\right)}{(1-b^2r^2)^{3/2}\left(1+\sqrt{1-b^2r^2}\right)^3}. \tag{5.46}$$

Recall $Q_r = g(Q)/r$ and $Q_{rr} = ((g'-1)g)(Q)/r^2$. We plug in again $br \leq c_0 < 1$, and we get :

$$|S_{r,b}(b;r)| \leq C(c_0)rb\left(|((1-g')g)(Q)(\vartheta)| + g(Q)(\vartheta)\right) = C(c_0)((1+|1-g'|)g)(Q)(\vartheta)br.$$

So when we integrate in b, we have :

$$\begin{aligned}|S_r(b;r)-Q_r(r)| &\leq \int_0^b |S_{r,b}(b;r)|db \leq C(c_0)b^2r \sup_{\theta\in I(b,r)} |(g+|1-g'|g)(Q)(\theta)| \\ &\leq C(c_0)b^2r\left(1+\sup_{\theta\in I(b,r)}|g'|(Q)(\theta)\right)\sup_{\theta\in I(b,r)}|g(Q)(\theta)|.\end{aligned}$$

And as Q takes its values in $[0,C^*]$, we get the second estimate. □

Proof of Lemma 5.3. We integrate the pointwise estimate of the previous lemma.

$$\begin{aligned}\|S(b;r)-Q(r)\|^2_{H([0,\alpha])} &= \int_0^\alpha \left((S(b;r)-Q(r))_r^2 + \frac{(S(b;r)-Q(r))^2}{r^2}\right)rdr \\ &\leq \int_0^\alpha \left(\left(D_0h(r)b^2r\right)^2 + \left(\frac{h(r)b^2r^2}{r}\right)^2\right)rdr \\ &\leq C(c_0)b^4\int_0^\alpha h^2(r)r^3dr \leq C(c_0)b^4\alpha^2\cdot\alpha^2h_2(\alpha).\end{aligned}$$

For the time-derivative estimates, we use directly (5.45), and we plug in $br \leq b\alpha \leq c_0 < 1$. This leads to :

$$|S_r(b;r)| \leq |Q_r(\vartheta)|(2+C(c_0)b^2r^2) \leq C(c_0)h(r)/r,$$

so that :

$$\|brS_r(b;r)\|^2_{L^2[0,\alpha]} \leq C(c_0)b^2\int_0^\alpha h^2(r)rdr.$$

For the third bound, we compute, using $rQ_r = g(Q)$:

$$\|Q\|_{H([\alpha;\infty))} = 2\int_\alpha^\infty g(Q)Q_rdr = 2\int_{Q(\alpha)}^{C^*} g \leq 2|C^*-Q(\alpha)|h(\alpha).$$ □

Proof of Lemma 5.4. Let us first do the computations for an affine interpolation :

$$\text{For } r \in [a,b], \tilde{v}(r) = (v(b)-v(a))\frac{r-a}{b-a} + v(a), \text{ i.e. } \tilde{v}_r(r) = \frac{v(b)-v(a)}{b-a}.$$

Then :

$$\int_a^b \tilde{v}_r^2 r dr = \frac{1}{2}\left(\frac{v(b)-v(a)}{b-a}\right)^2 (b^2-a^2) \leq \frac{1}{2}(v(b)-v(a))^2 \frac{b+a}{b-a},$$
$$\int_a^b \frac{\tilde{v}^2}{r} dr \leq \max(v(b),v(a))^2 \int_a^b \frac{dr}{r} \leq \max(v(b),v(a))^2 \ln\frac{b}{a}.$$

This already gives an extension in H. For a smoother extension, it is enough to regularize $\tilde{v}_r$ (locally near a and b) with a small variation of the L^2 norm, while keeping constant $\tilde{v}(b) - \tilde{v}(a) = \int_a^b \tilde{v}_r dr$. This is possible thanks to the density of smooth functions with given mean among L^2 functions with the same given mean. □

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