Construction of solutions to the L^2-critical KdV equation with a given asymptotic behaviour

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Abstract

We consider the critical Korteweg-de Vries equation :

$$u_t + (u_{xx} + u^5)_x = 0, \quad t, x \in \mathbb{R}.$$

Let $R_j(t,x) = Q_{c_j}(x - x_j - c_j t)$ $(j = 1, \dots, N)$ be N soliton solutions to this equation. Denote $U(t)$ the KdV linear group, and let $V \in H^1$ be with sufficient decay on the right, that is $(1 + x_+^{2+\delta_0})V \in L^2$ for some $\delta_0 > 0$.

We construct a solution $u(t)$ to the critical Korteweg-de Vries equation such that

$$\lim_{t\to\infty} \left\| u(t) - U(t)V - \sum_{j=1}^{N} R_j(t) \right\|_{H^1} = 0.$$

1 Introduction.

1.1 General setting

We consider the critical Korteweg-de Vries equation :

$$u_t + (u_{xx} + u^5)_x = 0, \quad t, x \in \mathbb{R}. \tag{1}$$

It is a special case of the generalized KdV equation :

$$u_t + (u_{xx} + u^p)_x = 0, \quad t, x \in \mathbb{R}, \tag{2}$$

where $p \geq 2$. The case $p = 2$ corresponds to the original equation introduced by Korteweg and de Vries [8] in the context of shallow water waves. For both $p = 2$ and $p = 3$, this equation has many applications to Physics : see for example Miura [20], Lamb [10].

There are two formally conserved quantities for solutions to (2) :

$$\int u^2(t) = \int u^2(0) \qquad (L^2 \text{ mass}), \tag{3}$$

$$E(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) = E(u(0)) \qquad (\text{energy}). \tag{4}$$

The local Cauchy problem for (2) has been intensively studied by many authors. Kenig, Ponce and Vega [6] proved the following existence and uniqueness result

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in $H^1(\mathbb{R})$: for $u_0 \in H^1(\mathbb{R})$, there exist $T = T(\|u_0\|_{H^1}) > 0$ and a solution $u \in C([0,T], H^1(\mathbb{R}))$ to (2) satisfying $u(0) = u_0$, which is unique in the class $Y_T \subset C([0,T], H^1(\mathbb{R}))$. Moreover, if T_1 denotes the maximal time of existence for u, then either $T_1 = +\infty$ (global solution) or $T_1 < \infty$ and $\|u(t)\|_{H^1} \to \infty$ as $t \uparrow T_1$ (blow-up solution). For such a solution, one has conservation of mass and energy. In the critical case, problem 1), this result is improved to local well-posedness in L^2 (see [6] and [7]).

The next problem is to know whether these solutions to (2) are global in time, or blow-up. In the case $2 \leq p < 5$ (sub-critical), all solutions in H^1 are global and uniformly bounded thanks to the conservation laws and the Gagliardo-Nirenberg inequality :

$$\forall v \in H^1(\mathbb{R}), \quad \int |v|^{p+1} \leq \kappa(p) \left(\int v^2\right)^{\frac{p+3}{4}} \left(\int v_x^2\right)^{\frac{p-1}{4}}. \tag{5}$$

The case $p = 5$ is L^2-critical, in the sense that mass remains unaffected by scaling. Indeed, $u_\lambda(t,x) = \lambda^{1/6} u(\lambda t, \lambda^{1/3} x)$ is also a solution to (1), and $\|u_\lambda\|_{L^2} = \|u\|_{L^2}$. Moreover, existence of finite time blow-up solutions was proved by Merle [19] and Martel and Merle [15]. Therefore $p = 5$ also appears as a critical exponent for the long time behaviour of solution to (2).

For $p > 5$ (super-critical case), numerics predict blow-up.

A fundamental property of (2) is the existence of a family of explicit traveling wave solutions. If Q denotes the only solution (up to translation) of :

$$Q > 0, \quad Q \in H^1(\mathbb{R}), \quad Q_{xx} + Q^p = Q, \quad \text{i.e.} \quad Q(x) = \left(\frac{3}{\cosh^2(2x)}\right)^{1/(p-1)},$$

then for $c > 0$ the soliton

$$R_{c,x_0} = c^{1/(p-1)} Q(\sqrt{c}(x - x_0 - ct)) \text{ is a solution to (2).}$$

Solitons are stable in H^1 in the sub-critical case $p \in [2,5)$ (see [13], and unstable in the $p > 5$ super-critical (see [22]) and $p = 5$ critical case (see [14]).

For $p = 2$ and $p = 3$, equation (2) is completely integrable, and thus has very special features. The inverse scattering transform method allows to solve the Cauchy problem in an appropriate space (for example if $u_0 \in H^4$ and $xu_0 \in L^1$) and the qualitative behaviour of solutions is well understood. For example, given u_0 smooth and with rapid decay, there exist N solitons R_{c_j,x_j} such that

$$\left\| u(t) - \sum_{j=1}^N R_{c_j,x_j}(t) \right\|_{L^\infty(x \geq -t^{1/3})} \leq \frac{C}{t^{1/3}} \quad (\text{as } t \to \infty).$$

See for example Schuur [21], Eckhaus and Schuur [4], Miura [20].

However, if $p \neq 2$ or 3, the inverse scattering transform method does not longer apply, and the description of solutions in the general, non-integrable case is a widely open problem, especially in the critical case. It can be decomposed in two types of problems.

Problem 1 : Asymptotic behavior. In the sub-critical case, given an initial data u_0, can we describe the behavior of the out-coming solution $u(t)$ to (2) ? In the *critical and super-critical cases, does $u(t)$ blow-up ? Can we determine the blow up rate and profile ?*

Problem 2 : Construction of a non-linear wave operator. Given some reasonable behavior at $t \to \infty$, can we find a solution $u(t)$ to (2) defined for large enough t, with this behavior ? Is there uniqueness for $u(t)$?

1.2 Recent results on Problems 1. and 2. for the critical KdV equation

From now on, we will focus only on equation (1), that is, the L^2-critical case.

Let us now develop some results which will be the base to our result. The first result deals with scattering for small initial data. One wants to prove that given an initial data u_0, small in an adequate functional space, the arising solution to the non-linear equation (1) behaves likes $U(t)v_0$, the solution to the linear equation $u_t + u_{xxx} = 0$, with initial data v_0 ($U(t)$ is the linear KdV group). The map $u_0 \mapsto v_0$ is called the scattering operator. The following result is an easy corollary of Kenig, Ponce and Vega [6].

Scattering operator. Given u_0 small enough in L^2, the out-coming solution $u(t)$ to (1) is global in time, and there is scattering, in the sense that there exists a function $V \in L^2$ so that

$$\|u(t) - U(t)V\|_{L^2} \to 0 \quad \text{as} \quad t \to \infty.$$

This is the description of solutions with initial data around 0 (in L^2), a result which can be understood as stability around 0.

The second point answers the question of behavior of solutions with initial data close to a soliton. As we are in the critical case, one does not have stability : contrary to the sub-critical case (see [14]), one has instability and blow-up. Let us cite a result of Merle [19] and Martel and Merle [16].

Blow-up solutions to (1). There exists $\alpha_0 > 0$ such that the following is true. Suppose

$$E(u) < 0 \quad \textit{and} \quad \int u(t)^2 \leq \int Q^2 + \alpha_0.$$

Then $u(t)$ blows-up in finite or infinite time $T \in (0, \infty]$. Furthermore, there exist $\lambda(t) > C(T-t)^{-1/3}$, $\varepsilon \in \{-1, 1\}$ and $x(t) \in \mathbb{R}$ such that

$$\varepsilon \lambda^{1/2}(t) u(t, \lambda(t)x + x(t)) \rightharpoonup Q \quad \text{in} \quad H^1\text{-weak} \quad \text{as} \quad t \uparrow T.$$

These results are related to Problem 1. Let us now turn to results concerning Problem 2. A surprising result of Martel [11] is the existence and uniqueness of N-solitons in the critical case :

Existence and uniqueness of the N-soliton. Let $p \in [2, 5]$. Let $N \in \mathbb{N}$, $0 < c_1 < \ldots < c_N$, and $x_1, \ldots, x_N \in \mathbb{R}$. There exist $T_0 \in \mathbb{R}$ and a unique function $u \in C([T_0, +\infty), \mathbb{R})$, which is a H^1 solution to (2), and such that

$$\left\| u(t) - \sum_{j=1}^{N} Q_{c_j}(\cdot - x_j - c_j t) \right\|_{H^1} \to 0 \quad \textit{as} \quad t \to \infty.$$

Furthermore, $u \in C^\infty([T_0,\infty)\times\mathbb{R})$ *and convergence takes place in* H^s *for all* $s \geq 0$*, with an exponential decay :*

$$\forall s \geq 0, \exists A_s \quad / \quad \left\| u(t) - \sum_{j=1}^N Q_{c_j}(\cdot - x_j - c_j t) \right\|_{H^1} \leq A_s e^{-\gamma t},$$

where $\gamma = \sigma_0\sqrt{\sigma_0}/32$ *and* $\sigma_0 = \min(c_1, c_2 - c_1, \dots, c_N - c_{N-1})$*.*

This result appears as a development of monotonicity properties and a dynamical argument, ideas which where used by Martel and Merle [13] and Martel, Merle and Tsai [17]. It is a surprise that the argument applies also in the critical case $p = 5$, although it fails in the proof of stability (failure which isn't due to a lack in the proof, but to true instability : see [14], [16]). The second surprise is the uniqueness of a solution behaving as a sum of N solitons.

The last result solves the case of a linear behavior, that is the existence of a wave operator (see [3]).

Large data wave operator. Let $V \in L^2$*. There exist* $T_0 \in \mathbb{R}$ *and a function* $u \in C([T_0,\infty), L^2)$ *solution to (1) such that*

$$\|u(t) - U(t)V\|_{L^2} \to 0 \quad \text{as} \quad t \to \infty.$$

Furthermore u is unique in an adapted class.

In the same way that the result of Martel [11] was based on considerations of Martel, Merle and Tsai [17], this result relies on the analysis of Kenig, Ponce and Vega [6].

1.3 Statement of the main result

Our goal in this article is to construct solutions which behave like a sum of a linear term $U(t)V$, and of N solitons, for the L^2-critical Korteweg-de Vries equation (1). Our main result is the following.

Theorem 1 (Nonlinear wave operator for (1)). *Let* $V \in H^1$ *have sufficient decay on the right, i.e. such that* $(1+x_+)^{2+\delta_0}V(x) \in L^2$ *for some* $\delta_0 > 0$ *(we denote* $x_+ = \max\{0,x\}$*).*

Let $N \in \mathbb{N}$*,* $0 < c_1 < \ldots < c_N$ *and* $x_1,\dots,x_N \in \mathbb{R}$*. Let* $R_j(t,x) = Q_{c_j}(x - x_j - c_j t)$*, for* $j = 1,\dots,N$*, be* N *solitons.*

Then there exists $u^* \in C([T_0,+\infty), H^1)$*, for some* $T_0 \in \mathbb{R}$*, solution to (1) and such that* $u^*(t)$ *is uniformly bounded in* H^1 *and*

$$\left\| u^*(t) - U(t)V - \sum_{j=1}^N R_j(t) \right\|_{H^1} \to 0 \quad \text{as} \quad t \to \infty. \tag{6}$$

Theorem 1 allows to work with large data (V large in L^2), which is both surprising and satisfactory. The decay on the right we assume for V is to ensure low interaction with the solitons. This result should be viewed as a step in the solving process of Problem 2.

***Remark* 1.** This result essentially units the linear approach contained in [7] and [3], and the solitons related approach, developed in [18] and [11]. The difficulty is to mix both methods together, so that they do not break down.

An important change in the method of proof when considering [11] is the following. Solitons have an exponential decay, and so integrability (in time) is always automatic. Here the linear term $U(t)V$ will interact with the solitons to produce a polynomial decay in time, which will require to be taken care of.

***Remark* 2.** This result is analogous to that obtained in [2], where a non-linear wave operator is constructed in the sub-critical case $p = 4$. However, in the sub-critical case, much more decay and smoothness are required on V. This is due to the fact that the linear scattering analysis of [7] is no longer available if $p \neq 5$.

In the sub-critical case,we have to relie on the scattering theory of Hayashi and Naumkin [5]. There it is proved scattering for small data $u_0 \in H^{1,1} = \{u \in H^1 | xu \in H^1\}$: for $p > 3$, given such a u_0 the out-coming solution $u(t)$ to (2) is global, satisfies the linear decay rate $\|u(t)\|_{L^\infty} \leq Ct^{-1/3}$, and there exists $V \in L^2$ such that $\|u(t) - U(t)V\|_{L^2} \to 0$ as $t \to \infty$. Their method is a very beautiful clock-work, but breaks down at some point when constructing the non-linear wave operator. To recover from this, the setting must be strengthen, and hence, the conditions on V must be reinforced.

Here, the methods of [11] and [7] can be smoothly adapted to take care of the interaction between non-linear terms (the solitons) and the linear term $(U(t)V)$, to provide an almost sharp result.

Indeed, our smoothness assumption H^1 is a natural setting to work with the solitons, and especially to have bounded energy. On the other side, notice that the decay assumption only concerns the L^2 level for V, and only decay on the right. The assumption on $U(t)V$ should be understood in this way : to handle the interaction of the solitons, we need one degree of decay on V so that its interference is low enough. To prevent the solitons from interfering too much when handling the linear term $U(t)V$, we need a second order of decay on V.

An optimal result for our framework would then be $(1 + x_+^2)V(x) \in L^2(dx)$. In view of this, our assumption appears to be almost optimal.

***Remark* 3.** The uniqueness of solutions to (1) with a given asymptotic behaviour of the form $U(t)V + R(t)$ is not clear. Remind that for $V = 0$, that is, the N-soliton, one has uniqueness in H^1 (see [11]) : it is linked to the fact the constructed solution is smooth and converges exponentially fast in H^s for all $s \geq 0$ ($s = 4$ would be enough). If V belong to H^1 but not more, this is not possible. However, one might be able to prove uniqueness for smoother V.

***Remark* 4.** There are some analogous results for the (critical) non-linear Schrödinger equation. See Bourgain and Wang [1], Krieger and Schlag [9], Merle [18]. In [1], a solution to the critical NLS equation with a given blow-up behaviour is constructed : thanks to the conformal transform, this is in fact equivalent to construct a solution to the critical NLS equation which behaves like the sum of a soliton and a linear term. High smoothness and low interaction with the soliton are required on the linear term.

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2 Strategy of the proof.

Following a usual convention, different positive constants might be denoted by the same letter C.

Let V as in the hypothesis of Theorem 1, $0 < c_1 < \ldots < c_N$ and $x_1, \ldots, x_N \in \mathbb{R}$. Denote the soliton with speed c_j and shift x_j

$$R_j(t,x) = Q_{c_j}(x - x_j - c_j t).$$

Define also $R(t) = \sum_{j=1}^N R_j(t)$.

Let S_n be an increasing sequence of time, so that $S_n \to \infty$ as $n \to \infty$ and

$$\lim_{n\to\infty} E(U(S_n) + R(S_n)) = \liminf_{t\to\infty} E(U(S_n) + R(S_n)). \tag{7}$$

(such a sequence obviously exists ; the condition on the energy appears when concluding the proof of Theorem 1). For $n > 0$, we define $u_n(t)$, the solution of

$$\begin{cases} u_{nt} + (u_{nxx} + u^5)_x = 0, \\ u_n(S_n) = U(S_n) + R(S_n). \end{cases} \tag{8}$$

Equivalently, we introduce $w_n(t)$ the error term

$$w_n(t) = u_n(t) - U(t)V - R(t),$$

so that $w_n(t)$ satisfies the equation

$$\begin{cases} w_{nt} + w_{nxxx} + \left(u^5 - \sum_{j=1}^N R_j^5(t)\right)_x = 0, \\ w_n(S_n) = 0. \end{cases} \tag{9}$$

As $u(S_n) \in H^1$, $u_n(t)$ is well defined, at least on a small interval of time around S_n.

The heart of the proof of Theorem 1 is the following result :

Proposition 1 (Uniform estimates). *There exist T_0, K_0 and a continuous function $\eta : [1,\infty) \to \mathbb{R}_*^+$, depending on V, with*

$$\eta(t) \downarrow 0 \quad \textit{as} \quad t \to \infty,$$

such that the following is true. For all n such that $S_n \geq T_0$, the solution $u_n(t)$ to (8) and the solution $w_n(t)$ to (9) belong to $C([T_0, S_n], H^1)$. Furthermore, we have the uniform decay estimate and control (in n) :

$$\forall t \in [T_0, S_n], \quad \|w_n(t)\|_{L^2} \leq \eta(t), \quad \textit{and} \quad \|w_n(t)\|_{H^1} \leq K_0. \tag{10}$$

The proof of this proposition requires several steps.

The first remark allows us to further assume smallness on $w_n(t)$, in order to get the decay (10).

Proposition 1' (Reduction of proof). *There exist* $\varepsilon_0 > 0$, $T_0 \geq 1$ *and a decreasing continuous function* $\eta : [1, \infty) \to \mathbb{R}_*^+$, *depending on* V, *with*

$$\eta(t) \downarrow 0 \quad as \quad t \to \infty,$$

such that the following is true. Introduce the norm

$$\|f(t,x)\|_{\mathcal{N}([A,B])} = \|f\|_{L_x^5 L_t^{10}(t\in[A,B])} + \sup_{t\in[A,B]} \|f(t)\|_{L_x^2}.$$

Let $n \in \mathbb{N}$ *so that* $S_n \geq T_0$. *Let* $I_n \in [T_0, S_n]$ *such that* $\|w_n\|_{\mathcal{N}([I_n,S_n])} \leq \varepsilon_0$. *Then in fact,*

$$\forall t \in [I_n, S_n], \quad \|w_n\|_{\mathcal{N}([t,S_n])} \leq \eta(t), \quad and \quad \|w_n(t)\|_{H^1} \leq K_0.$$

We introduce the $L_x^5 L_t^{10}$ space as it is necessary in the control of the linear term $U(t)V$: see [7] for further details.

Proof of Proposition 1 assuming Proposition 1'. This is a continuity argument. Let

$$T_0 = \inf\{\tau : \tau \geq 1 \text{ and } \eta(\tau) \leq \varepsilon_0\},$$

and define

$$I_n^* = \inf\left\{\tau : \tau \in [1, S_n], \text{ and } \|w_n\|_{\mathcal{N}([\tau,S_n])} \leq \varepsilon_0\right\}.$$

We now use the continuity the norm $L_x^5 L_t^{10} \cap C^0 H^1$ under the flow of (1), (see [7]). As $w_n(S_n) = 0$, we obtain that the set on which we do the infimum is non-empty, so that $I_n^* < S_n$.

Then of course, this allows us to apply Proposition 1' with $I_n = I_n^*$ so that

$$\forall t \in [I_n^*, S_n], \quad \|w_n\|_{\mathcal{N}([t,S_n])} \leq \eta(t), \quad \text{and} \quad \|w_n(t)\|_{H^1} \leq K_0. \tag{11}$$

By minimality of I_n^*, if $I_n^* > 1$, we also get that

$$\limsup_{t\downarrow I_n^*} \|w_n\|_{\mathcal{N}([t,S_n])} \geq \varepsilon_0.$$

In particular, this gives

$$\varepsilon_0 \leq \limsup_{t\downarrow I_n^*} \|w_n\|_{\mathcal{N}([t,S_n])} \leq \limsup_{t\downarrow I_n^*} \eta(t) \leq \eta(I_n^*).$$

So that $\eta(I_n^*) \geq \varepsilon_0$.

In any case, we get that $I_n^* \leq T_0$ (as η is decreasing) : (11) allows us to conclude. □

Proof of Proposition 1'.
Step 1 : Monotonicity and non-linear tools. We obtain L^2 estimates on the right. Let us introduce the cut-off speed

$$\sigma_0 \in (0, 1/2 \min\{c_1, c_2 - c_1, \ldots, c_N - c_{N-1}\}), \tag{12}$$

to be determined in the proof of the following Proposition 2 below, and the cut-off function

$$\psi(x) = \frac{2}{\pi} \arctan\left(\exp\left(-\frac{\sqrt{\sigma_0}}{2}x\right)\right), \qquad \psi_0(t,x) = \psi(x - \sigma_0 t - 2|x_1|). \tag{13}$$

$\psi_0(t)$ allows us to separate the solitons interaction from the $U(t)V$ interaction.

Proposition 2 (Interaction with the solitons). *There exist $\sigma_1 > 0$, ε_1, T_1, C_1 and K_0 such that the following is true. If $\sigma_0 \leq \sigma_1$, $\varepsilon_0 \leq \varepsilon_1$ and $T_0 \geq T_1$, then, for all $n \in \mathbb{N}$ and all $t \in [I_n, S_n]$,*

$$\|w(t)\|_{L^2(1-\psi_0(t))} \leq C_1 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{8}t} + C_1\|U(t)V\|_{L^2(1-\psi_0(t))} + C_1(S_n - t + 1)\|U(t)V\|_{L^2(1-\psi_0(S_n))} + C_1\int_t^{S_n}\|U(t)V\|_{L^2(1-\psi_0(t))}dt,$$

and

$$\|w_n(t)\|_{H^1} \leq K_0.$$

The control of the H^1-norm simply relies on uniform bounds of the energy, and on the smallness assumption on $\|w(t)\|_{L^2}$. The deep result is the first estimate.

Essentially we obtain a polynomial decay on $\|w_n(t)\|_{L^2(1-\psi_0(t))}$ (instead of an exponential decay in the case of solely soliton). However the good point is that we can choose this polynomial decay to be as fast as we want by lowering the interaction of $U(t)V$ with the solitons, that is, by requiring sufficient decay on the right for V : see Lemma 2.

Step 2 : Linear theory. Essentially we have to take care of the interaction of $U(t)V$ and w_n. For this, we use the linear estimates and the setting of [6] and [7].

Proposition 3 (Ineteraction with the linear term). *There exists $\varepsilon_2 > 0$, T_2, C_2 such that the following is true. Suppose that for some C and $\delta_0 > 0$, we have for all n such that $S_n \geq T_2$:*

$$\forall t \in [I_n, S_n], \quad \|w_n(t) + U(t)V\|_{L^2(1-\psi_0(t)} \leq \frac{C}{t^{1+\delta_0}}.$$

Then there exists C_2 such that if we denote :

$$\eta(t) = C_2\|U(\tau)V\|^5_{L^5_xL^{10}_\tau(\tau\geq t)} + C_2 e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|U(\tau)V\|_{L^5_xL^{10}_\tau(\tau\geq t)} + \frac{C_2}{t^{\delta_0}},$$

we have :

$$\|w_n(t)\|_{\mathcal{N}([I_n,S_n])} \leq \eta(t).$$

Of course, $\eta(t)$ decreases to 0 as $t \to \infty$, and so satisfies the conditions of Proposition 1'.

Finally, Proposition 2, Lemma 2 and estimates (42) and (43) ensure that the assumptions of Proposition 3 are fulfilled if V is chosen as in Theorem 1, that is $V \in H^1$ and $x_+^{2+\delta_0}V \in L^2$. Fix $\sigma_0 < \sigma_1$, $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$ and $T_0 = \min\{T_1, T_2\}$: this completes the proof of Proposition 1', and so, of Proposition 1 .

Proof of Theorem 1. From Proposition 1, we are able to prove some compactness property in L^2 on the sequence $u_n(T_0)$. The limit of a subsequence yields an initial data φ_0, from which $u^*(t)$ is the out-coming solution to (1). Then Proposition 1 allows to conclude that

$$\|u^*(t) - U(t)V - R(t)\|_{L^2} \to 0.$$

To obtain the H^1 convergence, we need another argument. We compare $E(U(S_n) + R(S_n))$ and $E(u^*(t))$, taking advantage of (7). By developping

$$E(u^*(t)) = E(w^*(t) + U(t)V + R(t)),$$

and studying carefully all the obtained terms, we finally prove that the error term $\|w_x^*(t)\|_{L^2} \to 0$ as $t \to \infty$: this completes the proof of Theorem 1.

The proof of Theorem 1 assuming Proposition 1 is done in Section 3. The rest of the proof completes the proof of Proposition 1' and thus that of Proposition 1. In Section 4., we give some preliminary estimates to be used both in Section 5. and Section 6. Section 5. is devoted the proof of Proposition 2. Finally, Proposition 3 is proved in Section 6.

3 Proof of Theorem 1 assuming Proposition 1

In this section, we assume Proposition 1 holds, and from this we conclude the proof of Theorem 1.

3.1 A compactness result linked with the monotonicity Lemma 5

From Proposition 1, we dispose of a sequence $u_n(t)$ defined on $[T_0, S_n]$ (we dropped the terms with $S_n < T_0$), solutions to (2), such that

$$u_n(S_n) = U(S_n)V + \sum_{j=1}^{N} R_j(S_n) = U(S_n) + R(S_n),$$

and that the following uniform estimates holds ($w_n(t) = u_n(t) - U(t)V - R(t)$) :

$$\forall n \in \mathbb{N},\ \forall t \in [T_0, S_n], \quad \|w_n(t)\|_{L^2} \leq \eta(t) \quad \text{and} \quad \|w(t)\|_{H^1} \leq K_0.$$

Claim. $u_n(T_0)$ is a compact sequence in the sense that

$$\lim_{A\to\infty} \sup_{n\in\mathbb{N}} \int_{|x|\geq A} {u_n}^2(T_0, x)dx = 0.$$

Proof of the Claim. Indeed, let $\varepsilon > 0$, and $T(\varepsilon)$ such that $\eta(T(\varepsilon)) \leq \sqrt{\varepsilon}$. Then

$$\int (u_n(T(\varepsilon)) - U(T(\varepsilon))V - R(T(\varepsilon))^2 \leq \varepsilon.$$

Let $A(\varepsilon)$ be such that $\int_{|x|\geq A(\varepsilon)} (U(T(\varepsilon))V + R(T(\varepsilon)))^2(x)dx \leq \varepsilon$; we get

$$\int_{|x|\geq A(\varepsilon)} {u_n}^2(T(\varepsilon), x)dx \leq 2\varepsilon.$$

Let $g \in C^3$ a function such that $g(x) = 0$ if $x \leq 0$, $g(x) = 1$ if $x \geq 2$, and furthermore $0 \leq g'(x) \leq 1$, $0 \leq g'''(x) \leq 1$.

Remind that if $f \in C^3$ does only depend on x, we have

$$\frac{d}{dt}\int {u_n}^2 f = -3\int {u_n}_x^2 f_x + \int {u_n}^2 f_{xxx} + \frac{2p}{p+1}\int {u_n}^{p+1} f_x.$$

(See Lemma 5 and its proof). For $C(\varepsilon)$ to be determined later, we then have

$$\frac{d}{dt}\int {u_n}^2(t,x) g\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right) = -\frac{3}{C(\varepsilon)}\int u_{nx}^2 g'\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right) + \frac{1}{C(\varepsilon)^3}\int {u_n}^2 g'''\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right) + \frac{2p}{(p+1)C(\varepsilon)}\int {u_n}^{p+1} g'\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right).$$

As $t \geq T_0$, u_n satisfies $\|u_n(t)\|_{H^1} \leq K_0 + \|V\|_{H^1} + \sum_{j=1}^N \|Q_{c_j}\|_{H^1} \leq C^0$. So that :

$$\left|\frac{d}{dt}\int {u_n}^2(t,x) g\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right)\right| \leq \frac{1}{C(\varepsilon)}\left(3\int u_{nx}^2(t) + \int {u_n}^2(t) + \frac{2p}{p+1}\|u_n\|_{L^\infty}^{p-1}\int {u_n}^2(t)\right) \leq \frac{1}{C(\varepsilon)}\left(3{C^0}^2 + \frac{2p}{p+1}2^{(p-1)/2}{C^0}^{p+1}\right).$$

Now choose $C(\varepsilon) = \max\left\{1, \frac{T(\varepsilon)-T_0}{\varepsilon}\left(3{C^0}^2 + \frac{2p}{p+1}2^{(p-1)/2}{C^0}^{p+1}\right)\right\}$, from which we derive

$$\left|\frac{d}{dt}\int {u_n}^2(t,x) g\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right)\right| \leq \frac{\varepsilon}{T(\varepsilon)-T_0}.$$

And after integration in time between T_0 and $T(\varepsilon)$,

$$\int_{x\geq 2C(\varepsilon)+A(\varepsilon)} {u_n}^2(T_0,x) \leq \int {u_n}^2(T_0,x) g\left(\frac{x-A(\varepsilon)}{C(\varepsilon)}\right) \leq 3\varepsilon.$$

Now considering $\frac{d}{dt}\int {u_n}^2(t,x) g\left(\frac{-A(\varepsilon)-x}{C(\varepsilon)}\right)$, we get in a similar way

$$\int_{x\leq -2C(\varepsilon)-A(\varepsilon)} {u_n}^2(T_0,x) \leq 3\varepsilon.$$

So that if we denote $A_\varepsilon = 2C(\varepsilon/6) + A(\varepsilon/6)$, we obtain

$$\forall n \in \mathbb{N}, \qquad \int_{|x|\geq A_\varepsilon} {u_n}^2(T_0,x) \leq \varepsilon,$$

as claimed. □

3.2 Construction of u^* and L^2 convergence to the profile

Now $u_n(T_0)$ is a bounded sequence in $H^1(\mathbb{R})$, and so converges weakly up to a subsequence, to some φ_0 in $H^1(\mathbb{R})$ (we suppose for convenience that the whole sequence converges weakly). The previous compactness result ensures that the convergence is strong in $L^2(\mathbb{R})$. Indeed, let $\varepsilon > 0$, and A such that $\int_{|x|\geq A} \varphi_0^2(x)dx \leq \varepsilon$ and

$$\forall n \in \mathbb{N}, \qquad \int_{|x|\geq A} {u_n}^2(T_0,x) \leq \varepsilon.$$

The injection $H^1([-A,A]) \to L^2([-A,A])$ is compact, so that $\int_{|x|\le A} |u_n(T_0,x) - \varphi_0(x)|^2 dx \to 0$. We thus derive that

$$\limsup_{n\to\infty} \|u_n(T_0) - \varphi_0\|^2_{L^2(\mathbb{R})} \le 4\varepsilon.$$

As this is true for all $\varepsilon > 0$, $u_n(T_0) \to \varphi_0$ in $L^2(\mathbb{R})$.

Denote $u^*(t)$ the solution to

$$\begin{cases} u^*_t + (u^*_{xx} + u^{*p})_x = 0, \\ u^*(T_0) = \varphi_0. \end{cases}$$

The Cauchy problem being well-posed in $L^2(\mathbb{R})$, u^* is well defined, at least for t in a neighborhood $\mathcal{V}$ of T_0. Now the flow is continuous in L^2 (in fact it is Lipschitz), so that for all $t \in \mathcal{V}$, $u_n(t) \to u^*(t)$ in L^2. As $(u_n(t))_n$ is a bounded sequence in H^1, this proves that the whole sequence converges weakly to $u^*(t)$ in H^1 :

$$\forall t \in \mathcal{V}, \quad \lim_{n\to\infty} u_n(t) = u^*(t) \quad \text{in } L^2(\mathbb{R}) - \text{strong and } H^1(\mathbb{R}) - \text{weak}. \tag{14}$$

Thus, we can take the limit in the estimates (10) (with t fixed), to get

$$\forall t \in \mathcal{V}, \quad \|u^*(t) - U(t)V - R(t)\|_{L^2} \le \eta(t), \quad \text{and} \quad \|u^*(t) - U(t)V - R(t)\|_{H^1} \le K_0.$$

This shows that $u^*(t)$ is H^1 uniformly bounded on $\mathcal{V}$, so that by the Cauchy problem theory and a standard continuity argument, u^* is defined for all $t \ge T_0$. Hence, $w^*(t)$ is uniformly bounded in H^1, and satisfies the expected L^2 decay estimate :

$$\|w^*(t)\|_{H^1} \le K_0 \quad \text{and} \quad \|w^*(t)\|_{L^2} \le \eta(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{15}$$

3.3 H^1 convergence of $u^*(t)$ to its profile

The H^1 convergence comes essentially from an analysis of the energy $E(u^*(t))$.

From (14), $u_n(t) \rightharpoonup u^*(t)$ H^1-weak and $u_n(t) \to u^*(t)$ in L^6 as $n \to \infty$, and we deduce that

$$\begin{aligned} E(u^*(T_0)) &\le \liminf_{n\to\infty} E(u_n(T_0)) \le \liminf_{n\to\infty} E(u_n(S_n)) \\ &\le \liminf_{n\to\infty} E(U(S_n)V + R(S_n)). \end{aligned}$$

Now, conservation of energy gives for $E(u^*(t)) = E(u^*(T_0))$, for $t \ge T_0$. By (7), and in view of the previous computation, we have

$$\begin{aligned} &\liminf_{t\to\infty} \big(E(U(t)V + R(t)) - E(u^*(t))\big) \\ &\qquad = \liminf_{t\to\infty} E(U(t)V + R(t)) - E(u^*(T_0)) \\ &\qquad = \lim_{n\to\infty} E(U(S_n)V + R(S_n)) - E(u^*(T_0)) \\ &\qquad \ge 0. \end{aligned} \tag{16}$$

Thus, let us estimate $E(U(t)V + R(t)) - E(u^*(t))$:

$$E(U(t)V + R(t)) - E(u^*(t))$$

$$
\begin{aligned}
&= E(U(t)V + R(t)) - E(w^*(t) + U(t)V + R(t)) \\
&= E(U(t)V + R(t)) - E(w^*(t)) - E(U(t)V + R(t)) - \int w_x^*(t)U(t)V_x \\
&\quad - \int w_x^*(t)R_x(t)) + \frac{1}{6}\sum_{k=1}^{5} C_6^k \int w^*(t)^k (U(t)V + R(t))^{6-k} \\
&= -\frac{1}{2}\int |w_x^*(t)|^2 - \int w_x^*(t)U(t)V_x - \int w_x^*(t)R_x(t) \\
&\quad + \frac{1}{6}\sum_{k=1}^{6} C_6^k \int w^*(t)^k (U(t)V + R(t))^{6-k}. \qquad (17)
\end{aligned}
$$

Remind (15) : by interpolation L^2-H^1, we get that for all $p \geq 2$, $\|w^*(t)\|_{L^p} \to 0$ as $t \to \infty$.

Let us first control the second line in (17) : for $k = 2, \dots, 6$,

$$
\left| \int w^*(t)^k (U(t)V + R(t))^{6-k} \right| \leq \|w^*(t)\|_{L^k}^k \|U(t)V + R(t)\|_{L^\infty}^{6-k} = o_{t\to\infty}(1).
$$

For $k = 1$, we have also

$$
\begin{aligned}
&\left| \int w^*(t)(U(t)V + R(t))^5 \right| \\
&\qquad \leq \|w^*(t)\|_{L^2} \|U(t)V + R(t)\|_{L^2} \|U(t)V + R(t)\|_{L^\infty}^4 = o_{t\to\infty}(1).
\end{aligned}
$$

Now,

$$
\left| \int w_x^*(t)R_x(t) \right| = \left| \int w^*(t)R_{xx}(t) \right| \leq \|w^*(t)\|_{L^2} \|R(t)\|_{H^2} = o_{t\to\infty}(1).
$$

The last term $\int w_x^*(t)U(t)V_x$ requires a little more attention. Consider the function $U(-t)(w^*(t))$: then $\|U(-t)(w^*(t))\|_{L^2} = \|w^*(t)\|_{L^2} \to 0$ as $t \to \infty$ and $\|U(-t)(w^*(t))\|_{H^1} = w^*(t)\|_{H^1}$ is uniformly bounded in t. Hence, the only possible weak limit of $U(-t)(w^*(t))$ in H^1 (as $t \to \infty$) is 0. This proves that :

$$
\begin{cases}
U(-t)(w^*(t)) \to 0 & \text{in } L^2 - \text{strong} \quad \text{as} \quad t \to \infty, \\
U(-t)(w_x^*(t)) \rightharpoonup 0 & \text{in } L^2 - \text{weak} \quad \text{as} \quad t \to \infty.
\end{cases}
$$

This proves that

$$
\int w_x^*(t)U(t)V_x = \int U(-t)(w_x^*(t))V_x = o_{t\to\infty}(1).
$$

We can conclude from (17) that

$$
E(U(t)V + R(t)) - E(u^*(t)) = -\frac{1}{2}\int |w_x^*(t)|^2 + o_{t\to\infty}(1),
$$

and in view of (16),

$$
\begin{aligned}
0 &\leq \liminf_{t\to\infty} \big(E(U(t)V + R(t)) - E(u^*(t))\big) \\
&\leq \liminf_{t\to\infty} \left(-\frac{1}{2}\int |w_x^*(t)|^2 + o_{t\to\infty}(1) \right)
\end{aligned}
$$

$$\leq -\frac{1}{2}\limsup_{t\to\infty}\int |w_x^*(t)|^2.$$

This proves that $\|w_x^*(t)\|_{L^2} \to 0$, and along with (15), we get that

$$\|u^*(t) - U(t)V - R(t)\|_{H^1} \to 0 \quad \text{as} \quad t \to \infty.$$

This concludes the proof of Theorem 1.

The following is devoted the proof of Proposition 1, or more precisely of Proposition 1'. We will now only work on the interval $[I_n, S_n]$.

4 Preliminaries

4.1 Cut-off functions and localized quantities

We already introduced $\sigma_0 \in (0, 1/2\min\{c_1, c_2 - c_1, \dots, c_N - c_{N-1}\})$, and the cut-off function :

$$\psi(x) = \frac{2}{\pi}\arctan\left(e^{-\frac{\sqrt{\sigma_0}}{2}x}\right). \tag{13}$$

We can check that $\lim_{+\infty}\psi = 0$, $\lim_{-\infty}\psi = 1$, and ψ is decreasing. Furthermore, by direct computations,

$$\psi'(x) = -\frac{\sqrt{\sigma_0}}{2\pi\cosh\left(\frac{\sqrt{\sigma_0}}{2}x\right)}, \quad \psi''' = \frac{\sigma_0}{4}\psi'(x)\left(1 - \frac{2}{\cosh\left(\frac{\sqrt{\sigma_0}}{2}x\right)}\right),$$

and so,

$$|\psi'''(x)| \leq -\frac{\sigma_0}{4}\psi'(x). \tag{18}$$

We introduce, for $j = 1, \dots, N-1$:

$$m_j(t) = \frac{c_j + c_{j+1}}{2}t + \frac{x_j + x_{j+1}}{2}, \quad m_0(t) = \sigma_0 t - 2|x_1|, \quad m_{-1}(t) = \frac{\sigma_0}{2}t - 2|x_1|.$$

So that we can define, for $j = -1, \dots, N-1$:

$$\psi_j(t,x) = \psi(x - m_j(t)), \quad \psi_N(t,x) = 1.$$

Then we set, for $j = 1, \dots, N-1$:

$$\phi_0(t) = \psi_0(t), \quad \phi_j(t) = \psi_j(t) - \psi_{j-1}(t), \quad \phi_N(t) = 1 - \psi_{N-1}(t).$$

By construction, $\sum_{k=1}^{j}\phi_k = \psi_j$. Finally, we define some local quantities related to mass and energy :

$$M_j(t) = \int u_t^2(t)\phi_j(t), \quad E_j(t) = \int\left(\frac{1}{2}u_x^2(t) - \frac{1}{p+1}u^{p+1}(t)\right)\phi_j(t),$$
$$F_j(t) = E_j(t) + \frac{1}{100}M_j(t).$$

For $j \geq 1$, the ϕ_j separates the solitons R_j from one another. $\psi_0(t)$ separates the solitons from the linear term $U(t)V$. The aim of $\psi_{-1}(t)$ is different : it provides an interval on which $U(t)V$ is small in H^1 an so in L^∞ (see Lemma 2 hereafter). This will be crucially used in the almost monoticity Lemma 5 (it is in fact the only place where $\psi_{-1}(t)$ plays a role).

Observe that $\|U(t)V\|_{L^\infty} \leq C\|V\|_{L^1}t^{-1/3}$, so that pointwise smallness on $U(t)V$ is automatic if $V \in L^1$. However, this hypothesis is not part of Theorem 1.

4.2 Preliminary bounds on $w_n(t)$

Notice that from the uniform bound on the energy, we get a uniform control on $w_n(t)$ for $t \in [I_n, S_n]$. This is the purpose of the following lemma. This preliminiary result will be very important in the proof of the almost monotonicity Lemma 5.

Lemma 1 (Bound on the H^1 norm of $w_n(t)$). *There exists K_0 independent of $\varepsilon_0 \in]0, \kappa(6)^{-1/4}]$ (remind (5)), such that*

$$\forall n \in \mathbb{N},\ \forall t \in [I_n, S_n], \quad \|w_n(t)\|_{H^1} \leq K_0. \tag{19}$$

In particular, the $\|w(t)\|_{L^\infty}$ can be made arbitrarily small :

$$\forall n \in \mathbb{N}, \forall t \in [I_n, S_n], \quad \|w(t)\|_{L^\infty} \leq \sqrt{\varepsilon_0 K_0}. \tag{20}$$

Remark that this lemma gives the second estimate of Proposition 1'.

Proof. We combine smallness of $w_n(t)$ in L^2 along with uniform bounds (in n) on $E(u_n)$. The energy is preserved so that $E(u_n(S_n)) = E(u_n(t))$. Then we have

$$\begin{aligned} E(u_n(S_n)) &= E(U(S_n)V + R(S_n)) \\ &\leq C\int |U(S_n)V_x|^2 + C\sum_{j=1}^{N}\int {Q_{c_j}}_x^2 + C\int |U(S_n)V|^6 + C\sum_{j=1}^{N}\int Q_{c_j}^6 \leq C. \end{aligned}$$

So that the energy $E(u_n(t))$ is uniformly bounded (in n). Now we have the following.

Claim. Let $f, \varepsilon \in H^1$, with $\|\varepsilon\|_{L^2} \leq \kappa(6)^{-1/4}$. Then there is a function $G : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|\varepsilon\|_{H^1} \leq 3E(f+\varepsilon) + G(\|f\|_{H^1}).$$

To conclude, it suffice to apply the claim for $\varepsilon = w_n(t)$ and $f = U(t)V + R(t)$ (whose H^1-norm is uniformly bounded in t).

Let us prove the claim. Indeed, we compute :

$$\begin{aligned} E(f+\varepsilon) &= \frac{1}{2}\int (f+\varepsilon)_x^2 - \frac{1}{6}\int (f+\varepsilon)^6 \\ &= \frac{1}{2}\int f_x^2 + \int f_x\varepsilon_x + \frac{1}{2}\int \varepsilon_x^2 - \frac{1}{6}\sum_{k=0}^{5} C_6^k \int \varepsilon^k f^{6-k} - \frac{1}{6}\int \varepsilon^6. \end{aligned}$$

Now, we have $\int f_x^2 \leq \|f\|_{H^1}^2$, $|\int f_x\varepsilon_x| \leq \|f\|_{H^1}\|\varepsilon_x\|_{L^2}$, $\int f^6 \leq \|f\|_{H^1}^6$, and $|\int \varepsilon f^5| \leq \|f\|_{H^1}^5\|\varepsilon\|_{L^2}$. For $k = 2, \ldots, 5$, we have the Gagliardo-Nirenberg inequality (whose sharp constant is $\kappa(k)$) :

$$\int \varepsilon^k f^{6-k} \leq \kappa(k)\|f\|_{L^\infty}^{6-k}\|\varepsilon\|_{L^2}^{k/2+1}\|\varepsilon_x\|_{L^2}^{k/2-1}.$$

For $k = 6$, the Gagliardo-Nirenberg inequality also applies, but gives an exponent 2 for $\|\varepsilon\|_{L^2}$:

$$\frac{1}{6}\int \varepsilon^6 \leq \frac{\kappa(k)}{6}\|\varepsilon\|_{L^2}^4\|\varepsilon_x\|_{L^2}^2 \leq \frac{1}{6}\int \varepsilon_x^2.$$

So that we get from the energy equality :

$$\frac{1}{2}\int \varepsilon_x^2 \le E(f+\varepsilon) + \|f\|_{H^1}^2 + \|f\|_{H^1}\|\varepsilon_x\|_{L^2} + \|f\|_{H^1}^6 + \|f\|_{H^1}^5\|\varepsilon\|_{L^2} \\ + \sum_{k=2}^{5} \|f\|_{L^\infty}^{6-k}\|\varepsilon\|_{L^2}^{k/2+1}\|\varepsilon_x\|_{L^2}^{k/2-1} + \frac{1}{6}\int \varepsilon_x^2.$$

This can be rewritten as ($\|\varepsilon\|_{L^2} \le 1$)

$$\frac{1}{3}\|\varepsilon_x\|_{L^2}^2 \le E(f+\varepsilon) + 2^5(\|f\|_{H^1} + \|f\|_{L^\infty}^6)(1+\|\varepsilon_x\|_{L^2}^{3/2}).$$

If $a^2 \le K_1 + K_2 a^{3/2}$, then obviously $a^2 \le K_1 + K_2^4$, so that we get

$$\|\varepsilon_x\|_{L^2}^2 \le 3E(f+\varepsilon) + 3\cdot 2^6(\|f\|_{H^1} + \|f\|_{L^\infty}^{24}). \qquad \square$$

4.3 Estimates of $U(t)V$ on the right

We now obtain bounds for $U(t)V$ on the right, which is will be crucial for the monotonicity Lemma 5, and also in Section 5 (analysis of the interaction of the linear term $U(t)V$).

Lemma 2 ($U(t)V$ estimates on the right). *Let $f \in L^2$, then*

$$\|U(t)f\|_{L^2(1-\psi_{-1}(t))} \le \|f\|_{L^2(1-\psi_{-1}(t/2))} \to 0 \quad as \quad t\to\infty. \tag{21}$$

In particular, if $f \in H^1$, then

$$\sup_{x\ge m_{-1}(t)} |U(t)f(x)|^2 \le 4\|f\|_{L^2}\|f_x\|_{L^2(1-\psi_{-1}(t/2))} \to 0 \quad as \quad t\to\infty. \tag{22}$$

Suppose that $(1+x_+^q)f(x) \in L^2(dx)$, for some $q>0$. Then there exists a constant $C = C(\sigma_0, x_1)$ independent of f such that

$$\forall t\ge 1, \quad \|U(t)f\|_{L^2(1-\psi_0(t))} \le \frac{C}{t^q}\|(1+x_+^q)f(x)\|_{L^2(dx)}. \tag{23}$$

If $(1+x_+^{1/2})f(x) \in L^2(dx)$, we have furthermore that

$$\int_{t\ge 0} \|U(t)f\|_{L^2(1-\psi_0(t))}^2 dt \le C\|(1+x_+^{1/2})f(x)\|_{L^2(dx)}^2 < \infty. \tag{24}$$

We will apply this result to V and V_x.

Proof. The key remark is that $U(t)$ "pushes" the L^2-mass on the left. Let $\varphi = \psi_{-1}$ or $\varphi = \psi_0$. We compute :

$$\begin{aligned}
&\frac{d}{d\tau}\int |U(2\tau-t)f|^2\varphi(\tau) \\
&= 2\int (U(2\tau-t)f)_\tau U(2\tau-t)f\varphi(\tau) + \int |U(2\tau-t)f|^2\varphi_\tau(\tau) \\
&= -4\int U(2\tau-t)f_{xxx}U(2\tau-t)f\varphi(\tau) + \int |U(2\tau-t)f|^2\varphi_\tau(\tau)
\end{aligned}$$

$$
\begin{aligned}
&= 4\int U(2\tau-t)f_{xx}U(2\tau-t)f_x\varphi(\tau) + 4\int U(2\tau-t)f_{xx}U(2\tau-t)f\varphi_x(\tau) \\
&\quad + \int |U(2\tau-t)f|^2\varphi_\tau(\tau) \\
&= -6\int |U(2\tau-t)f_x|^2\varphi_x(\tau) - 4\int U(2\tau-t)f_xU(2\tau-t)f\varphi_{xx}(\tau) \\
&\qquad + \int |U(2\tau-t)f|^2\varphi_\tau(\tau) \\
&= -6\int |U(2\tau-t)f_x|^2\varphi_x(\tau) + \int |U(2\tau-t)f|^2(2\varphi_{xxx}(\tau)+\varphi_\tau(\tau)).
\end{aligned}
$$

As $\psi_{xxx} \le \frac{\sigma_0}{4}|\psi_x|$, and $\psi_x < 0$, we have that, for $\varphi = \psi_{-1}$ or ψ_0,

$$
\varphi_x(\tau) < 0 \quad \text{and} \quad 2\varphi_{xxx}(\tau) + \varphi_\tau(\tau) \ge 0.
$$

So that $\tau \mapsto \int U(2\tau-t)f(x)^2\varphi_0(\tau,x)dx$ is an increasing function of τ. In particular, when comparing for $\tau = t$ and $\tau = t/2$ ($t \ge 0$), we have :

$$
\forall t \ge 0, \quad \int |U(t)f|^2\varphi(t) \ge \int f^2\varphi_0(t/2).
$$

As the flow $U(t)$ preserves the L^2-mass, we get in each case $\varphi = \psi_{-1}$ or ψ_0 :

$$
\forall t \ge 0, \quad \int |U(t)f|^2(x)(1-\psi_{-1}(t,x))dx \le \int f^2(x)(1-\psi_{-1}(t/2,x))dx, \tag{25}
$$

$$
\int |U(t)f|^2(x)(1-\psi_0(t,x))dx \le \int f^2(x)(1-\psi_0(t/2,x))dx. \tag{26}
$$

(25) immediatly gives (21). Let $x \ge m_{-1}(t)$. Then for $y \ge x$, $1-\psi_{-1}(t,y) \ge 1-\frac{2}{\pi}\arctan(1) = \frac{1}{2}$. Thus,

$$
\begin{aligned}
|U(t)f(x)|^2 &= -2\int_y^\infty U(t)f(y)U(t)f_x(y)dy \\
&\le 2\left(\int_y^\infty |U(t)f(y)|^2dy\right)^{1/2}\left(\int_y^\infty |U(t)f_x(y)|^2dy\right)^{1/2} \\
&\le 8\left(\int_y^\infty |U(t)f(y)|^2(1-\psi_{-1}(t,y))dy\right)^{1/2} \\
&\qquad\qquad \times\left(\int_y^\infty |U(t)f_x(y)|^2(1-\psi_{-1}(t,y))dy\right)^{1/2} \\
&\le 8\|U(t)f\|_{L^2(1-\psi_{-1}(t))}\|U(t)f_x\|_{L^2(1-\psi_{-1}(t))} \\
&\le 8\|f\|_{L^2(1-\psi_{-1}(t/2))}\|f_x\|_{L^2(1-\psi_{-1}(t/2))}.
\end{aligned}
$$

This is (22).

We will now use (26). Suppose that for some $q > 0$, $(1+x_+^q)f(x) \in L^2(dx)$. Then for $t \ge 1$,

$$
\int f^2(1-\psi_0(t/2)) = \int_{x\le\sigma_0 t/4} f^2(1-\psi_0(t/2)) + \int_{x\ge\sigma_0 t/4} f^2(1-\psi_0(t/2))
$$

$$\leq \sup_{x\leq\sigma_0 t/4}(1-\psi_0(t/2,x))\int f^2 + \left(\frac{\sigma_0 t}{4}\right)^{-2q}\int_{x\geq\sigma_0 t/4} x^{2q}f^2$$
$$\leq C(x_0)e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|f\|_{L^2}^2 + C(\sigma_0)t^{-2q}\|x_+^q f\|_{L^2}^2.$$

And we get

$$\forall t\geq 1,\quad \|U(t)f\|_{L^2(1-\psi_0(t))} \leq \frac{C}{t^q}\|(1+x_+)^q f\|_{L^2},$$

which is (23).

Suppose now that $(1+x_+^{1/2})f(x)\in L^2(dx)$. Then

$$\begin{aligned}\int_{t=0}^{\infty}\int_x |U(t)f(x)|^2(1-\psi_0(t,x))dxdt &\leq \int_{t=0}^{\infty}\int_x f^2(x)(1-\psi_0(t/2,x))dxdt\\ &\leq \int_x f^2(x)\int_{t=0}^{\infty}(1-\psi_0(t/2,x))dtdx\\ &\leq C\int f^2(x)\frac{1+x_+}{\sigma_0}dx\\ &\leq C\|(1+x_+^{1/2})f(x)\|_{L^2(dx)}^2,\end{aligned}$$

and this proves (24). □

5 Control of the interaction of $w_n(t)$ with the solitons

This section is devoted to the proof of Proposition 2. We develop arguments very similar to those of [17] and [16].

5.1 Modulation close to the asymptotic profile

Lemma 3. *There exist T_1 large enough and $\varepsilon_1>0$ small enough such that if $T_1\geq T_1$ and $\varepsilon_0\leq\varepsilon_1$, the following is true.*

There exist $2N$ C^1 functions $y_j,\gamma_j:[I_n,S_n]\to\mathbb{R}$ such that if we denote :

$$\tilde{R}_j(t,x)=Q_{\gamma_j(t)}(x-y_j(t)),\qquad \tilde{R}(t,x)=\sum_{j=1}^N\tilde{R}_j(t,x),$$
$$\tilde{w_n}(t)=u_n(t,x)-U(t)V-\tilde{R}(t,x),$$

we have for all $j=1,\ldots,N$:

$$\int\tilde{w_n}(t,x)\tilde{R}_{j\,x}(t,x)dx=0\quad\text{and}\quad\int\tilde{w_n}(t,x)\tilde{R}_j^{\,3}(t,x)dx=0.$$

Moreover, there exists C_{11} such that :

$$\|\tilde{w_n}(t)\|_{L^2}+\sum_{j=1}^N|\gamma_j(t)-c_j|+\sum_{j=1}^N|y_j(t)-x_j-c_jt|\leq C_{11}\varepsilon_0,\tag{27}$$

$$|y_j'(t) - c_j| + |\gamma_j'(t)| \leq C_{11} e^{-\frac{\sigma_0 \sqrt{\sigma_0}}{2} t} + C_{11} \|U(t)V\|_{L^2(1-\psi_0(t))} + C_{11} \left(\int \tilde{w_n}^2(t) e^{-\sqrt{\sigma_0}|x - c_j t|} \right)^{1/2}. \quad (28)$$

Proof. The existence of the modulation is essentially an application of the implicit function theorem. Consider the C^∞ functional

$$F : [I_n, S_n] \times L^2 \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n,$$
$$(t, u, (y_j)_j, (\gamma_j)_j) \mapsto (F_{1j}(t, u, (y_j)_j, (\gamma_j)_j), F_{2j}(t, u, (y_j)_j, (\gamma_j)_j)),$$

with

$$F_{1j}(t, u, (y_j)_j, (\gamma_j)_j) = \int (u - U(t)V - \tilde{R}(t)) \tilde{R}_{j\,x}(t) dx,$$
$$F_{2j}(t, u, (y_j)_j, (\gamma_j)_j) = \int (u - U(t)V - \tilde{R}(t)) \tilde{R}_j^{\,3}(t) dx,$$

locally on a neighborhood of the curve $y_j(t) = x_j + c_j t$, $\gamma_j(t) = c_j$, $u = U(t)V + R(t)$. To express y_j, γ_j in function of u, t, we apply the implicit function theorem stated in the Appendix : let us prove that $\partial_{y_j, \gamma_j} F$ is invertible at points $(t, U(t)V + R(t), x_j + c_j t)_j, (c_j)_j)$, compute $\partial_u F$, and do some uniform (in t) estimates.

For all t, α being y_j or γ_k we compute

$$\partial_\alpha F_{1j}(t) = -\int (\partial_\alpha \tilde{R}_j)(t) \tilde{R}_{j\,x}(t) + \int (u - U(t)V - \tilde{R}_j(t))(\partial_\alpha \tilde{R}_{j\,x})(t),$$
$$\partial_\alpha F_{2j}(t) = \int (\partial_\alpha \tilde{R}_j)(t) \tilde{R}_j^3(t) + 3 \int (u - U(t)V - \tilde{R}_j(t))(\partial_\alpha \tilde{R}_j)(t) \tilde{R}_j^2(t),$$

and

$$(\partial_{y_j} \tilde{R}_j)(t, x) = -\tilde{R}_{j\,x}(t, x),$$
$$(\partial_{\gamma_j} \tilde{R}_j)(t, x) = \frac{1}{4c_j} \tilde{R}_j(t, x) + \frac{1}{2c_j}(x - x_j - c_j t) \tilde{R}_{j\,x}(t, x).$$

Let u, y_j and γ_j be such that

$$\|u - U(t)V - R(t)\|_{L^2} + \sum_{j=1}^{N} |y_j| + |\gamma_j| \leq \varepsilon_0.$$

We get that

$$\left| \partial_{y_j} F_{1j} - \int Q_{c_j\,x}^{\,2} \right| \leq C\varepsilon_0,$$

(recall $\|Q_{c_j}\|_{L^2} = \|Q\|_{L^2}$) and for $k \neq j$, using the exponential decay :

$$|\partial_{y_k} F_{1j}(t)| \leq C e^{-\frac{\sigma \sqrt{\sigma_0}}{4} t} + C\varepsilon_0,$$
$$|\partial_{y_k} F_{2j}(t)| + |\partial_{\gamma_k} F_{1j}(t)| + |\partial_{\gamma_k} F_{2j}(t)| \leq C e^{-\frac{\sigma \sqrt{\sigma_0}}{4} t} + C\varepsilon_0.$$

Now Q is an even function, so that

$$|\partial_{\gamma_j} F_{1j}(t)| \leq C\delta, \qquad |\partial_{y_j} F_{2j}(t)| \leq C\varepsilon_0.$$

Finally, for $\partial_{\gamma_j} F_{2j}$, we have

$$\left|\partial_{\gamma_j} F_{2j} - \frac{1}{4c_j}\int {Q_{c_j}}^4\right| \leq C\varepsilon_0.$$

Hence, for T_1 large enough and $\varepsilon_1 > 0$ small enough, the conditions of the implicit function theorem are fufilled, and we obtain the existence and regularity of $y_j(t)$, $\gamma_j(t)$, along with the first estimate (27).

For the second estimate (28), we compute the equation satisfied by $\tilde{w}(t)$ and do the scalar product with every $\tilde{R}_{j\,x}$ and every $\tilde{R}_j^3$: the relations obtained in this way will yield the result. The equation satisfied by $\tilde{R}_k$ (using $-c_k R_{kx} + R_{kxxx} + R_k^5)_x = 0$) is now

$$\begin{aligned}
&\tilde{R}_{kt} + \tilde{R}_{kxxx}\\
&= (-y_k'(t) + c_k)\tilde{R}_{kx} + \frac{\gamma_k'(t)}{\gamma_k(t)}\left(\frac{\tilde{R}_k(t)}{4} + (x - y_k(t))\frac{\tilde{R}_{kx}(t)}{2}\right) - c_k\tilde{R}_{kx} + \tilde{R}_{kxxx}\\
&= (-y_k'(t) + c_k)\tilde{R}_{kx} + \frac{\gamma_k'(t)}{\gamma_k(t)}\left(\frac{\tilde{R}_k(t)}{4} + (x - y_k(t))\frac{\tilde{R}_{kx}(t)}{2}\right) - (\tilde{R}_k^5)_x.
\end{aligned}$$

So that with $\tilde{w_n} = u_n(t) - U(t)V - \tilde{R}(t)$, we get

$$\begin{aligned}
\tilde{w_n}_t + \tilde{w_n}_{xxx} = \sum_{k=1}^N (y_k'(t) - c_k)\tilde{R}_{kx} - \sum_{k=1}^N \frac{\gamma_k'}{\gamma_k}\left(\frac{\tilde{R}_k}{4} + (x - y_k(t))\frac{\tilde{R}_{kx}}{2}\right)\\
- \left((\tilde{w_n} + U(t)V + \tilde{R})^5 - \sum_{k=1}^N \tilde{R}_k^5\right)_x. \quad (29)
\end{aligned}$$

Now, if we express $\tilde{R}_j$ in terms of R_j, we get

$$\tilde{R}_{j\,xt} = -y_j'(t)\tilde{R}_{j\,xx} + \frac{\gamma_j'(t)}{\gamma_j(t)}\left(\frac{\tilde{R}_{j\,x}(t)}{4} + (x - y_j(t))\frac{\tilde{R}_{j\,xx}(t)}{2} + \frac{\tilde{R}_{j\,x}(t)}{2}\right).$$

And keeping in mind that $\frac{d}{dt}\int \tilde{w_n}\tilde{R}_{j\,x} = \int \tilde{w_n}\tilde{R}_{j\,x} = 0$, we get

$$\int \tilde{w_n}_t\tilde{R}_{j\,x} = -\int \tilde{w_n}\tilde{R}_{j\,xt} = \int \tilde{w_n}\left(y_j'(t) - \frac{\gamma_j'(t)}{\gamma_j(t)}\frac{x - y_j(t)}{2}\right)\tilde{R}_{j\,xx}.$$

We multiply (29) by $\tilde{R}_{j\,x}$, integrate in x, and do integration by parts :

$$\begin{aligned}
&(y_j'(t) - c_j)\int {\tilde{R}_{j\,x}}^2\\
&= -y_j'(t)\int \tilde{w}(t)\tilde{R}_{j\,xx} + \frac{\gamma_j'(t)}{2\gamma_j(t)}\int \tilde{w_n}(t)(x - y_k(t))\tilde{R}_{j\,xx} - \int \tilde{w_n}(t)\tilde{R}_{j\,xxxx}\\
&- \sum_{k,k\neq j}(c_k - y_k'(t))\int \tilde{R}_{j\,x}\tilde{R}_{kx} + \sum_{k=1}^N \frac{\gamma_k'}{\gamma_k}\int \tilde{R}_{j\,x}\left(\frac{\tilde{R}_k}{4} + (x - y_k(t))\frac{\tilde{R}_{kx}}{2}\right)\\
&\qquad - \int\left((\tilde{w_n} + U(t)V + \tilde{R})^5 - \sum_{k=1}^N \tilde{R}_k^5\right)\tilde{R}_{j\,xx}.
\end{aligned}$$

First consider the three first terms : as $Q_{xx} = Q - Q^p$, we can express $\tilde{R_j}_{xx}$ and $\tilde{R_j}_{xxxx}$ in terms of powers of $\tilde{R_j}$. Therefore, the integral part of these terms is bounded by

$$\int |\tilde{w_n}(t)|(1+|x-c_jt|)e^{-\sqrt{\sigma_0}|x-c_jt|} \leq C\left(\int |\tilde{w_n}(t)|^2 e^{-\sqrt{\sigma_0}|x-c_jt|}\right)^{1/2}.$$

For the fourth term, $\int |R_{j\,x} R_{k\,x}| \leq e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}$. This also apply to the fifth term, but for j-term, which vanishes :

$$\int \tilde{R_j}_x \left(\frac{\tilde{R_j}}{4} + (x-y_j(t))\frac{\tilde{R_j}_x}{2}\right) = 0.$$

And for the non-linear last term, when developing, the large terms cancel one another, so that we can control the rest by :

$$C\int (|\tilde{w_n}(t)| + |U(t)V|)e^{-\sqrt{\sigma_0}|x-c_jt|}.$$

Finally, we have altogether

$$\begin{aligned}|y_j'(t)-c_j| \leq{}& C\left(1+\left|\frac{\gamma_j'(t)}{\gamma_j(t)}\right|\right)\left(\int |\tilde{w_n}(t)|^2 e^{-\sqrt{\sigma_0}|x-c_jt|}\right)^{1/2} \\ &+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\sum_{k,k\neq j}|y_k'(t)-c_k| + Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\sum_{k,k\neq j}\left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right| \\ &+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C\|U(t)V\|_{L^2(1-\psi_0(t))}. \end{aligned} \tag{30}$$

Now, we have to do the same kind of argument on γ_j. As

$$\tilde{R_j}_t = -y_j'(t)\tilde{R_j}_x + \frac{\gamma_j'(t)}{\gamma_j(t)}\left(\frac{\tilde{R_j}(t)}{4} + (x-y_j(t))\frac{\tilde{R_j}_x(t)}{2} + \frac{\tilde{R_j}(t)}{2}\right),$$

we have

$$\int \tilde{w_n}_t \tilde{R}_j^3 = -3\int \tilde{w_n}\tilde{R_j}_t \tilde{R}_j^2 = 3\int \left((y_j' - \frac{\gamma_j'}{2\gamma_j}(x-y_j)\right)\left(\tilde{w_n}\tilde{R_j}_x \tilde{R}_j^2\right).$$

Let us multiply (29) by $\tilde{R}_j^3$. We obtain, after an integration by parts $\int (x-y_j(t))\tilde{R}_j\tilde{R_j}_x = -\frac{1}{2}\int \tilde{R}_j^2$,

$$\begin{aligned}\frac{1}{4}\frac{\gamma_j'(t)}{\gamma_j(t)}\int \tilde{R}_j^4 ={}& \frac{\gamma_j'(t)}{2\gamma_j(t)}\int \tilde{w_n}(t)(x-y_k(t))\tilde{R_j}^3 - \int \tilde{w_n}(t)(\tilde{R_j}^3)_{xxx} \\ &- \sum_{k=1}^{N}(c_k - y_k')\int \tilde{R_j}^3\tilde{R}_{kx} + \sum_{k\neq j}\frac{\gamma_k'}{\gamma_k}\int \tilde{R_j}^3\left(\frac{\tilde{R}_k}{4} + (x-y_k(t))\frac{\tilde{R}_{kx}}{2}\right) \\ &- 3\int\left((\tilde{w_n} + U(t)V + \tilde{R})^5 - \sum_{k,k\neq j}\tilde{R}_k^5\right)\tilde{R_j}_x\tilde{R_j}^2.\end{aligned}$$

Let us notice again that the only possibly large term (in the first sum) is in fact 0 ($\int \tilde{R}_j^3 \tilde{R}_{j\,x} = 0$). If we argue like before, we get

$$\left|\frac{\gamma_j'(t)}{\gamma_j(t)}\right| \le C\left(1+\frac{\gamma_j'(t)|}{\gamma_j(t)}\right)\left(\int \tilde{w_n}^2(t)e^{-\sqrt{\sigma_0}|x-c_jt|}\right)^{1/2}$$
$$+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\sum_{k,k\neq j}|y_k'(t)-c_k| + e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\sum_{k,k\neq j}\left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|$$
$$+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C\|U(t)V\|_{L^2(1-\psi_0(t))}. \tag{31}$$

We can now use our computations. Let us sum our $2N$ estimates (30) and (31) together :

$$\sum_{k=1}^N\left(|y_k'(t)-c_k| + \left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\right) \le C\left(1+\sum_{k=1}^N|y_k'(t)| + \sum_{k=1}^N\left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\right)\|\tilde{w_n}\|_{L^2}$$
$$+ Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\left(\sum_{k=1}^N|y_k'(t)| + \left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right|\right) + Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C\|U(t)V\|_{L^2(1-\psi_0(t))}.$$

So that for ε_1 small enough, as $\|\tilde{w_n}\|_{L^2} \le \varepsilon_0 \le \varepsilon_1$, and $t \ge T_1$ large enough, we get

$$\sum_{k=1}^N|y_k'(t)-c_k| + \left|\frac{\gamma_k'(t)}{\gamma_k(t)}\right| \le C.$$

Let us now go back to (30) : we get exactly what we want on $|y_j'(t)-c_j|$. In the same way, as $\gamma_k > \sigma_0$ for ε_0 small enough (first estimate), we get the result for $|\gamma_j'(t)|$ by plugging in (31). □

Notice that

$$\|w_n(t)-\tilde{w_n}(t)\|_{H^s} = \|R(t)-\tilde{R}(t)\|_{H^s} \le C(s)(|y_j(t)-c_jt-x_j| + |\gamma_j(t)-c_jt|). \tag{32}$$

We now turn to the extraction of the main terms in $\int {u_n}^2(t)$ and $E(u_n(t))$, which writes as follows : remind $\|Q_c\|_{L^2} = \|Q\|_{L^2}$ and $E(Q_c) = 0$). Let us denote for simplicity :

$$\tilde{v_n}(t) = \tilde{w_n}(t) + U(t)V = u_n(t) - \tilde{R}(t).$$

Lemma 4 (Main terms in M_j and E_j, $j \ge 1$). *We have, for all $t \in [I_n, S_n]$,*

(1) $$\left|M_j(t) - \left(\int Q^2 + 2\int \tilde{v_n}(t)\tilde{R}_j(t) + \int \tilde{v_n}^2(t)\phi_j(t)\right)\right| \le C_{12}e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t},$$

(2) $$\left|E_j(t) - \left[\frac{1}{2}\int(\tilde{v_n}_x^2(t) - 5\tilde{R}_j^4(t)\tilde{v_n}^2(t))\phi_j(t) - \gamma_j(t)\int \tilde{v_n}(t)\tilde{R}_j(t)\right)\right|$$
$$\le C_{12}e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C_{12}\varepsilon_0\int \tilde{v_n}^2(t)\phi_j(t),$$

(3) $$\left|\left(E_j(t) + \frac{\gamma_j(t)}{2}M_j(t)\right) - \frac{\gamma_j(t)}{2}\int Q^2 - \frac{1}{2}H_j(t)\right|$$
$$\le C_{12}e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C_{12}\varepsilon_0\int \tilde{v_n}^2(t)\phi_j(t),$$

where $H_j(t) = \int(\tilde{v_n}_x^2(t) - 5\tilde{R}_j^4(t)\tilde{v_n}^2(t) + \gamma_j(t)\tilde{v_n}^2(t))\phi_j(t)$.

Proof. (1) We compute $(u_n = \tilde{v_n} + \tilde{R})$:

$$M_j(t) = \int {u_n}^2 \phi_j(t) = \int \left(\tilde{v_n}^2 + 2\tilde{v_n}\tilde{R}(t) + \sum_{k=1}^{N} \tilde{R_k}^2(t) \right) \phi_j(t).$$

As $\phi_j(t)$ localized in the interval $[m_{j-1}(t), m_j(t)]$, like $\tilde{R}_j(t)$ we get $(k \neq j)$

$$\left| \int \tilde{R}_j^2(t)\phi_j(t) - \int Q^2_{\gamma_j(t)} \right| \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}, \quad \int \tilde{R}_k^2(t)\phi_j(t) \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}.$$

(2) In the same way,

$$\begin{aligned} E_j(t) &= \int \left(\frac{1}{2}(\tilde{v_n}_x^2(t) + 2\tilde{v_n}_x(t)\tilde{R}_x + \tilde{R}_x^2) - \frac{1}{6}(\tilde{v_n}(t) + \tilde{R}(t))^6 \right) \phi_j(t) \\ &= \int (\frac{1}{2}\tilde{v_n}_x^2(t) - \frac{5}{2}\tilde{R}^4\tilde{v_n}^2(t))\phi_j + \int (\frac{1}{2}\tilde{R}_x^2 - \frac{1}{6}\tilde{R}^6)\phi_j(t) \\ &\quad - \int \tilde{v_n}(t)(\tilde{R}_{xx} + \tilde{R}^5)\phi_j - \int \tilde{R}_x\tilde{v_n}(t){\phi_j}_x \\ &\quad + \int \left[\frac{(-(\tilde{v_n}(t) + \tilde{R})^6 + \tilde{R}^6)}{6} + \tilde{v_n}(t)\tilde{R}^p + \frac{5}{2}\tilde{R}^4\tilde{v_n}^2(t) \right] \phi_j. \end{aligned}$$

We keep the first integral untouched. The second one is $E(Q_{\gamma_j(t)})$ up to an exponential correction. For the third one, recall that $Q_{xx} + Q^5 = Q$, so that again

$$\int \tilde{v_n}(t)(\tilde{R}_{xx} + \tilde{R}^5)\phi_j = \gamma_j(t) \int \tilde{v_n}(t)\tilde{R}_j(t) + O(e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}).$$

The fourth one is exponentially small (with $\tilde{R}$ and ${\phi_j}_x$). Finally the fifth is of order at least 3 in v_n, so that we control it by

$$\int \tilde{v}(t)^k\phi_j(t) \leq \|\tilde{v}(t)\|^{k-2}_{L^\infty} \int \tilde{v}(t)^2\phi_j(t).$$

This gives the desired result.

(3) is the sum of (1) and (2). Notice that the scalar product $\int \tilde{v}(t)\tilde{R}_j(t)$ vanishes in H_j : the linear combination has been constructed for this. □

Proposition 4 (Positivity of a quadratic form). *There exists $\sigma_1 > 0$ small enough and $\lambda_1 > 0$ so that the following is true. For $\sigma_0 \leq \sigma_1$, there exists $T_1 = T_1(\sigma_0)$, so that for all $t \geq T_1$, for all $j = 1, \dots, N$, and for all $v \in H^1$,*

$$\begin{aligned} &\int (v_x^2 - 5\tilde{R}_j(t)^4v^2 + \gamma_j(t)v^2)\phi_j(t) \\ &\qquad \geq \lambda_1 \int (v_x^2 + v^2)\phi_j(t) - \frac{1}{\lambda_1}\left(\left(\int v\tilde{R}_j^3(t) \right)^2 + \left(\int v\tilde{R}_{j_x}(t) \right)^2 \right). \end{aligned}$$

Proof. A similar result can be found in [17, Lemma 4] and [16, Appendix A]. For the sake of completeness, the complete proof is done in the Appendix. □

From now on and throughout the rest of the proof, $\sigma_0 < \sigma_1$ is fixed.

5.2 Almost monotonicity properties and Abel transform

Lemma 5 (Monotocity formula [12]). *There exists T_1 large enough, ε_1 small enough, and $C_{13} > 0$ such that for all $j = 0, \dots, N$ and $t \in [I_n, S_n]$,*

$$\sum_{k=0}^{j}(M_k(S_n) - M_k(t)) \geq -C_{13}e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t},$$

$$\sum_{k=0}^{j}(F_k(S_n) - F_k(t)) \geq -C_{13}e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}.$$

Proof. This lemma is very similar to the monotonicity Lemma of [17] and [11]. The only difference is the presence of the term $U(t)V$: this will be taken care of essentially due to pointwise smallness of $U(t)V$ for $x \geq (\sigma_0/2)t$, that is (22).

Let us now do the computations. First notice

$$\sum_{k=0}^{j} M_k(t) = \int {u_n}_t^2(t)\psi_j(t), \quad \sum_{k=0}^{j} E_k(t) = \int \left(\frac{1}{2}{u_n}_x^2(t) - \frac{1}{6}{u_n}^6(t)\right)\psi_j(t).$$

For $j = N$, the result is the conservation of mass and energy. Otherwise we compute for $f(t,x) \in C^3$:

$$\begin{aligned}\frac{d}{dt}\int {u_n}^2 f - \int {u_n}^2 f_t &= 2\int {u_n}_t u_n f = -2\int ({u_n}_{xx} + {u_n}^5)_x u_n f \\ &= 2\int ({u_n}_{xx} + {u_n}^p)({u_n}_x f + u_n f_x) \\ &= \int \left(-3{u_n}_x^2 + \frac{5}{3}{u_n}^6\right) f_x - 2\int {u_n}_x u_n f_{xx} \\ &= \int \left(-3{u_n}_x^2 + \frac{5}{3}{u_n}^6\right) f_x + \int {u_n}^2 f_{xxx}.\end{aligned}$$

So that we get

$$\frac{d}{dt}\int {u_n}^2\psi_j(t) = -\int \left(3{u_n}_x^2 + m_j'(t){u_n}^2 - \frac{5}{3}{u_n}^6\right){\psi_j}_x + \int {u_n}^2 {\psi_j}_{xxx}.$$

But $m_j'(t) \geq \sigma_0$ so that by (18), and ${\psi_j}_x \leq 0$:

$$\frac{d}{dt}\int {u_n}^2\psi_j(t) \geq -\int \left(3{u_n}_x^2 + \frac{3}{4}\sigma_0 {u_n}^2 - \frac{5}{3}{u_n}^6\right)|{\psi_j}_x|(t).$$

It remains to bound the third term. We consider two cases. When $x \in I_j(t) = [m_j t - \frac{\sigma_0}{2}t, m_j(t) + \frac{\sigma_0}{2}t]$, ${\psi_j}_x$ is big but $R(t)$ and $U(t)V$ are small (recall (22)) so that u_n too. More precisely, for $x \in I_{j1}(t)$, $x \geq m_{-1}(t)$, and

$$\begin{aligned}\left|\frac{5}{3}{u_n}^4(t,x)\right| &\leq C(\|w_n(t)\|_{L^\infty}^4 + \|U(t)V\|_{L^\infty(x \geq m_{-1}(t))}^{p-1} + |R(t,x)|^4) \\ &\leq C(K_0^2\varepsilon_0^2 + \|V\|_{L^2(1-\psi_{-1}(t/2))}\|V_x\|_{L^2(1-\psi_{-1}(t/2))} + e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}) \\ &\leq \frac{\sigma_0}{4}, \end{aligned} \tag{33}$$

if T_1 is large enough ($t \geq T_1$), and ε_1 is small enough. On this interval, the second term is larger than the third :

$$\frac{5}{3}\int_{x\in I_j(t)} {u_n}^6|\psi_{j\,x}|(t) \leq \frac{\sigma_0}{4}\int {u_n}^2|\psi_{j\,x}|(t).$$

When $x \in \mathbb{R}\setminus I_j(t)$, then $x \notin [m_j(t) - \frac{\sigma_0}{2}t, m_j(t) + \frac{\sigma_0}{2}t]$, so that

$$|\psi_{j\,x}(t,x)| \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}.$$

Now by interpolation between L^2 and H^1, we have a uniform control $\int |u_n|^6 \leq C$:

$$\frac{5}{3}\int_{x\in\mathbb{R}\setminus I_j(t)} {u_n}^6|\psi_{j\,x}|(t) \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}.$$

So that finally

$$\frac{d}{dt}\int {u_n}^2\psi_j(t) \geq \int \left(3u_{n\,x}^2 + \frac{\sigma_0}{2}{u_n}^2\right)|\psi_{j\,x}(t)| - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} \geq -Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}. \tag{34}$$

We integrate this last estimate between t and S_n, and this gives the estimates on M_j.

For the estimates on F_j, we compute in a similar way

$$\begin{aligned}
&\frac{d}{dt}\int\left(u_{n\,x}^2 - \frac{1}{3}{u_n}^6\right)f - \int\left(u_{n\,x}^2 - \frac{1}{3}{u_n}^6\right)f_t\\
&= 2\int (u_{nxt}u_{nx} - {u_n}^5u_{nt})f = -2\int u_{nt}(u_{nxx} + {u_n}^p)f - 2\int u_{nt}u_{nx}f_x\\
&= -\int (u_{nxx} + {u_n}^5)^2 f_x + 2\int (u_{nxx} + {u_n}^5)_x u_{nx}f_x\\
&= -\int \left((u_{nxx} + {u_n}^5)^2 + 2u_{n\,xx}^2 - 10u_{n\,x}^2{u_n}^{p-1}\right)f_x - 2\int u_{nxx}u_{nx}f_{xx}\\
&= -\int \left((u_{nxx} + {u_n}^p)^2 + 2u_{n\,xx}^2 - 10u_{n\,x}^2{u_n}^{p-1}\right)f_x + \int u_{n\,x}^2 f_{xxx}.
\end{aligned}$$

So that

$$\begin{aligned}
&\frac{d}{dt}\int\left(u_{n\,x}^2 - \frac{1}{3}{u_n}^6\right)\psi_j(t)\\
&= -\int \left((u_{nxx} + {u_n}^5)^2 + 2u_{n\,xx}^2 - 10u_{n\,x}^2{u_n}^4\right)\psi_{j\,x}(t)\\
&\quad - m_j'(t)\int\left(u_{n\,x}^2 - \frac{1}{3}{u_n}^6\right)\psi_{j\,x}(t) + \int u_{n\,x}^2\psi_{j\,xxx}(t).
\end{aligned}$$

Again $m_j'(t) \geq \sigma_0$ and $|m_j'(t)| \leq c_N$, so that $\int u_{n\,x}^2\psi_{j\,xxx}(t) - \frac{\sigma_0}{4}\int u_{n\,x}^2\psi_{j\,x}(t) \geq 0$ and

$$\begin{aligned}
\frac{d}{dt}\int\left(u_{n\,x}^2 - \frac{{u_n}^6}{3}\right)\psi_j(t) \geq \frac{3}{4}\sigma_0\int u_{n\,x}^2|\psi_{j\,x}(t)|&\\
- \int\left(10u_{n\,x}^2|u_n|^4 - \frac{c_N}{3}|u_n|^6\right)|\psi_{j\,x}(t)|.&
\end{aligned} \tag{35}$$

To bound $10\int {u_n}_x^2|u_n|^{p-1}|{\psi_j}_x(t)|$, we proceed like before and get

$$10\int {u_n}_x^2|u_n|^4|{\psi_j}_x(t)| \geq -\frac{\sigma_0}{4}\int |{u_n}_x^2|{\psi_j}_x(t) - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}. \tag{36}$$

However for $\frac{c_N}{6}\int {u_n}^6|{\psi_j}_x(t)|$, some L^2 norm is needed (which is why we introduced F_j, as in [11]). Choosing ε_1 small enough and T_1 large enough, we can improve (33) to $\sigma_0/400$, and so obtain

$$\frac{c_N}{3}\int {u_n}^6 \geq -\frac{\sigma_0}{400}\int {u_n}^2|{\psi_j}_x(t)| - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}. \tag{37}$$

Now adding up (35) and $1/100\cdot$(34), and using (36) and (37), we get

$$\frac{d}{dt}\int\left({u_n}_x^2 - \frac{1}{3}{u_n}^6 + \frac{1}{200}{u_n}^2\right)\psi_j(t) \geq \frac{\sigma_0}{2}\int {u_n}_x^2|\psi_x(t)| - Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}.$$

And the estimate on F_j comes by integration between t and S_n. □

***Remark* 5.** Notice that it is possible to obtain almost monotonicity properties (on the left) related to other quantities than mass and energy. However, these are related to conservation laws : thus they translate to monotonicity properties on the right, and this is specially interesting and useful.

We can now conclude the proof of Proposition 2.

Proof of Proposition 2. We do some estimates on $\tilde{w_n}(t)$ first. The key point is the following resummation argument, which will allow us to use the monotonicity property. We compute

$$\begin{aligned}\sum_{j=1}^{N}\frac{1}{\gamma_j^2(t)}\left(E_j + \frac{\gamma_j(t)}{2}M_j\right) &= \sum_{j=1}^{N-1}\left(\left(\frac{1}{\gamma_j^2(t)} - \frac{1}{\gamma_{j+1}^2(t)}\right)\sum_{k=1}^{j}F_k\right)\\ &+ \sum_{j=1}^{N-1}\left(\frac{1}{2}\left(\frac{1}{\gamma_j(t)} - \frac{1}{\gamma_{j+1}(t)}\right)\left(1 - \frac{\sigma_0}{50}\left(\frac{1}{\gamma_j(t)} + \frac{1}{\gamma_{j+1}(t)}\right)\right)\sum_{k=1}^{j}M_k\right)\\ &+ \frac{1}{\gamma_N^2(t)}\sum_{k=1}^{N}F_k + \frac{1}{2\gamma_N(t)}\left(1 - \frac{\sigma_0}{50c_N}\right)\sum_{j=1}^{N}M_k.\end{aligned}$$

All the terms in the right hand side are positives, so that we can apply Lemma 5 :

$$\sum_{j=1}^{N}\frac{1}{\gamma_j^2(t)}\left(E_j(t) + \frac{\gamma_j(t)}{2}M_j(t)\right) - \sum_{j=1}^{N}\frac{1}{\gamma_j^2(t)}\left(E_j(S_n) + \frac{\gamma_j(t)}{2}M_j(S_n)\right) \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}.$$

Now we use fact 3. of Lemma 4 at time t and at time S_n (remind that $|\gamma_j(t) - c_j| \leq C\varepsilon_0$, so that $c_N + \varepsilon_0 \geq \gamma_j(t) \geq \sigma_0$) :

$$\sum_{j=1}^{N}\frac{1}{\gamma_j^2(t)}H_j(t)$$

$$\leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C\varepsilon_0 \int \tilde{v_n}^2(t) \sum_{j=1}^N \phi_j(t) + C\varepsilon_0 \int \tilde{v_n}^2(S_n) \sum_{j=1}^N \phi_j(S_n)$$
$$\leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t} + C\varepsilon_0 \|\tilde{v_n}(t)\|^2_{L^2(1-\psi_0(t))} + C\varepsilon_0 \|U(S_n)V\|^2_{L^2(1-\psi_0(S_n))}. \quad (38)$$

By Proposition 4, we have that for $j = 1, \dots, N$,

$$H_j(t) \geq \lambda_1 \int (\tilde{v_n}^2(t) + \tilde{v}_x^2(t))\phi_j(t) - \frac{1}{\lambda_1}\left(\left(\int \tilde{v}(t)\tilde{R}_j^3\right)^2 + \left(\int \tilde{v}(t)\tilde{R}_{j_x}\right)^2\right).$$

So that if we sum up those N inequalities, there exists $\lambda_0 > 0$ neither depending on σ_0 nor ε_0) such that

$$\sum_{j=1}^N \frac{1}{\gamma_j^2(t)} H_j(t)$$
$$\geq \lambda_0 \|\tilde{v_n}(t))\|^2_{H^1(1-\psi_0(t))} - \frac{1}{\lambda_0}\sum_{j=1}^N \left(\left(\int \tilde{v_n}(t)\tilde{R}_j^3(t)\right)^2 + \left(\int \tilde{v_n}(t)\tilde{R}_{j_x}(t)\right)^2\right)$$
$$\geq \lambda_0 \|\tilde{v_n}(t))\|^2_{H^1(1-\psi_0(t))} - \frac{1}{\lambda_0}\sum_{j=1}^N \left(\left(\int U(t)V\tilde{R}_j^3\right)^2 + \left(\int U(t)V\tilde{R}_{j_x}\right)^2\right)$$
$$\geq \lambda_0 \|\tilde{v_n}(t))\|^2_{H^1(1-\psi_0(t))} - \frac{C}{\lambda_0}\|U(t)V\|^2_{L^2(1-\psi_0(t))}. \quad (39)$$

Combining (39) and (38), provided that ε_0 is small enough so that $C_3\varepsilon_0 < \lambda_0/2$, we deduce

$$\frac{1}{C}\|\tilde{v_n}(t))\|^2_{H^1(1-\psi_0(t))} \leq e^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} + \|U(t)V\|^2_{L^2(1-\psi_0(t))} + \|U(S_n)V\|^2_{L^2(1-\psi_0(S_n))}.$$

We will only use the obtained bound on $\|\tilde{v_n}(t))\|_{L^2(1-\psi_0(t))}$. Recall $\tilde{v_n}(t) = \tilde{w_n}(t) + U(t)V$, thus

$$\|\tilde{w_n}(t))\|^2_{L^2(1-\psi_0(t))} \leq 2\|\tilde{v_n}(t))\|^2_{L^2(1-\psi_0(t))} + 2\|U(t)V\|^2_{L^2(1-\psi_0(t))}$$
$$\leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} + C\|U(t)V\|^2_{L^2(1-\psi_0(t))} + C\|U(S_n)V\|^2_{L^2(1-\psi_0(S_n))}. \quad (40)$$

Relying on estimate (40), we only need to go back to $w_n(t) = \tilde{w_n}(t) + R(t) - \tilde{R}(t)$. As we noted in (32),

$$\|w_n(t)\|_{L^2(1-\psi_0(t))} \leq \|R(t) - \tilde{R}(t)\|_{L^2} + \|\tilde{w_n}(t))\|_{L^2(1-\psi_0(t))}$$
$$\leq C\sum_{k=1}^N |y_j(t) - x_j - c_j t| + |\gamma_j(t) - c_j| + Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}$$
$$+ C\|U(t)V\|_{L^2(1-\psi_0(t))} + C\|U(t)V\|_{L^2(1-\psi_0(S_n))}. \quad (41)$$

Now, using the L^2_{loc} estimate of lemma 3, and (40) :

$$|y_j'(t) - c_j| + |\gamma_j'(t)|$$
$$\leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} + C\|U(t)V\|_{L^2(1-\psi_0(t))} + C\left(\int \tilde{w_n}^2(t)e^{-\sqrt{\sigma_0}|x-c_j t|}\right)^{1/2}$$

$$\leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{8}t} + C\|U(t)V\|_{L^2(1-\psi_0(t))} + C\|U(S_n)V\|_{L^2(1-\psi_0(S_n))}.$$

Let us integrate this between t and S_n. Remind the initial conditions $y_j(S_n) = x_j + c_jS_n$, $\gamma_j(S_n) = c_j$, we obtain

$$|y_j(t) - x_j - c_jt| + |\gamma_j(t) - c_j| \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t} + C\int_t^{S_n} \|U(t)V\|_{L^2(1-\psi_0(t))}dt + C(S_n - t)\|U(S_n)V\|_{L^2(1-\psi_0(S_n))}.$$

This, together with (41), concludes the proof of Proposition 2. □

6 Control of the interaction of w_n with $U(t)V$: the linear theory

This section is devoted to the proof of Proposition 3.

However, let us first link Proposition 2, Proposition 3, Lemma 2 and Proposition 1' together. We compute the decay we obtained on $\|w_n(t)\|_{H^1(1-\psi_0(t))}$.

$(1 + x_+)^{2+\delta_0}V \in H^1$, so that from Lemma 2,

$$\|U(t)V\|_{L^2(1-\psi_0(t))} \leq \frac{C}{t^{2+\delta_0}}. \tag{42}$$

From Proposition 2, we can then conclude that

$$\|w_n(t)\|_{L^2(1-\psi_0(t))} \leq \frac{C}{t^{1+\delta_0}}. \tag{43}$$

This ensures that the assumptions of Proposition 3 are fulfilled.

Thus, Proposition 1' follows from Proposition 2, Lemma 2 and Proposition 3.

6.1 Preliminary lemmas

First recall the fundamental linear estimate.

Lemma 6 ([7]). *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $B \in \mathbb{R}$. The following inequalities hold, as long as the right-hand side is bounded :*

$$\left\|\partial_x \int_\tau^B U(t-s)f(s,x)ds\right\|_{L^5_x L^{10}_\tau(\tau\in[t,B])} \leq \|f(\tau,x)\|_{L^1_x L^2_\tau(\tau\in[t,B])}, \tag{44}$$

$$\sup_{\tau\in[t,B]} \left\|\partial_x \int_\tau^B U(t-s)f(s,x)ds\right\|_{L^2_x} \leq \|f(\tau,x)\|_{L^1_x L^2_\tau(\tau\in[t,B])}. \tag{45}$$

Proof. In [7], the proof of the first estimate is done without restriction in time, that is

$$\left\|\partial_x \int_\tau^B U(t-s)f(s)ds\right\|_{L^5_x L^{10}_\tau} \leq \|f\|_{L^1_x L^2_\tau}.$$

Now as $s \in [\tau, B] \subset [t, B]$, we get

$$\left\|\partial_x \int_\tau^B U(t-s)f(s)ds\right\|_{L^5_x L^{10}_\tau(\tau\in[t,B])}$$

$$= \left\| \partial_x \int_\tau^B U(t-s)(f(s)\mathbb{1}_{s\in[\tau,B]})ds \right\|_{L^5_x L^{10}(\tau\in[t,B])}$$
$$\leq \left\| \partial_x \int_\tau^B U(t-s)(f(s)\mathbb{1}_{s\in[\tau,B]})ds \right\|_{L^5_x L^{10}}$$
$$\leq \|f(s)\mathbb{1}_{s\in[\tau,B]}\|_{L^1_x L^2_t} = \|f\|_{L^1_x L^2_\tau(\tau\in[t,B])}.$$

The proof of the second estimate with no restriction on time is done in [6] : the restricted one is done analogously. □

Now, let us prove a lemma which will handle the interference of the solitons when we will control the interaction of w_n with the linear term $U(t)V$.

Lemma 7 (Weak interference of solitons). *Let $A \geq 1$, $B \geq A$, $\delta_0 > 0$, and $f : [A,B]\times\mathbb{R} \to \mathbb{R}$. Suppose that*

$$f \in L^5_x L^{10}_t(t\in[A,B]) \quad \textit{and} \quad \forall t \in [A,B], \quad \|f(t)\|_{L^2(1-\psi_0(t))} \leq \frac{C}{t^{1+\delta_0}}.$$

Then there exists C (independent of A and B) such that

$$\forall t \in [A,B], \quad \|fR\|_{L^1_x L^2_\tau(\tau\in[t,B])} \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|f\|_{L^5_x L^{10}_\tau(\tau\in[t,B])} + \frac{C}{t^{\delta_0}}. \qquad (46)$$

***Remark* 6.** Observe that this result is almost optimal with respect to the decay rate required on $\|f(t)\|_{L^2(1-\psi_0(t))}$. Indeed, suppose that $f(t,x) = \frac{1}{t^\alpha}Q(x-t)$. Then

$$\|f(\tau,x)Q(x-t)\|_{L^1_x L^2_\tau(\tau\in[t,B])} = \int \left(\int_{\tau\in[t,B]} Q^3(x-t)\frac{d\tau}{\tau^{2\alpha}} \right)^{1/2} dx$$
$$\sim \int_{x\in[t,B]} \frac{dx}{x^\alpha} \sim \frac{C}{t^{1-\alpha}}.$$

Thus, in order to have a decay estimate, we have to impose $\alpha > 1$, and we lose one order of decay.

Proof. First notice that it is enough to obtain the result for a single soliton. Indeed, to conclude for the N-soliton case, it suffices to see

$$\|fR\|_{L^1_x L^2_\tau(\tau\in[t,B])} \leq \sum_{j=1}^N \|fR_j\|_{L^1_x L^2_\tau(\tau\in[t,B])}.$$

The idea is to split the double integral into two pieces, depending whether $|x - c_j\tau - x_j| \geq (x-x_j)/2$ or not. Denote

$$A(x) = \left\{ \tau : \tau\in[t,B], |x-c_j\tau - x_j| \geq \frac{|x-x_j|}{2} \right\}, \quad \text{and} \quad B(x) = [t,B]\setminus A(x).$$

Then

$$\|fR_j(s)\|_{L^1_x L^2_\tau(\tau\in[t,B])} = \int_x \left(\int_{\tau\in[t,B]} |fR_j|^2(\tau,x)d\tau \right)^{1/2} dx$$

$$
\begin{aligned}
&= \int_x \left(\int_{\tau \in A(x)} |fR_j|^2(\tau,x) d\tau \right)^{1/2} dx \\
&\quad + \int_x \left(\int_{\tau \in B(x)} |fR_j|^2(\tau,x) d\tau \right)^{1/2} dx \\
&= I + II.
\end{aligned}
$$

We estimate separately I and II.

For I, we are "away" from the soliton, and we use its decay to go from $L^1_x L^2_t$ to $L^5_x L^{10}_t$, with an exponentially small constant. Recall $R_j(t,x) = Q_{c_j}(x - c_j t - x_j)$. Remark that as $Q(y)$ is even and decreasing (to 0) for $y \geq 0$,

$$
\sup_{\tau \in A(x)} |R_j(\tau,x)| = Q_{c_j}\left(\frac{x - x_j}{2} \right).
$$

So that using Hölder's inequality in the τ integral with exponents $\frac{1}{2} = \frac{1}{10} + \frac{2}{5}$, we get

$$
\begin{aligned}
I &\leq \int_x Q_{c_j}^{1/2}\left(\frac{x - x_j}{2} \right) \left(\int_{\tau \in A(x)} |f|^2(\tau,x) Q_{c_j}(x - c_j\tau - x_j) d\tau \right)^{\frac{1}{2}} dx \\
&\leq \int_x Q_{c_j}^{1/2}\left(\frac{x - x_j}{2} \right) \left(\int_{\tau \in A(x)} |f(\tau,x)|^{10} Q_{c_j}(x - c_j\tau - x_j) d\tau \right)^{\frac{1}{10}} \\
&\qquad \times \left(\int_{\tau \in A(x)} Q_{c_j}(x - c_j\tau - x_j) d\tau \right)^{\frac{2}{5}} dx.
\end{aligned}
$$

Now let (τ, x) such that $\tau \in A(x)$. We claim that $|x - c_j\tau - x_j| \geq \frac{c_j}{3}\tau \geq \frac{c_j}{3}t$.

Indeed : first suppose $x - x_j \geq c_j\tau$. As $\tau \geq t \geq A \geq 0$, $x - x_j \geq 0$, and we have $x - c_j\tau - x_j \geq (x - x_i)/2$. Thus $x - x_j \geq 2c_j\tau$, so that $|x - c_j\tau - x_j| = x - c_j\tau - x_j \geq c_j\tau$.

Else if $x - x_j \leq c_j\tau$: if $x - x_j \leq 0$, as $\tau \geq 0$, we get $|x - c_j\tau - x_j| = c_j\tau - (x - x_j) \geq c_j\tau$. Else if $x - x_j \geq 0$, then $c_j\tau - (x - x_j) \geq (x - x_j)/2$, so that $\frac{2}{3}c_j\tau \geq (x - x_j)$. Thus, $c_j\tau - (x - x_j) \geq \frac{1}{3}c_j\tau$. This proves our claim.

And we get (with $y = x - c_j t - x_j$, $dy = c_j d\tau$)

$$
\begin{aligned}
\left(\int_{\tau \in A(x)} Q_{c_j}(x - c_j\tau - x_j) d\tau \right)^{2/5} &\leq \left(\int_{|y| \geq c_j/3t} Q_{c_j}(y) \frac{dy}{c_j} \right)^{2/5} \\
&\leq Ce^{-\frac{2c_j\sqrt{c_j}}{15}t}.
\end{aligned}
$$

Applying again Hölder's inequality (in the x integral) with exponent $1 = \frac{4}{5} + \frac{1}{5}$, we can thus estimate

$$
\begin{aligned}
I &\leq Ce^{-\frac{2c_j\sqrt{c_j}}{15}t} \int_x Q_{c_j}^{1/2}\left(\frac{x - x_j}{2} \right) \\
&\qquad \times \left(\int_{\tau \in A(x)} |f|^{10}(\tau,x) Q_{c_j}(x - c_j\tau - x_j) d\tau \right)^{\frac{1}{10}} dx \\
&\leq Ce^{-\frac{2c_j\sqrt{c_j}}{15}t} \left(\int_x Q_{c_j}^{5/8}\left(\frac{x - x_j}{2} \right) dx \right)^{4/5}
\end{aligned}
$$

$$\times\left(\int_x\left(\int_{\tau\in A(x)}|f|^{10}(\tau,x)Q_{c_j}(x-c_j\tau-x_j)d\tau\right)^{1/2}dx\right)^{1/5}$$
$$\le Ce^{-\frac{2c_j\sqrt{c_j}}{15}t}\|Q_{c_j}^{1/2}\|_{L^{5/4}}\|fR_j^{1/10}\|_{L^5_xL^{10}_\tau(\tau\in[t,B])}$$
$$\le Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|f\|_{L^5_xL^{10}_\tau(\tau\in[t,B])}. \tag{47}$$

For II, we have the full bump of the soliton, but $x-x_j\ge\frac{1}{3}c_j\tau$, so we can use our decay on the right. This decay is in $L^\infty_tL^2_x$, so we have to interchange integrals : we will decompose the x integral in intervals $x\sim 2^j$, so that when applying the Cauchy-Schwarz inequality (to have $L^2_xL^2_t$, and then apply the Fubini-Tonelli Theorem), we don't pay too high a price.

Notice that for (τ,x) such that $\tau\in B(x)$, $|(x-x_j)-c_j\tau|\le|x-x_j|/2$, so that $x-x_j\ge 0$ because $\tau\ge 0$. This implies that we can restrict ourselves to $x\ge x_j$ in the integral in x. Let $L_p=2^p-1+x_j$. Then

$$II=\sum_{p\in\mathbb{N}}\int_{x=L_p}^{L_{p+1}}\left(\int_{\tau\in B(x)}|fR_j|^2(\tau,x)d\tau\right)^{1/2}dx,$$

and by Cauchy-Schwarz inequality,

$$\le\sum_{p\in\mathbb{N}}2^{p/2}\left(\int_{x=L_p}^{L_{p+1}}\int_{\tau\in B(x)}|fR_j|^2(\tau,x)d\tau dx\right)^{1/2}.$$

Now let (x,τ) such that $x\in[L_p,L_{p+1}$ and $\tau\in B(x)$. Then $|(x-x_j)-c_jt|\le|x-x_j|/2$ and $x-x_j\ge 0$ implies that

$$c_j\tau\in[(x-x_j)-(x-x_j)/2,(x-x_j)+(x-x_j)/2]=[(x-x_j)/2,3(x-x_j)/2],$$

so that $\tau\in[(2^p-1)/(2c_j),3(2^{p+1}-1)/(2c_j)]$ and of course $\tau\in[t,B]$. This means that :

$$\{(\tau,x):x\in[L_p,L_{p+1}],\tau\in B(x)\}$$
$$\subset[L_p,L_{p+1}]\times\left(\left[\frac{2^p-1}{2c_j},\frac{3(2^{p+1}-1)}{2c_j}\right]\cap[t,B]\right),$$

which is a rectangle : thus we can interchange integrals.

$$\int_{x=L_p}^{L_{p+1}}\int_{\tau\in B(x)}|fR_j|^2(\tau,x)d\tau dx\le\int_{x=L_p}^{L_{p+1}}\int_{\substack{\tau:\tau\in[t,B],\\ \tau\in\left[\frac{2^p-1}{2c_j},\frac{3(2^{p+1}-1)}{2c_j}\right]}}|fR_j|^2(\tau,x)d\tau dx$$
$$\le\int_{\substack{\tau:\tau\in[t,B],\\ \tau\in\left[\frac{2^p-1}{2c_j},\frac{3(2^{p+1}-1)}{2c_j}\right]}}\int_{x=L_p}^{L_{p+1}}|fR_j|^2(\tau,x)dxd\tau.$$

Define K_1 the maximal index such that $(2^{K_1}-1)/(2c_j)\le t$, and K_2 the maximal index such that $(2^{K_2}-1)/(2c_j)\le B$. We can now use our decay estimate on $\|f(t)\|_{L^2(1-\psi_0(t))}$:

$$II\le\sum_{p=K_1}^{K_2}2^{p/2}\left(\int_{\tau\in\left[\frac{2^p-1}{2c_j},\frac{3(2^{p+1}-1)}{2c_j}\right]}\int_x|fR_j|^2(\tau,x)dxd\tau\right)^{1/2}$$

$$+ 2^{(K_2+1)/2} \int_{\tau:\tau\in\left[\frac{2^{K_2}-1}{2c_j},B\right]} \int_x |fR_j|^2(\tau,x)dxd\tau\Bigg)^{1/2}$$

$$\leq \sum_{p=K_1}^{K_2} 2^{p/2}\Bigg(\int_{\tau=\frac{2^p-1}{2c_j}}^{\frac{3(2^{p+1}-1)}{2c_j}} \|f(\tau)\|^2_{L^2(1-\psi_0(\tau))}d\tau\Bigg)^{1/2}$$

$$+ 2^{(K_2+1)/2} \int_{\tau=\frac{2^{K_2}-1}{2c_j}}^{B} \|f(\tau)\|^2_{L^2(1-\psi_0(\tau))}d\tau\Bigg)^{1/2}$$

$$\leq C\sum_{p=K_1}^{K_2+1} 2^{p/2}\Bigg(\int_{\tau=\frac{2^p-1}{2c_j}}^{\frac{3(2^{p+1}-1)}{2c_j}} \frac{d\tau}{\tau^{2+2\delta_0}}\Bigg)^{1/2}$$

$$\leq C\sum_{p=K_1}^{K_2+1} 2^{p/2}(2^{p-1}-1)^{-1/2-\delta_0}.$$

As $2^{p/2}(2^{p-1}-1)^{-1/2-\delta_0} \leq C2^{-p\delta_0}$ and $(2^{K_1}-1)/(2c_j) \geq t/2$, which means $C2^{K_1} \geq t$, we get

$$II \leq C\sum_{p=K_1}^{K_2+1} 2^{-p\delta_0} \leq C2^{-K_1\delta_0} \leq Ct^{-\delta_0}. \tag{48}$$

Summing up (47) and (48) yields the result (46). □

Lemmas 6 and 7 will be used with $f = w_n + U(t)V$, $A = T_0$ and $B = S_n$.

6.2 Proof of Proposition 3

Proof of Proposition 3. From (9) and Duhamel formula, $w_n(t)$ satisfies the following integral formulation :

$$w_n(S_n) = U(S_n-t)w_n(t) + \partial_x \int_t^{S_n} \Big((w_n(\tau)+U(\tau)V+R(\tau))^5 - \sum_j R_j^5(\tau)\Big)d\tau.$$

Compose by $U(t-S_n)$, as recall that $w_n(S_n)=0$, so that

$$w_n(t) = -\partial_x \int_t^{S_n} U(t-\tau)\Big((w_n(\tau)+U(\tau)V+R(\tau))^5 - \sum_j R_j^5(\tau)\Big)d\tau$$

$$= -\sum_{k=1}^{5} C_5^k \partial_x \int_t^{S_n} U(t-\tau)\Big((w_n(\tau)+U(\tau)V)^k R^{5-k}(\tau)\Big)d\tau$$

$$- \partial_x \int_t^{S_n} U(t-\tau)\Big(R^5(\tau) - \sum_{j=1}^{N} R_j^5(\tau)\Big)ds. \tag{49}$$

We now use the $L_x^5 L_t^{10}$ setting of [7]. According to (49), with estimates (44) and (45), we have

$$\|w_n(\tau,x)\|_{C^0([t,S_n],L_x^2)} + \|w_n(\tau,x)\|_{L_x^5L_\tau^{10}(\tau\in[\tau,S_n])}$$
$$\leq \|(w_n+U(t)V)^5\|_{L_x^1L_\tau^2(\tau\in[t,S_n])}$$

$$+ C\sum_{k=1}^{4} C_5^k \|(w_n(\tau)+U(\tau)V)^k R^{5-k}(\tau)\|_{L^1_x L^2_\tau(\tau\in[t,S_n])}$$

$$+ C\Big\| R^5(\tau) - \sum_{j=1}^{N} R_j^5(\tau)\Big\|_{L^1_x L^2_\tau(\tau\in[t,S_n])}. \tag{50}$$

First consider the last term. Recall the simple inequality

$$|z-a|+|z-b| \geq 2\left|z-\frac{a+b}{2}\right| + \frac{|a-b|}{2}.$$

As $|R_j(t,x)| \leq Ce^{-\frac{\sqrt{\sigma_0}}{2}|x-x_j-c_jt|}$, we get that for $i \neq j$,

$$|R_i(t,x)R_j(t,x)| \leq Ce^{-\sqrt{\sigma_0}\left|x-\frac{x_i+x_j}{2}-\frac{c_i+c_j}{2}t\right|}e^{-\frac{\sqrt{\sigma_0}}{4}|c_i-c_j|t}.$$

As $|c_j - c_j| \geq 2\sigma_0$, we obtain

$$\Big\| R^5(\tau,x) - \sum_{j=1}^{N} R_j^5(\tau,x)\Big\|_{L^1_x L^2_\tau(\tau\geq t)} \leq C(\sigma_0)e^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}. \tag{51}$$

Now consider the purely linear interaction ($k=5$), that is the first term in (50) :

$$\begin{aligned}\|(w_n(\tau)+U(\tau)V)^5\|_{L^1_x L^2_\tau(\tau\in[t,S_n])} &= \|w_n(\tau)+U(\tau)V\|^5_{L^5_x L^{10}_\tau(\tau\in[t,S_n])}\\ &\leq C\|w\|^5_{\mathcal{N}([t,S_n])} + C\|U(\tau)V\|^5_{L^5_x L^{10}_\tau(\tau\geq t)}.\end{aligned} \tag{52}$$

It remains to control in (50) the terms with an interaction between $w_n + U(t)V$ and the solitons.

From (42) and (43), Lemma 7 applies to all the remaining terms in (50) (i.e. $k=1,2,3,4$), to give

$$\begin{aligned}&\|(w_n(\tau)+U(\tau)V)^k R^{5-k}(\tau)\|_{L^1_x L^2_\tau(\tau\in[t,S_n])}\\ &\leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|w_n(\tau)+U(\tau)V\|_{L^5_x L^{10}_\tau(\tau\in[t,S_n])} + \frac{C}{t^{\delta_0}}\\ &\leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|w_n\|_{\mathcal{N}([t,S_n])} + Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|U(\tau)V\|_{L^5_x L^{10}_\tau(\tau\geq t)} + \frac{C}{t^{\delta_0}}.\end{aligned} \tag{53}$$

(recall that $\|w_n(t)\|_{H^1}$ is uniformly bounded, like $\|U(t)V\|_{H^1}$, so that $\|w_n(\tau)+U(\tau)V\|_{L^\infty_{x,\tau}(\tau\in[I_n,S_n])} \leq C$ uniform in n). Summing up (51), (52) and (53), and plugging it in (50), we obtain

$$\begin{aligned}\|w\|_{\mathcal{N}([t,S_n])} &= \|w_n(\tau,x)\|_{C^0([t,S_n],L^2_x)} + \|w_n(\tau,x)\|_{L^5_x L^{10}_\tau(\tau\in[t,S_n])}\\ &\leq C\|w_n\|^5_{\mathcal{N}([t,S_n]} + Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|w_n\|_{\mathcal{N}([t,S_n])}\\ &\quad + C\|U(\tau)V\|^5_{L^5_x L^{10}_\tau(\tau\geq t)} + Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\|U(\tau)V\|_{L^5_x L^{10}_\tau(\tau\geq t)} + \frac{C}{t^{\delta_0}}.\end{aligned} \tag{54}$$

Then for ε_0 small enough so that $C\varepsilon_0^4 \leq 1/3$, and T_0 large enough so that $Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}T_0} \leq 1/3$, we get that

$$\forall t \in [I_n, S_n], \quad \|w_n\|_{\mathcal{N}([t,S_n])} \leq \eta(t), \tag{55}$$

where

$$\eta(t) = C\|U(\tau)V\|^5_{L^5_x L^{10}_\tau(\tau\geq t)} + Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|U(\tau)V\|_{L^5_x L^{10}_\tau(\tau\geq t)} + \frac{C}{t^{\delta_0}}$$

satisfies the conditions of Proposition 1'. □

Appendix

We state a version of the implicit function theorem, to be used in the proof of Lemma 3.

Implicit function Theorem with parameter. Let E, F, G, H be Banach spaces, and $f : E \times F \times G \to H$ a C^1 function. Let U be an open set in E. We suppose there exist C^1 functions $x_0, y_0 : U \to F, G$ such that for all $t \in U$, $f(t, x_0(t), y_0(t)) = 0$, and that $\partial_y f(t, x_0(t), y_0(t))$ is invertible.

Furthermore, we suppose that there exist $\delta_0 > 0$, $\eta_0 > 0$ such that

$$\sup_{\substack{t\in U\\ x\in B(x_0(t),\delta_0)\\ y\in B(y_0(t),\eta_0)}} \|\partial_y^{-1} f(t, x_0(t), y_0(t))\| \|\partial_y f(t, x_0(t), y_0(t)) - \partial_y f(t, x, y)\| = k < 1,$$

and

$$\sup_{\substack{t\in U\\ x\in B(x_0(t),\delta_0)\\ y\in B(y_0(t),\eta_0)}} \|\partial_y^{-1} f(t, x_0(t), y_0(t))\| \|\partial_x f(t, x, y)\| = C < \infty.$$

Then there exist $\delta_1, \eta_1 > 0$ such that the following holds.

Define the tubular neighborhoods $V = \{(t, x) | t \in U,\ x \in B((x_0(t), \delta_1)\}$ and $W = \{(t, y) | t \in U,\ y \in B((y_0(t), \eta_1)\}$.

Then there exists a C^1 function $G : V \to W$, $G(t, x) = (t, g(t, x))$ such that

$$\forall t, \forall x, y \in B(x_0(t), \eta_1) \times B(y_0(t), \eta_1), \qquad f(t, x, y) = 0 \iff y = g(t, x).$$

Furthermore, $g(t, \cdot)$ is $(C + 1)/(1 - k)$-Lipschitz (in particular, we can choose $\eta_1 \leq (C + 1)/(1 - k) \cdot \delta_1$).

Let us conclude with the proof of Proposition 4.

Proof of Proposition 4. See [17, Lemma 4] and [16, Appendix A], for the proof of a very similar result. The main idea is to use localization arguments on a definite positive operator. Indeed, recall that there exists $\lambda > 0$ such that for all $v \in H^1$,

$$\int (v_x^2 - 5Q^4 v^2 + v^2) \geq \lambda \|v\|^2_{H^1} - \frac{1}{\lambda}\left(\left(\int vQ^3\right)^2 + \left(\int vQ_x\right)^2\right). \tag{56}$$

By translation, and denoting

$$\phi(t,x) = \psi(x - (m_{j-1}t - \gamma_j(t))) - \psi(x - (m_j t - \gamma_j(t))),$$

we want to prove that for all $v \in H^1$,

$$\int (v_x^2 - 5Q^4v^2 + v^2)\phi(t) \geq \lambda_1 \int (v_x^2 + v^2)\phi(t) - \frac{1}{\lambda_1}\left(\left(\int vQ^3\right)^2 + \left(\int vQ_x\right)^2\right).$$

For simplicity, let us denote

$$H(v,w) = \int (v_x w_x - 5Q^4 vw + vw), \quad H_\phi(v,w) = \int (v_x w_x - 5Q^4 vw + vw)\phi(t).$$

We begin by the following perturbation lemma concerning (56).

Lemma 8. *There exists $\delta > 0$ (depending only on $\lambda > 0$) such that if $|(v|Q^3)| + (v|Q_x)| \leq \delta \|v\|_{H^1}$, then*

$$H(v,v) = \int v_x^2 - 5Q^4v^2 + v^2 \geq \frac{\lambda}{2}\int (v_x^2 + v^2).$$

Proof. See [23], where an analoguous case is treated. Let us decompose in L^2 : $v = v_1 + aQ^3 + bQ_x = v_1 + v_2$, $(v_1|Q^3) = (v_1|Q_x) = 0$, so that by hypothesis,

$$|a| + |b| \leq \delta \|v\|_{H^1}.$$

If $\delta < 1/2$, we deduce that

$$\frac{\sqrt{3}}{2}\|v\|_{H^1} \leq \|v_1\|_{H^1} \leq 2\|v\|_{H^1}.$$

Now $H(v,v) = H(v_1,v_1) + H(v_2,v_2) + 2H(v_1,v_2)$. By (56),

$$H(v_1,v_1) \geq \lambda \|v_1\|_{H^1}^2 \geq \frac{3\lambda}{4}\|v\|_{H^1}^2.$$

If $C\delta^2 \leq \lambda/8$ (where C is the continuity norm of H),

$$|H(v_2,v_2)| \leq C(|a|^2 + |b|^2) \leq C\delta^2 \|v\|_{H^1}^2 \leq \frac{\lambda}{8}\|v\|_{H^1}^2,$$

and if $4C\delta \leq \lambda/8$,

$$2|H(v_1,v_2)| \leq C\|v_1\|_{H^1}\|v_2\|_{H^1} \leq 4C(|a|+|b|)\|v\|_{H^1} \leq 2C\delta\|v\|_{H^1}^2 \leq \frac{\lambda}{8}\|v\|_{H^1}^2.$$

Finally :

$$H(v,v) \geq \left(\frac{3}{4} - \frac{1}{8} - \frac{1}{8}\right)\lambda\|v\|_{H^1}^2 \geq \frac{\lambda}{2}\|v\|_{H^1}^2. \qquad \square$$

Lemma 9. *There exists $\sigma_1 > 0$ such that the following is true. Given $\sigma_0 < \sigma_1$, there exists $T_3 = T_3(\sigma)$, such that if $(v, Q^3) = (v, Q_x) = 0$, then*

$$H_\phi(v,v) \geq \frac{\lambda}{4}\int (v_x^2 + v^2)(\phi).$$

Proof. Notice that $|(\phi)_x| \leq C\sqrt{\sigma_0}\phi$, the constant C not depending on t, x, σ_0 (it is a computation involving ψ). Let $v \in H^1$ such that $(v|Q^3) = (v|Q_x) = 0$. Now let us compute with $v\sqrt{\phi}$:

$$\begin{aligned} H_\phi(v,v) &= \int (v_x^2 - 5Q^4v^2 + v^2)\phi \\ &= \int (v\sqrt{\phi})_x^2 - 5Q^4(v\sqrt{\phi})^2 + (v\sqrt{\phi})^2) - \int vv_x\phi_x - \int v^2(\sqrt{\phi})_x^2 \\ &= H(v\sqrt{\phi}, v\sqrt{\phi}) - \int vv_x\phi_x - \int v^2(\sqrt{\phi})_x^2. \end{aligned}$$

Thanks to the orthogonality properties on v,

$$\begin{aligned} \left| \int v\sqrt{\phi}Q^3 \right| &= \left| \int v(1-\sqrt{\phi})Q^3 \right| \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|v\|_{L^2} \quad \text{and,} \\ \left| \int v\sqrt{\phi}Q_x \right| &= \left| \int v(1-\sqrt{\phi})Q_x \right| \leq Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{4}t}\|v\|_{L^2}. \end{aligned}$$

Thus, as soon as $t \geq T_3 = T_3(\sigma_0)$ is large enough, we are in the setting of the previous lemma, and so

$$H(v\sqrt{\phi}) \geq \frac{\lambda}{2}\|v\sqrt{\phi}\|_{H^1}^2 = \frac{\lambda}{2}\left(\int (v_x^2 + v^2)\phi + \int vv_x\phi_x + \int v^2(\sqrt{\phi})_x^2 \right).$$

Thus :

$$H_\phi(v,v) \geq \frac{\lambda}{2}\int (v_x^2+v^2)\phi + \left(\frac{\lambda}{2}-1\right)\int vv_x\phi_x + \left(\frac{\lambda}{2}-1\right)\int v^2(\sqrt{\phi})_x^2.$$

We need to control the last two terms. First,

$$\left| \int vv_x\phi_x \right| \leq C\sqrt{\sigma_0}\int |vv_x|\phi \leq C\sqrt{\sigma_0}\int (v_x^2+v^2)\phi,$$

and in the same way, as $|(\sqrt{\phi})_x| \leq C\sqrt{\sigma_0}\sqrt{\phi}$,

$$\int v^2(\sqrt{\phi})_x^2 \leq C\sigma_0 \int v^2\phi.$$

Choose $\sigma_1 \leq \frac{1}{8C}\frac{\lambda_1}{1-\lambda_1/2}$, then for $t \geq T(\sigma_0)$ large enough, we get

$$H_\phi(v,v) \geq \frac{\lambda}{4}\int (v_x^2+v^2)(\phi),$$

as claimed. □

We can now conclude the proof of Proposition 4. Let $v \in H^1$. Let us write the L^2 decomposition $v = v_1 + aQ + bQ_x$, and develop :

$$\begin{aligned} H_\phi(v,v) = H_\phi(v_1,v_1) + 2aH_\phi(v_1,Q) + 2bH_\phi(v_1,Q_x) \\ + a^2H_\phi(Q,Q) + 2abH_\phi(Q,Q_x) + b^2H_\phi(Q_x,Q_x). \end{aligned}$$

By hypothesis, $H_\phi(v_1,v_1)\geq \lambda_2\int(v_x^2+v^2)\phi(t)$. Then, notice that $|a|=|(v|Q)|\leq C\|v\|_{L^2}$ and $|b|\leq\|v\|_{L^2})$. Thus

$$\|v_1\|_{H^1}\leq C\|v\|_{H^1}.$$

Now $H(v_1,Q)=(v_1,\lambda_Q Q)=0$ as Q (and Q_x) are eigenfunctions, and H_ϕ is symmetric : we deduce $H_\phi(v_1,aQ)=H_{1-\phi}(v_1,aQ)$ etc. And we get, by continuity of H,

$$|H_\phi(v_1,aQ)|+|H_\phi(bQ_x,aQ)|\leq C\|v\|_{H^1}^2\|(|Q_x|+Q)(1-\phi(t))\|_{L^\infty}.$$

And of course $\|(|Q_x|+Q)(1-\phi(t))\|_{L^\infty}\leq Ce^{-\sigma_0\sqrt{\sigma_0}t}$. As $|\phi|\leq 1$, $|H_\phi(Q,Q)|\leq \|Q\|_{H^1}^2$ and $|H_\phi(Q_x,Q_x)|\leq\|Q_x\|_{H^1}^2$. This gives

$$H_\phi(v,v)\geq\lambda_2\int({v_1}_x^2+v_1^2)\phi(t)-Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\|v\|_{H^1}^2-C(a^2+b^2).$$

Now,

$$\int({v_1}_x^2+v_1^2)\phi(t)\geq\int(v_x^2+v^2)\phi(t)-C(a^2+b^2)$$
$$-2a\int(v_xQ_x+vQ)\phi(t)-2b\int(v_xQ_{xx}+vQ_x)\phi(t).$$

But

$$\left|a\int(v_xQ_x+vQ)\phi(t)\right|\leq Ca\left(\int(v_x^2+v^2)\phi(t)\right)^{1/2}\leq C^2a^2+\frac{1}{4}\int(v_x^2+v^2)\phi(t).$$

Doing the same for $\int(v_xQ_{xx}+vQ_x)\phi(t)$, we get

$$\int({v_1}_x^2+v_1^2)\phi(t)\geq\frac{1}{2}\int(v_x^2+v^2)\phi(t)-C(a^2+b^2),$$

so that finally,

$$H_\phi(v,v)\geq\lambda_2/2\int(v_x^2+v^2)\phi(t)-Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}t}\|v\|_{H^1}^2-C(a^2+b^2).$$

Choosing T_1 so that $Ce^{-\frac{\sigma_0\sqrt{\sigma_0}}{2}T_1}\leq\lambda_2/4$, as $t\geq T_1$, this gives :

$$H_\phi(v,v)\geq\lambda_2/2\int(v_x^2+v^2)\phi(t)-C(a^2+b^2).$$ □

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