# Instability of non-constant harmonic maps for the $1+2$-dimensional equivariant wave map system

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**Abstract**

In this paper we study (1+2)-dimensional equivariant wave maps to a surface $N$. This is, after reduction, solutions $u : \mathbb{R}_t \times \mathbb{R}_r \to \mathbb{R}$ to the following initial value problem :

$$\begin{cases} u_{tt} - u_{rr} - \frac{1}{r}u_r &= -\frac{f(u)}{r^2}, \\ (u, u_t)|_{t=0} &= (u_0, u_1). \end{cases}$$

($f$ depends on $N$). We consider the existence of a finite energy harmonic map $Q$ (a stationary solution), and show that when it exists, $Q$ is instable in the energy space.

Our result applies in particular to the case of wave maps to the sphere $\mathbb{S}^2$, and to the critical Yang-Mills equations in dimension 4.

## 1 Introduction

### 1.1 Recall of known results

Let us introduce a function $g \in C^1$, such that $g(0) = 0$, and $f = g \cdot g'$. We consider the following initial value problem on function $u : \mathbb{R}_t \times \mathbb{R}_r^+ \to \mathbb{R}$ :

$$\begin{cases} u_{tt} - \Delta u &= -\frac{f(u)}{r^2}, \\ (u, u_t)|_{t=0} &= (u_0, u_1). \end{cases} \tag{1}$$

(We denote $\Delta u = u_{rr} + u_r/r = 1/r(ru_r)_r$ the radial Laplacian in $\mathbb{R}^2$). This problem has the following geometric interpretation. Let $N$ be a surface of revolution with polar coordinates $(\rho, \theta) \in [0, \infty) \times \mathbb{S}^1$. Let $ds^2$ be the metric on $N$ :

$$ds^2 = d\rho^2 + g^2(\rho) d\theta^2. \tag{2}$$

A wave map is a function $U : \mathbb{R}^{1+2} \to N$ satisfying the system :

$$\begin{cases} \Box U^\alpha + \Gamma^\alpha_{\beta\gamma}(U) \partial_a U^\beta \partial^a U^\gamma = 0, \\ (U, U_t)|_{t=0} = (U_0, U_1). \end{cases} \tag{3}$$

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($\Gamma^{\alpha}_{\beta\gamma}$ are the Christoffel symbols for $(N, ds^2)$, $\alpha = \rho$ or $\theta$). Denote $(r, \phi)$ the usual polar coordinates on $\mathbb{R}^2$. We are concerned with the corotationnal equivariant case, that is, we impose that $U$ has the form

$$\rho = u(t, r), \quad \theta = \phi.$$

The wave map system (3) then simplifies to a single nonlinear scalar equation for $u : \mathbb{R}_t \times \mathbb{R}_r^+ \to \mathbb{R}$, which is (1) (see the book by Shatah and Struwe [4] for further details). Of course, any result on $u$ has a reformulation for $U$.

At least formally, one has conservation of one quantity, that is energy :

$$E(u) = \int \left( u_t^2 + u_r^2 + \frac{g^2(u)}{r^2} \right) rdr = E(u_0, u_1). \tag{4}$$

(In fact $E(u) = \|U_t\|_{L^2}^2 + \|\nabla U\|_{L^2}^2$). Let us introduce :

$$H = \left\{ (u, v) \middle| \; \|(u, v)\|_H^2 \stackrel{\text{def}}{=} E(u, v) = \int \left( v^2 + u_r^2 + \frac{g(u)^2}{r^2} \right) rdr < \infty \right\}. \tag{5}$$

$H$ appears as an energy space, in which it is natural to study the solutions to (1). In [6], Shatah and Tahvildar-Zadeh proved local in time existence and uniqueness of solutions to (1) arising from initial data in the energy space :

**Local existence in $H$ [6, Theorem 1.1].** *Let $(u_0, u_1) \in H$. Then there exist $T > 0$ and a unique solution $u$ to Problem (1) such that :*

$$(u, u_t) \in L^\infty([0, T), H), \qquad u \in L^q([0, T), \dot{B}^{1/2}_{10/3, 10/3}(\mathbb{R}^+, rdr)).$$

Let us define the usual notation :

$$E(u, a, b) \stackrel{\text{def}}{=} \int_a^b \left( u_t^2 + u_r^2 + \frac{g^2(u)}{r^2} \right) rdr.$$

One way to express the finite speed of propagation is the fact that the energy is decreasing on light cones :

$$\forall R \geq 0,\ \forall |\tau| \leq R, \quad E(u(t), 0, R - |\tau|) \leq E(u(t + \tau), 0, R). \tag{6}$$

One should also notice that Problem (1) has a natural scaling :

$$u(t, r) \text{ is a wave map} \quad \Longleftrightarrow \quad u_\lambda(t, r) \stackrel{\text{def}}{=} u(\lambda t, \lambda r) \text{ is a wave map.}$$

A straightforward computation gives :

$$E(u_\lambda) = \int \left( \lambda^2 u_t^2 + \lambda^2 u_r^2 + \lambda^2 \frac{g^2(u)}{(\lambda r)^2} \right) rdr = E(u).$$

The energy remains unaffected by scaling : Problem (1) is thus said to be (scaling-)critical for the energy.

The main remaining open question for this problem is global well-posedness.

The result by Shatah and Tavildar-Zadeh [6] solves in particular the case of small energy data : there exists a constant $\varepsilon_0 > 0$ such that if $E(u) = E(u_0, u_1) < \varepsilon_0$, then $u$ is global in time. Another direct consequence of the proof of Theorem 1.1 of [6] is the following condition for blow-up : $u$ blows-up at time $T$ only if

$$\liminf_{t \uparrow T} E(u(t), 0, T-t) \geq \varepsilon_0. \tag{7}$$

Remind that due to radial symmetry, concentration of energy can only happen at point $r = 0$.

Under some assumptions on $N$, the energy can not concentrate, and this is enough to ensure global existence in time : in [5], Shatah and Tavildar-Zadeh proved non-concentration under the assumption $g' \geq 0$ (geodesical convexity). This condition was later laxed by Grillakis to $g(\rho) + g'(\rho)\rho > 0$, and finally by Struwe in [7], to $g > 0$ for $\rho > 0$. On the other hand, in [1], Bizoń and al. gave strong evidence of blow-up for system (1), in the case $N = \mathbb{S}^2$, and $g = \sin$.

In [7], Struwe proved further that if $u$ does blow up (say at time $t = 0$), then for a subsequence $t_n$, there exists a scaling parameter $\lambda(t_n)$ such that $\lambda(t_n)|t_n| \to \infty$ and :

$$u(t_n + t/\lambda(t_n), r/\lambda(t_n)) \to Q(r) \quad \text{in} \quad H^1_{\mathrm{loc}}(\mathbb{R}^{1+2}), \tag{8}$$

where $Q$ is a non-constant harmonic map, i.e. a stationary solution to (1) :

$$\Delta Q = \frac{f(Q)}{r^2}. \tag{9}$$

This proves in particular that if there is no harmonic map, then there is no blow up : this is the case when $g > 0$ for $\rho > 0$. Furthermore, [7, Theorem 1] has two corollaries :

1. Let $Q$ be a non-constant harmonic map with least energy. Suppose $E(u) \leq E(Q)$ : then $u$ is global in time.
2. Blow-up (at $T$) is characterized by $\liminf_{t\uparrow T} E(u(t), 0, T-t) \geq E(Q)$. In particular, if the initial data is such that $E((u_0, u_1), 0, R) \leq E(Q)$, then the corresponding wave map $u(t)$ is defined at least up to time $t = R$ (remind that the energy is decreasing on cones).

These results can be reformulated by saying that one can choose $\varepsilon_0 = E(Q)$.

## 1.2 Statement of the results

For a single function $u$, not depending on time, we shall often note $\|u\|_H$ instead of $\|(u,0)\|_H$, as well as $E(u,a,b)$ for $E((u,0),a,b)$.

From now on, $Q$ will denote a non-constant harmonic map, i.e. $Q$ satisfies (9). $Q$ is important for the qualitative study of (1). Indeed, notice that our first criterion (7) for blow-up states the impossibility to apply the local well-posedness result. Due to [7], it is in fact equivalent to the formation of a singularity with the well-defined blow-up profile $Q$, which is a descriptive result.

As we shall see, the existence of $Q$ is equivalent to the vanishing of $g$ at some point. The problematic is now : if $Q$ exists, does blow-up occur ? In particular, for initial data in a neighborhood of $Q$, does one has blow-up ?

Our goal in this paper is to prove the instability of $Q$ in the energy space. This result should be related to previous works on instability of stationary states for other equations. In particular, we refer to [3] for the critical non-linear Schrödinger equation :

$$\begin{cases} \mathrm{i}u_t + \Delta u + |u|^{4/N} u = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^N, \end{cases}$$

and [2] for the critical generalized Korteweg-de Vries equation :

$$\begin{cases} u_t + (u_{xx} + u^5)_x = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}. \end{cases}$$

In this context, proving instability for $Q$ should be viewed as a first step toward understanding the blow-up mechanism.

More precisely we can build wave maps, which are in an arbitrary neighborhood of $Q$ at time 0, and which change profile to a certain $Q(\lambda\cdot)$, $\lambda \neq 1$ (in fact $\lambda > 1$) :

**Theorem 1.** – *[Geometric instability of $Q$ in $H$] Suppose $g$ vanishes at some point $C^* > 0$ (isolated zero). Then there exists a non-constant finite energy harmonic map $Q$ for Problem (1). Moreover, for any $\lambda_0 > 1$, there exist a sequence of initial data $(u_n^0, u_n^1)$ such that*

$$\|(u_n^0, u_n^1) - (Q,0)\|_H \to 0 \quad \text{as} \quad n \to \infty, \tag{10}$$

*and if we denote the arising wave maps $u_n$ (solutions to (1)), $u_n$ is defined at least up to some time $t_n$ such that*

$$\|(u_n, u_{nt})(t_n) - (Q(\lambda_0\cdot),0)\|_H \to 0 \quad \text{as} \quad n \to \infty. \tag{11}$$

We shall call this change of profile (from $Q$ to $Q(\lambda_0\cdot)$) *geometric instability*. Notice, as proven in Corollary 4, that one must have $t_n \to \infty$. As a direct consequence, we have :

**Corollary 1.** *Let $Q$ be a finite energy harmonic map for Problem (1). Then $Q$ is unstable in $H$. For some constant $C(\lambda_0) > 0$ independent of $n$ :*

$$\sup_{t\in[0,t_n]} \|(u_n, u_{nt}) - (Q,0)\|_H \geq C(\lambda_0).$$

As Problem (1) has no symmetry besides scaling, this result also proves orbital instability.

Of course, as mentioned earlier, this result can be rewritten as geometrical instability of harmonic maps for the wave map system (3).

Theorem 1 deals in particular with the following two special cases.

**Corollary 2** (Wave maps to the 2-sphere)**.** *It corresponds to the case $g = \sin$ : $C^* = \pi$. The equation takes the form (cf. [1] for more details) :*

$$u_{tt} - \Delta u = -\frac{\sin 2u}{2r^2}.$$

*And the harmonic solution is explicit : $Q(r) = 2\arctan(r)$. Then $Q$ is geometrically unstable.*

$Q$ can be seen as the minimal (in the sense of energy) connection between the north and south poles. Theorem 1 is here in agreement with the numerical investigation of [1].

**Corollary 3** (Critical Yang-Mills equation in dimension 4)**.** *It corresponds to $g(\rho) = (1-\rho^2)$ : $C^* = 1$, which gives the equation :*

$$u_{tt} - \Delta u = \frac{2u(1-u^2)}{r^2}.$$

*Then the harmonic map $Q(r) = \tanh(\ln r) = \frac{r^2-1}{r^2+1}$ is geometrically unstable.*

Here, one should think of a slight modification of our setting, as $\tilde{u} = u+1$ is the wave map - with $g$ replaced by $\tilde{g}(\rho) = g(\rho-1) = \rho(2-\rho)$ : the main difference will be that $\rho \in [-1, \infty)$.

The proof is organized as follows : in Section 2.1 we characterize harmonic maps and their properties. In Section 2.2, we study special, infinite energy wave maps, which are in fact related to harmonic maps : the self similar solutions. Then, in Section 2.3, we regularize these solutions to obtain wave maps with initial data in a neighborhood of $Q$. Finally in Section 2.4, using finite speed of propagation, we exhibit a family of initial data satisfying Theorem 1.

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# 2 Proofs

## 2.1 On stationary solutions to (1)

Recall that the existence of a non constant harmonic map implies that $g$ vanishes at some point (besides 0) : it is a consequence of [7]. In the following, we suppose that there exists a least positive real number $C^*$ such that $g(C^*) = 0$. Without loss of generality, we can suppose that $g(\rho) > 0$ for $\rho \in (0, C^*)$. Finally, as to avoid degeneracy of the energy, we assume that $g$ has only isolated zeros. We shall denote $G(\rho) = \int_0^\rho |g(\rho')| d\rho'$ : $G$ is increasing.

**Proposition 1.** *For any $g$ such that $g(C^*) = 0$, there exists a non constant finite energy harmonic map $Q$, i.e. a solution to (9). Furthermore :*

1. *Regularity for $Q$. Any harmonic map $Q$ is of class $C^2$ and satisfies one of the equations :*

$$rQ_r = g(Q), \quad \text{or} \quad rQ_r = -g(Q).$$

   *As a consequence, $Q$ is monotone, and joins 2 consecutive zeros of $g$.*

2. *Variational characterization of $Q$. Suppose that $Q$ joins 0 to $C^*$. Then $Q$ is of minimal energy for this property : for a function $v$, such that $v(0) = 0$, and $v(r) \to C^*$ as $r \to \infty$, then*

$$E(v) \leq E(Q) \Longrightarrow v(r) = Q(\lambda r) \text{ for some } \lambda > 0.$$

We will use many times the following simple inequality :

**Lemma 1** (Pointwise inequality)**.** *For a finite energy function $v$ :*

$$E(v, \alpha, \beta) = \int_\alpha^\beta \left( v_r^2 + \frac{g^2(v)}{r^2} \right) r dr \geq 2 \int_\alpha^\beta |g|(v) v_r dr \geq 2|G(v(\beta)) - G(v(\alpha))|. \quad (12)$$

*Proof.* First, let us prove the existence of a harmonic map. Let $Q$ be a maximal (in the sense of Cauchy-Lipschitz) solution to

$$rQ_r = g(Q), \quad Q(1) = C^*/2. \quad (13)$$

Suppose $Q$ is defined on $(a, b)$. Then :

$$\int_a^b \left( Q_r^2 + \frac{g^2(Q)}{r^2} \right) r dr = 2 \int_a^b Q_r g(Q) dr = 2(G(Q(b)) - G(Q(a))).$$

Now, as $u = 0$ and $u = C^*$ are solutions, by uniqueness, we always have $Q(r) \in (0, C^*)$ for $r > 0$. This proves both that $Q$ defined for $r \in [0, \infty)$ and that $Q$ is of finite energy $2G(C^*)$. If we differentiate (13), we obtain :

$$\Delta Q = \frac{1}{r}(rQ_r)_r = \frac{f(Q)}{r^2}.$$

As $Q_r(1) \neq 0$, $Q$ is not constant : it is the desired harmonic map.

Let us now prove the properties for any harmonic map. We denote $r = e^x$, so that $r\partial/\partial r = \partial/\partial x$, and thus equation 9 writes :

$$Q_{xx} = f(Q). \tag{14}$$

The Theorem of Cauchy-Lipschitz allows us to solve (14) ; this ensures that $Q$ is $C^2$ where it is defined. Multiply (14) by $Q_x$ and integrate between $a$ and $b$ :

$$[Q_x^2]_a^b = [g^2(Q(x))]_a^b. \tag{15}$$

On another side, for any finite energy function $v(r)$, in view of (12), $G(v)$ satisfies the Cauchy criterion as $\alpha, \beta \to 0$, and as $\alpha, \beta \to \infty$. Thus $G(v)$ admits limits $G(v)(0)$ and $G(v)(\infty)$, at $r = 0$ and $r \to \infty$ respectively. As $G$ is increasing and continuous, it is a homeomorphism, and in fact $v$ admits limits in 0 and $\infty$, $v(0)$ and $v(\infty)$. Of course, as $v$ is of finite energy, we must have :

$$g(v(0)) = g(v(\infty)) = 0.$$

Therefore, $Q$ admits limits at $r = 0$ and at $\infty$ (or in the $x$ variable, at $\pm\infty$), that are zeros of $g$. In (15), fix $b$, and let $a \to -\infty$. The left hand side has a limit, so that $Q_x$ has a limit $\ell$ at $-\infty$. If $\ell \neq 0$, then $Q$ has no limit at $-\infty$, a contradiction. Hence $Q_x \to 0$, and finally we get

$$Q_x^2(b) = g^2(Q(b)).$$

We already know that $Q$ joins two zeros of $g$, say $\alpha$ and $\beta$ : if they are not consecutive, then for a certain $c$, $g(Q(c)) = 0$, so $Q_x(c) = 0$. By uniqueness, $Q = Q(c) =$ constant : a contradiction, so $a$ and $b$ are two consecutive zeros. The same argument shows that for all $x \in \mathbb{R}$, $Q(x) \in (a, b)$, and that $g(Q)$ has a constant sign : as a consequence, $Q_x$ does not vanish, hence $Q$ is monotone. We then deduce (for sign reasons) that either :

$$\forall x, \quad Q_x = g(Q) \quad \text{ or } \quad \forall x, \quad Q_x = -g(Q).$$

Finally, let us prove the the minimizing property. Up to rescaling of $v$, we can suppose that $v(1) = Q(1)$. Now, using (12) :

$$E(Q, 0, 1) = 2|G(Q(1)) - G(Q(0))| = 2|G(v(1)) - G(v(0)| \leq E(v, 0, 1).$$

In the same way, $E(Q,1,\infty) \leq E(v,1,\infty)$, and we obtain $E(v) \geq E(Q)$. Thus $E(v) = E(Q)$ (note that this already shows that $Q$ is minimizing). More precisely, using 12, we have for all $r$ :

$$\begin{aligned} E(v) &= E(v,0,r) + E(v,r,\infty) \\ &\geq (|G(v(r)) - G(0)| + |G(C^*) - G(v(r))|) \geq E(Q) = E(v). \end{aligned}$$

So there is in fact equality everywhere. As a consequence, $v$ is non-decreasing (so as not to lose any energy), and takes its values in $(0, C^*)$ for $r > 0$. The energy equality also gives :

$$v_r^2 = g^2(v)/r^2.$$

As before, using the fact that $v$ is non-decreasing, we get : $v_r = g(v)/r$. As $v(1) = Q(1)$, $v = Q$. □

From now on, $Q$ will denote a harmonic map joining 0 to $C^*$. As $g > 0$ on $(0, C^*)$ :

$$rQ_r = g(Q). \tag{16}$$

**Proposition 2** (Decomposition)**.** *There exists $\alpha_0 > 0$ and an increasing function $\delta : [0, \alpha_0] \to \mathbb{R}^+$, with $\delta(\alpha) \to 0$ as $\alpha \to 0$, such that the following is true.*

*Suppose $v$ is a function of finite energy, with $v(0) = 0$ and $v(r) \to C^*$ as $r \to \infty$ and such that :*

$$E(v) = E(Q) + \alpha < E(Q) + \alpha_0.$$

*Then there exist $\lambda \in \mathbb{R}_*^+$, $\epsilon \in H$, such that :*

$$v(r) = Q(\lambda r) + \epsilon(r), \quad \|\epsilon\|_H \leq \delta(\alpha).$$

*Proof.* Follows from the variational characterization of $Q$ : cf. Appendix A. □

Let $u$ be a wave map with energy lower than $E(Q) + \alpha_0$. Such a decomposition exists for all $u(t)$ : this gives two functions $\lambda(t)$ and $\epsilon(t,r)$ (the proof of Proposition 2 shows that they are continuous functions of $t$) such that

$$u(t,r) = Q(\lambda(t)r) + \epsilon(t,r).$$

Now blow up occurs only when there is concentration of at least $E(Q)$ energy. As $\epsilon(t,r)$ has a small energy, it cannot lead to blow up : only the $Q(\lambda(t)r)$ can do it, which it equivalent to $\lambda(t) \to \infty$. On the other side, if $\lambda(t) \to \infty$, then the initial blow-up criterion is fulfilled, and the result of [7] applies.

Finally $u$ blows up at time $T$ if and only if $\lambda(t) \to \infty$ as $t \uparrow T$. In this setting, Theorem 1 gives the existence of a wave map whose scaling parameter $\lambda(t)$ goes from $\lambda(0) = 1$ to $\lambda(t_n) = \lambda_0 > 0$ : this can be seen as a first step toward existence of blow up.

Another consequence of Proposition 2 is the following :

**Corollary 4.** *Let $T > 0$ and $\varepsilon > 0$. Then there exists $\eta > 0$ such that for all initial data $(u_0, u_1)$ satisfying $\|(u_0, u_1) - (Q, 0)\|_H \le \eta$, the arising wave map $u$ is defined at least up to time $T$ and :*

$$\sup_{t \in [0,T]} \|(u, u_t) - (Q, 0)\|_H \le \varepsilon.$$

In particular, this shows that one can not expect instability of $Q$ for bounded times, and in Theorem 1, $t_n \to \infty$. The proof is postponed to Appendix A.

## 2.2 Self-similar solutions on light cones

The proof of Theorem 1 relies on the study of self-similar solutions. A self similar solution is a solution $u$ to (1), defined for $t < 0$, with the ansatz :

$$u(t, r) = v\left(\frac{r}{|t|}\right).$$

It of course blows up at time $t = 0$. Let us first exhibit self similar wave maps.

**Corollary 5.** *Define, for $\xi < 1$ :*

$$P_\alpha(\xi) = Q\left(\frac{2\alpha\xi}{1 + \sqrt{1 - \xi^2}}\right).$$

*Then $P_\alpha$ generates a self-similar solution (on the light cone $r < |t|$).*

*Proof.* It is mainly computations with change of variables. We plug the ansatz $u(t, r) = v(-r/t)$ (and $-r/t = \xi$) in (1), and write the equation for $v(\xi)$ :

$$u_t = \frac{r}{t^2} v'(\xi), \quad u_{tt} = \frac{r^2}{t^4} v''(\xi) - \frac{2r}{t^3} v'(\xi), \quad u_r = -\frac{1}{t} v'(\xi), \quad u_{rr} = \frac{1}{t^2} v''(\xi).$$

So $v$ satisfies :

$$\left(\frac{r^2}{t^4} - \frac{1}{t^2}\right) v''(\xi) + \left(-\frac{2r}{t^3} + \frac{1}{rt}\right) v'(\xi) = -\frac{f(v(\xi))}{r^2}.$$

Now we multiply by $t^2$, replace $-r/t$ by $\xi$, and again multiply by $\xi^2$ : we get the equation for $v$ in the $\xi$ variable :

$$\xi^2(\xi^2 - 1)\frac{d^2 v}{d\xi^2} + \left(2\xi^3 - \xi\right) \frac{dv}{d\xi} = -f(v). \tag{17}$$

We would like to make the first order term vanish. For this, let us now do the change of variable $\chi = \ln\left(\xi/(1+\sqrt{1-\xi^2})\right)$ i.e. $\xi = 1/\cosh\chi$ (diffeomorphism for $\xi \in (0,1)$, that is $r < |t|$), then :

$$\frac{dv}{d\xi} = \frac{1}{\xi\sqrt{1-\xi^2}}\frac{dv}{d\chi}, \quad \frac{d^2v}{d\xi^2} = \frac{2\xi^2-1}{\xi^2(1-\xi^2)^{3/2}}\frac{dv}{d\chi} + \frac{1}{\xi^2(1-\xi^2)}\frac{d^2v}{d\chi^2}.$$

We plug this last relation in (17), the equation simplifies to

$$\frac{d^2v}{d\chi^2} = f(v).$$

But this is simply equation (14), whose solutions are

$$Q(2\alpha\exp\chi) = Q\left(\frac{2\alpha\xi}{1+\sqrt{1-\xi^2}}\right),$$

according to Proposition 1 (recall $x = \exp r$). □

This computation motivates the definition, for $b > 0$ :

$$S(b;r) \stackrel{\text{def}}{=} P_{1/b}(br) = Q\left(\frac{2r}{1+\sqrt{1-b^2r^2}}\right) \quad \text{for } r \leq \frac{1}{b}. \tag{18}$$

Indeed if we define

$$\mathcal{S}(b;t,r) \stackrel{\text{def}}{=} P_{1/b}\left(\frac{r}{|t|}\right) = S\left(b;\frac{r}{b|t|}\right), \tag{19}$$

$\mathcal{S}(b;t,r)$ satisfies system (1) with initial data (at time $t = -1/b$) :

$$\left\{\begin{array}{l} \mathcal{S}(b;-\frac{1}{b},r) = S(b;r), \\ \mathcal{S}_t(b;-\frac{1}{b},r) = brS_r(b;r). \end{array}\right. \tag{20}$$

$\mathcal{S}(b;t,r)$ is defined in the interior of the cone $\{(t,r)|r \leq b|t|\}$ ; it blows up at time $t = 0$, and its life-span is $1/b$. Observe that from the proof of Corollary 5, it follows that $S(b;r)$ satisfies the equation :

$$(b^2r^2-1)\Delta S(b;r) + b^2rS_r(b;r) = -\frac{f(S(b;r))}{r^2}.$$

One important thing to notice is the following :

*Claim.*

$$\left\{\begin{array}{ll} \forall A \geq 0, & \lim_{b\to 0}\|(S(b;r), brS_r(b;r)) - (Q,0)\|_{H([0,A])} \to 0, \\ \text{but } \forall b > 0, & \|S(b,r)\|_H = +\infty. \end{array}\right.$$

This is due to the specific form of $P_\alpha$. If the convergence were in $H(\mathbb{R}^+)$, we would obtain a family of blowing up wave maps, whose initial data would converge to $Q$ (in $H$). But these blowing up wave maps are always of infinite energy. Indeed, if we compute at point $\xi = 1$ (c.f. (45)) :

$$S_r\left(b; \frac{1}{b} - \varepsilon\right) \sim Q'\left(\frac{1}{b}\right) \cdot \sqrt{\frac{b}{\varepsilon}},$$

(as $Q'(1/b) \neq 0$), and $S_r^2(b) \sim C\frac{b}{\varepsilon}$ gives a logarithmic divergence when integrating in $rdr$, and thus an infinite energy. However, if we were to forget this infinite energy tail, $\mathcal{S}(b;t,r) = S(b; r/(b|t|)$ has the "profile" $Q(\lambda(t)r)$ with $\lambda(t) = 1/(b|t|)$. We recover the fact that $\lambda(t) \to \infty$ as $t \uparrow 0$, that is, blow-up. Furthermore $1/|t|$ is the self-similar blow-up rate. Notice that this rate can never be the blow-up rate of a finite energy wave map (cf. (8)).

In higher dimension, self-similar solutions are some examples of blowing up wave maps with smooth initial conditions (see [4, ch. 7]). In dimension 2, no blowing up wave map of finite energy is known.

The next step is to regularize $S(b;r)$ to obtain initial conditions in $H$.

## 2.3 Construction of approximation of self-similar profiles in the energy space

For simplicity in writing expressions in throughout this section, let us first introduce the following notations.

Let $c_0 < 1$. *Notation.* We define :

- $C(c_0)$, a constant that may change form line to line, but which does only depend on $c_0 < 1$ (as $c_0 \uparrow 1$, $C(c_0$ may tend to $+\infty$).
- the interval $I(b,r) = [r, 2r/(1+\sqrt{1-b^2r^2})]$,
- the functions $h(r) = \sup_{\rho\in[Q(r),C^*]} |g(\rho)| = \sup_{\theta\in[r,\infty)} |g(Q)(\theta)|$, and $h_2(r) = \frac{1}{r^2}\int_0^r h^2(s)sds$.

Observe that if $g$ is decreasing in a neighborhood of $C^*$, (which is always the case if $g'(C^*) < 0$), $h(r) = g(Q)(r)$ in this neighborhood. In any case, $h$ decreases to 0 as $r \to \infty$. In particular, $h_2(r) \to 0$ as $r \to \infty$. This gives the existence of a constant $A \geq 10$ such that for $\alpha \geq A$ :

$$h(\alpha) \leq 0.01, \quad \text{and} \quad h_2(\alpha) \leq 0.01.$$

This section is devoted to the proof of the following proposition :

**Proposition 3.** *Let $\alpha, b > 0$ such that $b\alpha \leq c_0 < 1$, and $\alpha \geq A$. There exist $C^2$ functions $S^0(\alpha,b;r)$ and $S^1(\alpha,b;r)$, defined for $r \in \mathbb{R}^+$, and satisfying :*

$$\begin{cases} S^0(\alpha,b;r) = S(b;r) & \text{if} \quad 0 \leq r \leq \alpha, \\ S^0(\alpha,b;r) = Q(r) & \text{if} \quad r \geq \alpha(1+h(\alpha)), \end{cases} \tag{21}$$

$$\begin{cases} S^1(\alpha,b;r) = brS_r(b;r) & \text{if} \quad 0 \leq r \leq \alpha, \\ S^1(\alpha,b;r) = 0 & \text{if} \quad r \geq \alpha(1+h(\alpha)). \end{cases} \tag{22}$$

*With the following estimate :*

$$\|(S^0(\alpha,b;r),S^1(\alpha,b;r))-(Q(r),0)\|_H^2 \leq C(c_0)(b^2\alpha^2h_2(\alpha)+h(\alpha)). \quad (23)$$

*Remark.*

1. These modified profiles are simply truncated self-similar wave maps (at point $\alpha$), that were smoothly reconnected to $Q$ (at point $\alpha(1+h(\alpha))$).
2. As we shall see through the proof, the contribution before truncation is in $h_2(\alpha)$, and the contribution for joining is in $h(\alpha)$, which is often worse (one should think as $h_2 \sim h^2$). So the global contribution is in $h(\alpha)$.

*Proof.* The proof goes as follows. As the values of $(S^0(\alpha,b;r),S^1(\alpha,b;r))$ are given on $[0,\alpha]$, we only have to compute the desired estimate on this interval : this is done in lemma 2 (pointwise estimate) and 3 ($H$ estimate). On $[\alpha,\alpha(1+h(\alpha))]$, we need both the construction of a smooth reconnection, and the estimates that goes along with it : this is lemma 4. The part $[\alpha(1+h(\alpha)),\infty)$ does not add any contribution. The proofs are mainly computational, and are postponed to Appendix B.

**Lemma 2.** *If $br \leq c_0$, then :*

$$|S(b;r)-Q(r)| \leq C(c_0)b^2r^2h^2(r), \quad (24)$$
$$|S_r(b;r)-Q_r(r)| \leq C(c_0)b^2rh(r). \quad (25)$$

**Lemma 3.** *For $b\alpha \leq c_0$ and $\alpha \geq A$, then :*

$$\|S(b;r)-Q(r)\|_{H([0,\alpha])}^2 \leq C(c_0)b^4\alpha^4h_2(\alpha), \quad (26)$$
$$\|brS_r(b;r)\|_{L^2([0,\alpha])}^2 \leq C(c_0)b^2\alpha^2h_2(\alpha), \quad (27)$$
$$\|Q\|_{H([\alpha;\infty))} \leq 2|C^*-Q(\alpha)|h(\alpha). \quad (28)$$

**Lemma 4** (Joining lemma)**.** *Let $v: I=[0,a]\cup[b,\infty)\to\mathbb{R}$. Then there exists $\tilde{v}:\mathbb{R}_+\to\mathbb{R}$ extending $v$, as smooth as $v$, such that :*

$$\|\tilde{v}\|_H^2 \leq \|v\|_{H(I)}^2+(v(b)-v(a))^2\frac{b+a}{b-a}+2\max(v(a),v(b))^2\ln\frac{b}{a}. \quad (29)$$

To construct the $S^i(\alpha,b;r)$, we only have choice on the interval $r\in[\alpha,\alpha(1+h(\alpha))]$. For $S^0(\alpha,b;r)$ we use the joining Lemma 4 to obtain a smooth reconnection. For $S^1(\alpha,b;r)$, the density of smooth functions in $L^2$ allows us to have a smooth reconnection such that :

$$\|S^1(\alpha,b;r)\|_{L^2([\alpha,\alpha(1+h(\alpha))])}^2 \leq h(\alpha).$$

Let us now turn to estimate (23). First, by estimate (27) we have :

$$\|S^1(\alpha,b;r)\|_{L^2}^2 \le C(c_0)b^2\alpha^2 h_2(\alpha) + h(\alpha). \tag{30}$$

Let us now focus on $S^0(\alpha,b;r) - Q(r)$. For $r \ge \alpha(1+h(\alpha)$, there is no contribution. On $[0,\alpha]$, we use lemma 3 : it gives the bound

$$C(c_0)b^4\alpha^4 h_2(\alpha) \le C(c_0)b^2\alpha^2 h_2(\alpha). \tag{31}$$

On $[\alpha,\alpha(1+h(\alpha))]$, in order to use estimate (29), we compute (with Rolle's theorem, using $h(\alpha)\le 0.01)$) :

$$\begin{aligned}
|S(b;\alpha) - Q(\alpha(1+h(\alpha))| &= \left| Q\left(\frac{2\alpha}{1+\sqrt{1-b^2\alpha^2}}\right) - Q\left(\alpha(1+h(\alpha))\right)\right| \\
&\le \alpha \left| \frac{2}{1+\sqrt{1-b^2\alpha^2}} - (1+h(\alpha))\right| \sup_{\vartheta\ge\alpha} |Q_r(\vartheta)| \\
&\le \alpha(1+h(\alpha))h(\alpha)/\alpha \le h(\alpha).
\end{aligned}$$

So the second term on the right hand side of (29) can be estimated by ($\alpha \ge 10$):

$$4h(\alpha)^2 \frac{\alpha(2+h(\alpha))}{\alpha h(\alpha)} \le 4h(\alpha).$$

And for the third term of (29), it remains to notice that $S(b;\alpha)^2, Q(\alpha(1+h(\alpha))^2 \le C^{*2}$ and the well known : $\ln(1+h(\alpha)) \le h(\alpha)$. Let's also allow $h(\alpha)$ for the time-derivative estimate on this interval (using the density of regular function in $L^2$). So the contribution of $\|S^0(\alpha,b;r) - Q(r)\|_H$ on $[\alpha,\alpha(1+h(\alpha))]$ is bounded by :

$$(2C^{*2}+4)h(\alpha). \tag{32}$$

Summing up the contributions (31) and (32), we get :

$$\|S^0(\alpha,b;r) - Q(r)\|_H \le C(c_0)b^2\alpha^2 h_2(\alpha) + Ch(\alpha). \tag{33}$$

(33) and (30) give the estimate (23). □

## 2.4 Finite speed of propagation and conclusion

Before proving Theorem 1, let us remind a consequence of the finite speed of propagation for system (1).

**Proposition 4.** *Let $(u_0,u_1)$ and $(v_0,v_1)$ be a couple of initial data (at time $t=0$) with finite energy on the interval $[0,R)$. Let $u,v$ be the respective solutions to (1) arising from them. Suppose that :*

$$\forall r \in [0,R), \quad (u_0,u_1)(r) = (v_0,v_1)(r).$$

*Suppose that $u$ is defined up to time $T>0$, and let $T_0 = \min\{R,T\}$. Then $v$ does not blow up before time $T_0$, and for $t\in[0,T_0]$ we have :*

$$\forall r \in [0,R-|t|), u(t,r) = v(t,r).$$

*Remark.* Of course, if $(u_0, u_1)$ and $(v_0, v_1)$ coincide on $(a, b)$, with $a > 0$, then at time $t \in [(a-b)/2, (b-a)/2]$, $u(t)$ and $v(t)$ coincide on the interval $r \in (a + |t|, b - |t|)$.

*Proof.* It relies on the proof of Theorem 1.1 of [6] : more precisely, this Theorem is a direct consequence of the following claim, which is what is proved indeed in [6]

*Claim :* There exists $\varepsilon_0 > 0$ such that the following is true. Let $(u_0, u_1)$, initial data at time $t_0$, have energy less than $\varepsilon_0$ on $B(R_0, C) = (R_0 - C, R_0 + C)$. Then exists a unique solution $u$ to (1) on the full cone of dependence $K(R_0, C) = \{(t, r) || r - R_0| < C - |t - t_0|\}$.

Proposition 4 is also a direct consequence of this claim. Let $T_1 \geq 0$ be the greatest time for which $u$ and $v$ coincide on the truncated cone

$$\{(t, r) | t \in [0, T_1] \text{ and } 0 \leq r < R - t\}.$$

First, $T_1 > 0$ : indeed, we divide $[0, R]$ into finitely many overlapping intervals $(a_n, b_n)$ such that on every interval, the energy of $(u_0, u_1)$ is less than $\varepsilon_0$. Thanks to the claim, on every cone of dependence associated to $(a_n, b_n)$, $v$ (exist and) coincides with $u$. As the $(a_n, b_n)$ overlap and are in finite number, there exists $\delta > 0$ such that $u$ and $v$ coincide on the truncated cone

$$\{(t, r) | t \in [0, \delta] \text{ and } 0 \leq r < R - t\}.$$

Thus, $T_1 \geq \delta > 0$. Now, if $T_1 < T_0$ : $u(T_1) = v(T_1)$, $u_t(T_1) = v_t(T_1)$ on $[0, R - T_1)$, we can repeat the same argument at time $T_1$ to obtain a greater time for the truncated cone on which $u$ and $v$ coincide, and this contradicts the maximality of $T_1$. Hence $T_1 = T_0$. $\square$

*Proof of Theorem 1.* The idea is the following. We know the evolution of $\mathcal{S}(b)$ : given initial data $S^i(\alpha, b; r)$, we deduce from Proposition 4 what happens at time $t$ on the space interval $r \in [0, \alpha - t]$.

The problem is now to choose $\alpha = \alpha(b)$ large enough so that our control takes place for large enough times (so that the scaling parameter $\lambda$ changes), but not too large so that the initial data is close to $(Q, 0)$ in $H$ : there we need the estimates of Proposition 3.

Choose a fixed $c_0 < 1$, and set :

$$\lambda_0 = \frac{1}{1 - c_0^2} > 1 \quad \text{ and for } b > 0, \quad \alpha = \alpha(b) = \frac{c_0}{b}.$$

Define $R(b,t;r)$ as the wave map arising from the regularized initial conditions with $\alpha(b) = 2c_0/b$ :

$$\begin{cases} R(b;0,r) = S^0(c_0/b,b;r), \\ R_t(b,0,r) = S^1(c_0/b,b;r). \end{cases} \tag{34}$$

First, $(R(b;0,r), R_t(b;0,r))$ coincide with $(\mathcal{S}(b,-1/b,r), \mathcal{S}_t(b,-1/b,r))$ on the interval $[0,c_0/b]$, and is of finite total energy. Thus, Proposition 4 ensures that $R(b;t,r)$ is defined at least up to time $T = c_0/b$ (as $\mathcal{S}(b;t,r)$ is defined on an interval of length $1/b \geq 2c_0/b$), and :

$$\forall t \in \left[0;\frac{c_0}{b}\right], \forall r \leq \frac{c_0}{b} - t, \quad R(b;t,r) = \mathcal{S}\left(b;-\frac{1}{b}+t,r\right) = S\left(b;\frac{r}{1-bt}\right).$$

In particular, for $t(b) = c_0(1-c_0)/b$ :

$$\forall r \leq \frac{c_0^2}{b}, \quad R(b;t(b),r) = S\left(b;\lambda_0 r\right). \tag{35}$$

Now, we use the estimate (23) (for $t = 0$) and we obtain :

$$\|(R(b;0,r), R_t(b;0,r) - (Q(r),0)\|_H^2 \leq C(h(c_0/b) + h_2(c_0/b)) \to 0 \text{ as } b \to 0.$$

This is the first condition (10). It remains to estimate what happens at time $t(b) = c_0(1-c_0)/b$, i.e. to bound :

$$\|(R(b;t(b),r), R_t(b;t(b),r)) - (Q(\lambda_0 r),0)\|_H.$$

Consider separately the contributions on $[0,c_0^2/b]$ and on $[c_0^2/b,\infty)$. On the first interval, estimates of Lemma 3 give the bound

$$\|(R(b;t(b),r), R_t(b;t(b),r) - (Q(\lambda_0 r),0)\|_{H([0,c_0^2/b])}^2 \leq Ch_2(c_0^2/b). \tag{36}$$

On the second interval, we work out separately

$$\|(R(b;t(b),r), R_t(b;t(b),r)\|_{H([c_0^2/b,\infty))} \quad \text{and} \quad \|Q(\lambda_0 r)\|_{H([c_0^2/b,\infty))}. \tag{37}$$

Let us focus on the first term of (37). In view of the initial conditions (34),

$$E(R(b)) \leq E(Q) + C(h_2(c_0/b) + h(c_0/b)).$$

On the other side, we use our control on the interval $[0,c_0/b]$ :

$$\begin{aligned} \|(S(b;\lambda_0 r), b\lambda_0 r S(b;\lambda_0 r))\|_{H([0,c_0^2/b])} &\geq \|Q\|_{H([0,c_0^2/b])} \\ &\geq E(Q) - 2(C^* - Q(c_0^2/b))h(c_0^2/b). \end{aligned}$$

(First inequality because $S(b;\lambda_0 r) \geq Q(r)$ and the pointwise inequality, second inequality due to (28)). Hence we have (as $h$ is decreasing) :

$$\begin{aligned}\|(R(b;t(b),r), R_t(b;t(b),r))\|_{H([c_0^2/b,\infty))}& \\ &= E(R(b)) - \|(S(b;\lambda_0 r), b\lambda_0 r S(b;\lambda_0 r))\|_{H[0,c_0^2/b]} \\ &\leq E(Q) + C(h(c_0^2/b) + h_2(c_0^2/b)) - (E(Q) - Ch(c_0^2/b)) \\ &\leq C(h(c_0^2/b) + h_2(c_0^2/b)).\end{aligned} \tag{38}$$

For the second term of (37), using (28), we obtain :

$$\|Q(\lambda_0 r)\|_{H([c_0^2/b,\infty))} \leq 2(C^* - Q(\lambda_0 c_0^2/b))h(c_0^2/b) = o(h(c_0^2/b)). \tag{39}$$

Finally adding up (36), (38) and (39), we have :

$$\begin{aligned}\|(R(b;t(b),r), R_t(b;t(b),r)) - (Q(\lambda_0 r),0)\|_H& \\ &\leq C(h(c_0^2/b) + h_2(c_0/b) + h_2(c_0^2/b)) \to 0 \quad \text{as} \quad b \to 0.\end{aligned}$$

This exactly the second condition (11). To conclude, choose a sequence $(b_n)$ decreasing to 0, and define the sequence :

$$u_n(t,r) = R(b_n;t,r), \quad t_n = (1-c_0)c_0/b_n.$$

The previous computations show that the initial data $(u_n(0), u_{nt}(t)) \to (Q,0)$ in $H$ and that $(u_n(t_n), u_{nt}(t_n)) \to (Q(\lambda_0 \cdot), 0)$ in $H$ as $n \to \infty$. This is true for any $c_0 < 1$, and thus for any $\lambda_0 = 1/(1-c_0^2) > 1$. The instability in $H$ is then straightforward. □

*Remark.*

1. As $\|(R(b;0,r), R_t(b,0,r) - (Q,0)\|_H \to 0$, $E(R(b)) \to E(Q)$ : for $b$ small enough, $R(b)$ admits a decomposition and a scaling factor $\lambda_b(t)$ for all $t$ up to an possible blow-up time. Theorem 1 simply says that $\lambda_b(c_0/b) \to \lambda_0$.

2. The proof of Theorem 1 also gives the time to leave a neighborhood of $Q$. Suppose we start in a $\delta$-neighborhood of $Q$ : then the corresponding $b$ is such that $h(c_0/b) + h_2(c_0/b) \sim \delta$. We have to wait time $c_0/2b$ to leave the neighborhood, that is approximately $h^{-1}(\delta)$. Let us give an example.

   In the case of Corollary 2 ($g = \sin$), studied in [1], we have : $C^* = \pi$, and $g'(\pi) = -1$, so $h(r) \sim 1/r$, $h_2(r) \sim \ln r/r^2$, so $\|\tilde{Q}_b - Q\|_H \sim 1/\sqrt{b}$. Thus the time to exit a $\delta$-neighborhood is $O(1/\delta^2)$.

   In the case of Corollary 3 ($g(\rho) = (1-\rho^2)$, we have : $C^* = 1$, $g'(1) = -2$, so $h(r) \sim 1/r^2$, $h_2(r) \sim 1/r^2$, so $\|\tilde{Q}_b - Q\|_H \sim 1/b$, and the time to exit a $\delta$-neighborhood is $O(1/\delta)$.

## Appendix A : Decomposition of a wave map with energy close to $E(Q)$

**Proposition 2** (Decomposition). *There exist $\alpha_0 > 0$ and an increasing function $\delta : [0, \alpha_0] \to \mathbb{R}^+$, with $\delta(\alpha) \to 0$ as $\alpha \to 0$, such that the following is true.*

*Suppose $v$ is a function of finite energy, with $v(0) = 0$ and $v(r) \to C^*$ as $r \to \infty$ and such that :*

$$E(v) = E(Q) + \alpha < E(Q) + \alpha_0$$

*Then there exist $\lambda \in \mathbb{R}^+_*$ and $\epsilon \in H$ such that :*

$$v(r) = Q(\lambda r) + \epsilon(r), \qquad \text{and} \qquad \|\epsilon\|_H \leq \delta(\alpha).$$

*Proof.* Recall that $G(s) = \int_0^s |g|$, $E(Q) = 2G(C^*)$, and that $G$ is increasing. Denote $D^*$ the unique point such that $G(Q(D^*)) = G(C^*)/2$ (or equivalently $E(Q, 0, D^*) = E(Q)/2$.

We proceed by contradiction. Suppose that there exist $\delta_0 > 0$ and a sequence of finite energy functions $v_n$, such that

$$E(v_n) \leq E(Q) + \frac{1}{n}, \quad v_n(0) = 0, \quad v_n(r) \to C^* \quad \text{as} \quad r \to \infty,$$

and :

$$\forall n \in \mathbb{N}, \ \ \forall \lambda > 0, \quad \|v_n - Q(\lambda \cdot)\|_H \geq \delta_0. \tag{40}$$

Set $w_n(r) = v_n(r/\lambda_n)$, where $\lambda_n$ is such that $w_n(D^*) = Q(D^*)$ (this is possible because $v_n$ is continuous). Using scaling invariance, we have :

$$E(w_n) = E(v_n) \leq E(Q) + \frac{1}{n}.$$

$(w_{nr})$ is $L^2(rdr)$-bounded, and so, on $(0, \infty)$, $(w_n)$ is locally $C^{1/2}$-continuous, and so locally equicontinuous. Furthermore :

$$|G(w_n(a)) - G(w_n(b))| \leq \int_a^b |g(w_n(\rho) w_{nr}(\rho)| d\rho \leq E(w_n, a, b).$$

Apply with $a \to 0$, $b = r$, and then $a = r$, $b \to \infty$. As $g$ is not uniformly 0 outside $[0, C^*]$, we deduce that for $N$ large enough, the $(w_k)_{k \geq N}$ are uniformly bounded in $C^0([0, \infty))$ by a constant $K$. So one can apply Ascoli's theorem : for any compact set $X$ of $(0, \infty)$, $(w_n|_X)_n$ has a compact closure in $C^0(X)$. Let us construct a diagonal extraction. Let $m \in \mathbb{N}$, and suppose we already constructed an extraction $\phi_1 \circ \cdots \circ \phi_m : \mathbb{N} \to \mathbb{N}$ to obtain a converging subsequence

$$w_{\phi_1 \circ \cdots \circ \phi_m(n)}|_{[1/m, m]} \to w \text{ in } C^0([1/m, m]) \text{ as } n \to \infty.$$

We can then construct $\phi_{m+1}$ so that the convergence of $w_{\phi_1\circ\cdots\circ\phi_m\circ\phi_{n+1}(n)}$ takes place in $C^0([1/(m+1), m+1])$ (by applying Ascoli's theorem on $[1/(m+1), m+1]$). Now, define :

$$\varphi(n) = \phi_1 \circ \cdots \circ \phi_n(n).$$

For any $m$, and $n \geq m$, $w_{\varphi(n)}$ is a subsequence of $w_{\phi_1\circ\cdots\circ\phi_m(n)}$, and hence converges in the space $C^0([1/m, m])$. Let us denote again $w \in C^0(R^+_*)$ the common limit. For convenience, we can consider the subsequence as the initial sequence, and thus drop the $\varphi$. We obtained :

$$\forall m \in \mathbb{N}, \quad w_n \to w \;\; C^0([1/m, m]).$$

We can also suppose that $w_{nx} \rightharpoonup w_x$ weakly in $L^2(rdr)$. In particular, $w$ is continuous, $w(D^*) = Q(D^*)$. By weak limit :

$$\int w_r^2 rdr \leq \liminf {w_n}_r^2 rdr.$$

And by Fatou lemma (of course, there is a.e. convergence) :

$$\int g^2(w)dr/r \leq \liminf \int g^2(w_n)dr/r.$$

So that $E(w) \leq E(Q)$. Moreover $w(D^*) = Q(D^*)$. As $w$ is of finite energy, $w$ admits limits in $r = 0$ and $r \to \infty$, where $g$ vanishes : using the pointwise inequality between $0$, $D^*$ and $\infty$, we deduce that the only possibility for these limits are $0$ and $C^*$. Let us now prove that :

$$\forall r \geq D^*, \;\; w(r) \geq Q(D^*), \quad \text{and} \quad \forall r \leq D^*, \;\; w(r) \leq Q(D^*). \tag{41}$$

We prove only one of these inequalities, the second one can be deduced in the same way. We again proceed by contradiction. Let $r < D^*$ such that $w(r) > Q(D^*)$ : denote $\varepsilon = G(w(r)) - G(C^*)/2 > 0$. Due to uniform convergence on compact sets, there exists $N$ such that for $n \geq N$, $G(w_n(r)) - G(C^*)/2 \geq \varepsilon$. Then :

$$\begin{aligned} E(Q) + \frac{1}{n} \geq E(w_n) &\geq E(w_n, 0, r) + E(w_n, r, D^*) + E(w_n, D^*, \infty) \\ &\geq 2\big[G(w_n(r)) + |G(w_n(r)) - G(C^*)/2| + G(C^*)/2\big] \\ &\geq 2(G(C^*) + 2\varepsilon) \geq E(Q) + 4\varepsilon. \end{aligned}$$

This is impossible if $n \geq 1/(4\varepsilon)$, and proves (41). Thus, we conclude that $w(0) = 0$ and $w \to C^*$ as $r \to \infty$. Together with $E(w) \leq E(Q)$ and $w(D^*) = Q(D^*)$, the variational characterization of $Q$ allows to conclude that $w = Q$.

Let us now prove that $\|w_n - Q\|_H = E(w_n - Q) \to 0$. First, $E(w_n) \to E(Q)$, so that $\lim \|w_{nr}\|_{L^2(rdr)} = \|Q_r\|_{L^2(rdr)}$, and the weak convergence $w_{nr} \rightharpoonup Q_r$ is in fact strong-$L^2(rdr)$ :

$$\|w_{nr} - Q_r\|_{L^2(rdr)} \to 0. \tag{42}$$

Let us now consider $\int g^2(w_n - Q)dr/r$. Let $\varepsilon > 0$. Define $c, d > 0$ such that $E(Q,0,c) \leq \varepsilon/16$ and $E(Q,d,\infty) \leq \varepsilon/16$. The convergence of $w_n$ to $Q$ in $C^0([c,d])$ gives the existence of $N \geq 4/\varepsilon$ such that

$$\forall n \geq N, \quad \int_c^d g^2(w_n - Q)\frac{dr}{r} \leq \varepsilon/2.$$

(because $g$ is continuous at 0). Again by convergence in $C^0([c,d])$, we can choose $N$ such that :

$$\forall n \geq N, \quad |G(w_n(c)) - G(Q(c)| + |G(w_n(d)) - G(Q(d))| \leq \varepsilon/16.$$

Hence (pointwise inequality), $E(w_n, c, d) \geq E(Q, c, d) - \varepsilon/8 \geq E(Q) - \varepsilon/4$. In view of $N \geq 4/\varepsilon$, we obtain

$$E(w_n, 0, c) + E(w_n, d, \infty) \leq \varepsilon/2.$$

So that for $n \geq N$ :

$$\int g^2(w_n - Q)\frac{dr}{r} \leq \int_0^c + \int_c^d + \int_d^\infty \leq E(w_n, 0, c) + \varepsilon/2 + E(w_n, d, \infty) \leq \varepsilon.$$

This together with (42) proves that $\|w_n - Q\|_H = \|v_n - Q(\lambda_n \cdot)\|_H \to 0$ as $n \to \infty$ : a contradiction with (40). □

**Corollary 4.** *Let $T > 0$ and $\varepsilon > 0$. Then there exists $\eta > 0$ such that for all initial data $(u_0, u_1)$ satisfying $\|(u_0, u_1) - (Q, 0)\|_H \leq \eta$, the arising wave map $u$ is defined at least up to time $T$ and :*

$$\sup_{t \in [0,T]} \|(u, u_t) - (Q, 0)\|_H \leq \varepsilon.$$

*Proof.* We will choose $\eta > 0$ small enough later.

First, let us notice that we can use the decomposition. Observe indeed that the energy $E(u_0, u_1) \leq E(Q) + C\eta^2 \leq E(Q) + \alpha_0$, if $\eta$ is so small that $C\eta^2 \leq \alpha_0$. Hence, $u_0$ has limits at $r = 0$ and as $r \to \infty$, that are zeros for $g$. As 0 and $C^*$ are isolated zeros of $g$, we necessarily have $u_0(0) = 0$, $u_0(r) \to C^*$ as $r \to \infty$, if we choose $\eta > 0$ small enough.

The local existence Theorem gives a maximal time $T^* > 0$ of existence : we can consider $u(t)$ the arising wave map. By conservation of energy $E(u) \leq E(Q) + C\eta^2$, we get :

$$\int u_t^2 r dr \leq C\eta^2. \tag{43}$$

In particular, as $u(t,0)$ and $\lim_{r\to\infty} u(t,r)$ are always zeros for $g$, we obtain that for all $t < T^*$, $u(t,0) = u_0(0) = 0$ and $\lim_r u(t,r) = \lim_r u_0(r) = C^*$ (see [7, Lemma 1]). Therefore, we can apply Proposition 2 : for all $t < T^*$, there exists $\lambda(t)$, $\epsilon(t,r)$ such that

$$u(t,r) = Q(\lambda(t)r) + \epsilon(t,r), \quad \text{and} \quad \|\epsilon(t)\|_H \leq \delta(C\eta^2).$$

We can choose $\lambda(0) = 1$. Now, as noticed earlier, blow-up for $u$ is characterized by $\lambda(t) \to \infty$. From now on, we will consider $T' \leq T^*$ maximal such that :

$$\forall t < T', \quad \lambda(t) \leq 2 + \varepsilon.$$

Then (if $\delta(C\eta^2)$ is small enough) :

$$\|u(t) - Q\|_H \leq C|\lambda(t) - 1|. \tag{44}$$

Now fix $a \in (0, 1/2)$, such that for $r \in [1-a, 1+a]$, $Q_r(r) \geq Q_r(1)/2$. Using (43), we have :

$$\int_0^t \int_{1-a}^{1+a} u_t dr dt \leq \int_0^t \sqrt{\int u_t^2 r dr} \sqrt{\int_{1-a}^{1+a} \frac{dr}{r}} dt \leq C\eta t.$$

On the other side :

$$\begin{aligned}
\int_0^t \int_{1-a}^{1+a} u_t dr dt &= \int_{1-a}^{1+a} (u(t,r) - u(0,r)) dr \\
&= \int_{1-a}^{1+a} \left(Q(\lambda(t)r) - Q(r) + \epsilon(t,r) - \epsilon(0,r)\right) dr.
\end{aligned}$$

Now, as $\|\epsilon(t)\|_{L^\infty} \leq C(\|\epsilon(t)\|_H) \leq c(\eta)$ (for some function $c$ such that $c \to 0$ at 0) :

$$\int_{1-a}^{1+a} (\epsilon(t,r) - \epsilon(0,r)) dr \leq 2c(\eta).$$

And :

$$\left| \int_{1-a}^{1+a} Q(\lambda(t)r) - Q(r) dr \right| \geq \frac{1}{2} |\lambda(t) - 1| \int_{1-a}^{1+a} Q_r(1) r dr \geq C|\lambda(t) - 1|.$$

Combining our two expressions for the integral, we deduce :

$$|\lambda(t) - 1| \leq C(\eta t + c(\eta)).$$

And finally, using (44) and $\|u_t\|_L^2 \leq \eta$ :

$$\|(u(t), u_t(t) - (Q, 0)\|_H \leq C(\eta t + \eta + c(\eta)).$$

It is enough to choose $\eta \leq \alpha_0/C$ so that $C(\eta T + \eta + c(\eta)) < \varepsilon$. Indeed, the previous computations are then valid up to $T'$. Now if $T' < T$, we have that for all $t \in [0, T']$, $\lambda(t) \leq 1 + \varepsilon < 2 + \varepsilon$, therefore the solution can be continued (blow-up hasn't occurred) and the continuity of $\lambda(t)$ contradicts the maximality of $T'$. □

*Remark.* The proof gives a more accurate result : the time to leave a $\eta$ neighborhood is at least $O(1/\eta)$. This is coherent with the computations for the sequence $u_n$ of Theorem 1.

## Appendix B

Here, we prove the computational lemmas needed for Proposition 3.

*Proof of Lemma 2.* Let us note $\vartheta = \frac{2r}{1+\sqrt{1-b^2r^2}}$. We compute explicitly :

$$S_b(b;r) = Q_r(\vartheta)\frac{2br^3}{\left(1+\sqrt{1-b^2r^2}\right)^2\sqrt{1-b^2r^2}}.$$

So if we plug in $br \leq 0.01$ and $Q_r = g(Q)/r$ :

$$|S_b(b;r)| \leq C(c_0)Q_r(\vartheta)br^3 \leq C(c_0)g(Q)(\vartheta)br^2.$$

Now

$$|S(b;r) - Q(r)| \leq \int_0^b |S_b(b;r)|db \leq C(c_0)\sup_{\theta\in I(r,b)}|(g(Q))(\theta)|b^2r^2.$$

For the second estimate, we again compute explicitly :

$$S_r(b;r) = Q_r(\vartheta)\left(\frac{2}{1+\sqrt{1-b^2r^2}} + \frac{2r}{(1+\sqrt{1-b^2r^2})^2}\frac{b^2r}{\sqrt{1-b^2r^2}}\right). \quad (45)$$

Now, we can compute :

$$\begin{aligned} S_{r,b}(b;r) = Q_{rr}(\vartheta)&\frac{4r^3b}{\left(1+\sqrt{1-b^2r^2}\right)^3(1-b^2r^2)} \\ &- Q_r(\vartheta)\frac{2br^2\left(2b^2r^2-3-3\sqrt{1-b^2r^2}\right)}{(1-b^2r^2)^{3/2}\left(1+\sqrt{1-b^2r^2}\right)^3}. \quad (46) \end{aligned}$$

Recall $Q_r = g(Q)/r$ and $Q_{rr} = ((g'-1)g)(Q)/r^2$. We plug in again $br \leq c_0 < 1$, and we get :

$$\begin{aligned} |S_{r,b}(b;r)| &\leq C(c_0)rb\left(|((1-g')g)(Q)(\vartheta)| + g(Q)(\vartheta)\right) \\ &= C(c_0)((1+|1-g'|)g)(Q)(\vartheta)br. \end{aligned}$$

So when we integrate in $b$, we have :

$$\begin{aligned} |S_r(b;r) - Q_r(r)| &\leq \int_0^b |S_{r,b}(b;r)|db \\ &\leq C(c_0)b^2r\sup_{\theta\in I(b,r)}|(g+|1-g'|g)(Q)(\theta)| \\ &\leq C(c_0)b^2r\left(1+\sup_{\theta\in I(b,r)}|g'|(Q)(\theta)\right)\sup_{\theta\in I(b,r)}|g(Q)(\theta)|. \end{aligned}$$

And as $Q$ takes its values in $[0, C^*]$, we get the second estimate. □

*Proof of Lemma 3.* We integrate the pointwise estimate of the previous lemma.

$$\begin{aligned}\|S(b;r)-Q(r)\|^2_{H([0,\alpha])} &= \int_0^\alpha \left( (S(b;r)-Q(r))_r^2 + \frac{(S(b;r)-Q(r))^2}{r^2} \right) rdr \\ &\le \int_0^\alpha \left( \left(D_0 h(r) b^2 r\right)^2 + \left(\frac{h(r)b^2r^2}{r}\right)^2 \right) rdr \\ &\le C(c_0)b^4 \int_0^\alpha h^2(r) r^3 dr \le C(c_0) b^4 \alpha^2 \cdot \alpha^2 h_2(\alpha).\end{aligned}$$

For the time-derivative estimates, we use directly (45), and we plug in $br \le b\alpha \le c_0 < 1$. This leads to :

$$|S_r(b;r)| \le |Q_r(\vartheta)|(2 + C(c_0)b^2r^2) \le C(c_0)h(r)/r,$$

so that :

$$\|brS_r(b;r)\|^2_{L^2[0,\alpha]} \le C(c_0)b^2 \int_0^\alpha h^2(r)rdr.$$

For the third bound, we compute, using $rQ_r = g(Q)$ :

$$\|Q\|_{H([\alpha;\infty))} = 2\int_\alpha^\infty g(Q)Q_r dr = 2\int_{Q(\alpha)}^{C^*} g \le 2|C^* - Q(\alpha)|h(\alpha). \qquad \square$$

*Proof of Lemma 4.* Let us first do the computations for an affine interpolation :

$$\text{For } r \in [a,b], \tilde{v}(r) = (v(b)-v(a))\frac{r-a}{b-a} + v(a), \text{ i.e. } \tilde{v}_r(r) = \frac{v(b)-v(a)}{b-a}.$$

Then :

$$\begin{aligned}\int_a^b \tilde{v}_r^2 r dr &= \frac{1}{2}\left(\frac{v(b)-v(a)}{b-a}\right)^2 (b^2-a^2) \le \frac{1}{2}(v(b)-v(a))^2 \frac{b+a}{b-a}, \\ \int_a^b \frac{\tilde{v}^2}{r} dr &\le \max(v(b),v(a))^2 \int_a^b \frac{dr}{r} \le \max(v(b),v(a))^2 \ln\frac{b}{a}.\end{aligned}$$

This already gives an extension in $H$. For a smoother extension, it is enough to regularize $\tilde{v}_r$ (locally near $a$ and $b$) with a small variation of the $L^2$ norm, while keeping constant $\tilde{v}(b) - \tilde{v}(a) = \int_a^b \tilde{v}_r dr$. This is possible thanks to the density of smooth functions with given mean among $L^2$ functions with the same given mean. $\square$

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