

Homomorphic Images of Branch Groups, and Serre’s Property (FA)

Thomas Delzant and Rostislav Grigorchuk

Dedicated to the memory of Sasha Reznikov

Abstract. It is shown that a finitely generated branch group has Serre’s property (FA) if and only if it does not surject onto the infinite cyclic group or the infinite dihedral group. An example of a finitely generated self-similar branch group surjecting onto the infinite cyclic group is constructed.

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Introduction

The study of groups acting on trees is a central subject in geometric group theory. The Bass-Serre theory establishes a dictionary between the geometric study of groups acting on trees and the algebraic study of amalgams and HNN extensions. A central topic of investigation is the fixed point property for groups acting on trees, introduced by J.-P. Serre in his book as the property (FA) [Ser80]. A fundamental result due to Tits states that a group without a free subgroup on two generators which acts on a tree by automorphisms fixes either a vertex or a point on the boundary or permutes a pair of points on the boundary; see [Tit77, PV91]. The group $SL(3, \mathbb{Z})$, and more generally, groups with Kazhdan’s property (T), in particular lattices in higher rank Lie groups have the property (FA) ([dlHV89, Mar91]). A natural problem is to understand the structure of the class of (FA)-groups (the class of groups having the property (FA)). There is an algebraic characterization of enumerable (FA)-groups, due to J.-P. Serre. ([Ser80], Theorem I.6.15, page 81).

An enumerable group has the property (FA) if and only if it satisfies the following three conditions:

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- (i) it is not an amalgam,
- (ii) it is not indicable (i.e., admits no epimorphism onto $\mathbb{Z}$),
- (iii) it is finitely generated.

But even such a nice result does not clarify the structure of the class of (FA)-groups, as the first of these properties is usually difficult to check.

The class of (FA)-groups contains the class of finite groups and is closed under quotients. As every infinite finitely generated group surjects onto a just-infinite group (i.e., an infinite group with all proper quotients finite) a natural problem is to describe just infinite (FA)-groups.

In [Gri00] the class (JINF) of just infinite groups is divided in three subclasses: the class (B) of branch groups, the class (HJINF) of finite extensions of finite powers of hereditary just infinite groups and the class (S) of finite extensions of finite powers of simple groups. For example, the group $SL(3, \mathbb{Z})$ belongs to the class (JINF); all infinite finitely generated simple torsion groups constructed in [OI'79] are (FA)-groups and belong to the class (S).

A precise definition of a branch group is given in Section 1. Roughly speaking a branch group is a group which acts faithfully and level transitively on a spherically homogeneous rooted tree, and for which the structure of the lattice of subnormal subgroups mimics the structure of the tree. Branch groups may enjoy unusual properties. Among them one can find finitely generated infinite torsion groups, groups of intermediate growth, amenable but not elementary amenable groups and other surprising objects. Profinite branch groups are also related to Galois theory and other topics in Number Theory [Bos00].

In this article we discuss fixed point properties for actions of branch groups on Gromov hyperbolic spaces, in particular on $\mathbb{R}$ -trees, and apply Bass-Serre theory to branch groups. Recall (see [Ser80]) that a group G is an amalgam (resp. an HNN extension) if it can be written as a free product with amalgamation $G = A *_C B$, with $C \neq A, B$ (resp. $G = A *_t C_{t-1} = C'$). We say that this amalgam (resp. HNN extension) is *strict* if the index of C in A is at least 3 and the index of C in B is at least 2 (resp. the indexes of C and C' in A are at least 2).

One of the corollaries of Theorem 3 is:

Theorem 1. *Let G be a finitely generated branch group. Then G is not a strict amalgam or HNN. Therefore a branch group cannot be an amalgam unless it surjects onto $\mathbb{D}_\infty$. It has Serre's property (FA) if and only if it is not indicable and has no epimorphism onto $\mathbb{D}_\infty$.*

We say that a group is (FL) if it has no epimorphism onto $\mathbb{Z}$ or $\mathbb{D}_\infty$. A f.g. group is (FL) if and only if it fixes a point whenever it acts isometrically on a line.

All proper quotients of branch groups are virtually abelian [Gri00]. A quotient of a branch group may be infinite: the full automorphism group of the binary rooted tree is a branch group and its abelianization is the infinite cartesian product of copies of a group of order two. It is more difficult to construct examples of finitely generated branch groups with infinite quotients (especially in the restricted setting of self-similar groups). The corresponding question was open since 1997 when the

second author introduced the notion of a branch group. Perhaps the main difficulty was psychological, as he (and some other researches working in the area) was sure that all finitely generated branch groups are just infinite. Now we know that this is not correct and the second part of the paper (Section 3) is devoted to a construction of an example of an indicable finitely generated branch group (thus providing an example of a finitely generated branch group without the property (FA)). This example is the first example of a finitely generated branch group defined by a finite automaton that is not just infinite. Another example is related to Hanoi Towers group on 3 pegs H (introduced in [GŠ06] and independently in [Nek05]). Hanoi Towers group H is a 3-generated branch group [GŠ07] that has a subgroup of index 4 (the Apollonian group) which is also a branch group and is indicable (this is announced in [GNŠ06]). The group H itself is not indicable (it has finite abelianization), but it surjects onto $\mathbb{D}_\infty$, as was recently observed by Zoran Šunić. Thus H is the first example of a finitely generated branch group defined by a finite automaton that surjects onto $\mathbb{D}_\infty$.

The example of an indicable branch group presented in this paper is an elaboration of the 3-generated torsion 2-group $G = \langle a, b, c, d \rangle$ firstly constructed in [Gri80a] and later studied in [Gri89, GM93, Gri98a, Gri99] and other papers (see also the Chapter VIII of the book [dlH00] and the article [CSMS01]).

Let L be the group generated by the automaton defined in Figure 1.

Theorem 2. *The group L is a branch, contracting group that surjects onto $\mathbb{Z}$.*

Starting from this example, a construction of a branch group surjecting onto $\mathbb{D}_\infty$ has been proposed by Dan Segal. Let L be an indicable branch group and $l : L \rightarrow \mathbb{Z}$ an epimorphism. It is easy to see that the semi-direct product $H = \mathbb{Z}/2\mathbb{Z} \ltimes (L \times L)$ is branch. Furthermore, it surjects onto $\mathbb{D}_\infty = \mathbb{Z}/2\mathbb{Z} \ltimes \mathbb{Z}$ by the unique morphism l' whose restriction to $\mathbb{Z}/2\mathbb{Z}$ is the identity and such that $l'(g, h) = l(g) - l(h)$.

An interesting question is to understand which virtually abelian group can be realized as a quotient of a finitely generated branch group. This question is closely related to the problem of characterization of finitely generated branch groups having the Furstenberg-Tychonoff fixed ray property (FT) [Gri98b]) (existence of an invariant ray for actions on a convex cone with compact base). The problem of indicability of branch groups is also related to the recent work of D.W. Morris [Mo] who studied the action of an amenable group by homomorphisms on the line.

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1. Basic definitions and some notation

Let T be a tree, G be a group acting on T (without inversion of edges) and T^G the set of fixed vertices of T .

Definition 1. A group G has the property (FA) if for every simplicial tree T on which G acts simplicially and without inversion, $T^G \neq \emptyset$.

The class of (FA)-groups possesses the following properties.

- (i) The class of (FA)-groups is closed under taking quotients.
- (ii) Let G be a group with the property (FA). If G is a subgroup of an amalgamated free product $G_1 *_A G_2$ or an HNN extension $G = G_1 *_A$, then G is contained in a conjugate of G_1 or G_2 .
- (iii) The class of (FA)-groups is closed under forming extensions.
- (iv) If a subgroup of finite index in a group G has the property (FA), then the group G itself has the property (FA).
- (v) Every finitely generated torsion group has the property (FA).

The class of (FA)-groups has certain nice structural properties and is interesting because of the strong embedding property given by (ii) and by the fact that the eigenvalues of matrices in the image of a linear representation $\rho : G \rightarrow GL_2(k)$ are integral over $\mathbb{Z}$ for any field k (Prop. 22, [Ser80]).

The property (i), the existence of just infinite quotients for finitely generated infinite groups and the trichotomy from [Gri00] mentioned in the introduction make the problem of classification of finitely generated just infinite (FA)-groups worthwhile. We are reduced to the classification of finitely generated (FA)-groups in each of the classes (B), (HJINF) and (S). Below we solve this problem, in a certain sense, for the class (B).

If a group G has a quotient isomorphic to $\mathbb{Z}$, then it acts by translations on a line and cannot be an (FA)-group. Similarly, if G surjects onto the infinite dihedral group $\mathbb{D}_\infty$, then it acts on the line via the obvious action of $\mathbb{D}_\infty$. This suggests the following definition (the first part being folklore):

Definition 2. a) A group is called *indicable* if it admits an epimorphism onto $\mathbb{Z}$.
 b) A group has property (FL) (fixed point on line) if every action of G by isometry on a line fixes a point. If G is finitely generated this means that G has no epimorphism onto $\mathbb{Z}$ or $\mathbb{D}_\infty$.

In this article we will often use two other notions: the notion of a hyperbolic space and that of a branch group.

For the definition and the basic properties of Gromov hyperbolic spaces we refer the reader to [CDP90]. The theory of CAT(0)-spaces is described in [BH99]. For the definition and the study of basic properties of branch groups we refer the reader to [Gri00, BGŠ03].

Let us recall the main definition and a few important facts and notations that will be often used later.

Definition 3. A group $\mathcal{G}$ is an *algebraically branch group* if there exists a sequence of integers $\bar{k} = \{k_n\}_{n=0}^\infty$ and two decreasing sequences of subgroups $\{R_n\}_{n=0}^\infty$ and $\{V_n\}_{n=0}^\infty$ of $\mathcal{G}$ such that

- (1) $k_n \geq 2$, for all $n > 0$, $k_0 = 1$,

(2) for all n ,

$$R_n = V_n^{(1)} \times V_n^{(2)} \times \dots \times V_n^{(k_0 k_1 \dots k_n)}, \quad (1.1)$$

where each $V_n^{(j)}$ is an isomorphic copy of V_n ,

(3) for all n , the product decomposition (1.1) of R_{n+1} is a refinement of the corresponding decomposition of R_n in the sense that the j -th factor $V_n^{(j)}$ of R_n , $j = 1, \dots, k_0 k_1 \dots k_n$ contains the j -th block of k_{n+1} consecutive factors

$$V_{n+1}^{((j-1)k_{n+1}+1)} \times \dots \times V_{n+1}^{(jk_{n+1})}$$

of R_{n+1} ,

(4) for all n , the groups R_n are normal in $\mathcal{G}$ and

$$\bigcap_{n=0}^{\infty} R_n = 1,$$

(5) for all n , the conjugation action of $\mathcal{G}$ on R_n permutes transitively the factors in (1.1),

and

(6) for all n , the index $[\mathcal{G} : R_n]$ is finite.

A group G is a *weakly algebraically branch group* if there exists a sequence of integers $\bar{k} = \{k_n\}_{n=0}^{\infty}$ and two decreasing sequences of subgroups $\{R_n\}_{n=0}^{\infty}$ and $\{V_n\}_{n=0}^{\infty}$ of G satisfying the conditions (1)–(5).

There is a geometric counterpart of this definition.

Let $(\mathcal{T}, \emptyset)$ be a spherically homogeneous rooted tree, where $\emptyset$ is the root and $\mathcal{G}$ be a group acting on $(\mathcal{T}, \emptyset)$ by automorphisms preserving the root. Let v be a vertex, and $\mathcal{T}_v$ be the subtree consisting of the vertices w such that $v \in [w, \emptyset]$ (geodesic segment joining w with the root). The rigid stabilizer $\text{rist}_{\mathcal{G}}(v)$ of a vertex v consists of elements acting trivially on $\mathcal{T} \setminus \mathcal{T}_v$. The rigid stabilizer of the n -th level, denoted $\text{rist}_{\mathcal{G}}(n)$, is the group generated by the rigid stabilizers of the vertices on level n .

The action of $\mathcal{G}$ on $\mathcal{T}$ is called *geometrically branch* if it is faithful, level transitive, and if, for any n , the rigid stabilizer $\text{rist}_{\mathcal{G}}(n)$ of n -th level of the tree has finite index in $\mathcal{G}$.

Observe that, in the level transitive case, the rigid stabilizers of the vertices of the same level are conjugate in $\mathcal{G}$. In this case $\text{rist}_{\mathcal{G}}(n)$ is algebraically isomorphic to the product of copies of the same group (namely the rigid stabilizer of any vertex on the given level). Hence the rigid stabilizers of the levels and vertices play the role of the subgroups R_n and V_n of the algebraic definition. A geometrically branch group is therefore algebraically branch. The algebraic definition is slightly more general than the geometric one but at the moment it is not completely clear how big the difference between the two classes of groups is. Observe that in Section 2 we will assume that the considered groups are algebraically branch, while in sections 3 and 4 we construct examples of geometrically branch groups.

When constructing these examples, we will deal only with actions on a rooted binary tree and our notation and the definition below are adapted exactly for this case. Let $\mathcal{G}$ be a branch group acting on a binary rooted tree $\mathcal{T}$. The vertices of $\mathcal{T}$ are labeled by finite sequences of 0 and 1. Let $\mathcal{T}_0, \mathcal{T}_1$ be the two subtrees consisting of the vertices starting with 0 or 1, respectively.

Notation. If $A, B, C \subset \text{Aut}(\mathcal{T})$ are three subgroups, we write $A \succeq B \times C$ if A contains the subgroup $B \times C$ of the product $\text{Aut}(\mathcal{T}_0) \times \text{Aut}(\mathcal{T}_1)$ via the canonical identification of $\text{Aut}(\mathcal{T})$ with $\text{Aut}(\mathcal{T}_i)$.

Recall that a level transitive group $\mathcal{G}$ acting on a regular rooted binary tree is called regular branch over its normal subgroup H if H has finite index in $\mathcal{G}$, $H \succeq H \times H$ and if moreover the last inclusion is of finite index.

A level transitive group $\mathcal{G}$ is called weakly regular branch over a subgroup H if H is nontrivial and $H \succeq H \times H$.

Definition 4. A group $\mathcal{G}$ acting on the rooted binary tree $(\mathcal{T}, \emptyset)$ is called *self-replicating* if, for every vertex u , the image of the stabilizer $\text{st}_{\mathcal{G}}(u)$ of u in $\text{Aut}(\mathcal{T}_u)$ (the automorphism group of the rooted tree $\mathcal{T}_u$) coincides with the group $\mathcal{G}$ after the canonical identification of $\mathcal{T}$ with $\mathcal{T}_u$.

Obviously a self-replicating group is level transitive if and only if it is transitive on the first level (see also Lemma A in [Gri00]).

We will use the notations $\langle R \rangle^{\mathcal{G}}$ for the normal closure in $\mathcal{G}$ of a subset $R \subset \mathcal{G}$, $x^y = y^{-1}xy$, $[x, y] = x^{-1}y^{-1}xy$. Given two subgroups, A, B in a group $\mathcal{G}$, $[A, B]$ is the subgroup of $\mathcal{G}$ generated by the commutators $[a, b]$ of elements in A and B , and $[A, B]^{\mathcal{G}}$ its normal closure. If $\mathcal{G}$ is a group, $\gamma_2(\mathcal{G})$ denote the second member of its lower central series.

2. Fixed point properties of branch groups

Let X be a Gromov hyperbolic metric space, and ∂X its Gromov's boundary. Recall (see [Gro87] or [CDP90] Chap. 9 for instance) that a subgroup G of the group $\text{Isom}(X)$ of isometries of X is called elliptic if it has a bounded orbit (or equivalently if every orbit is bounded), parabolic if it has a unique fixed point on ∂X but is not elliptic, and loxodromic if it is not elliptic and if there exists a pair w^+, w^- of points in ∂X preserved by G . A group which is either elliptic, or parabolic or loxodromic is called elementary; this terminology is inspired by the theory of Kleinian groups. There are no constraints on the algebraic structure of elementary groups due to the following remark.

Remark. Every f.g. group G can be realized as a *parabolic* group of isometries of some proper geodesic hyperbolic space: if C is the Cayley graph of G , $C \times \mathbb{R}$ admits a G -invariant hyperbolic metric ([Gro87], 1.8.A, note that this construction is equivariant). One can also construct a finitely generated group acting on a tree

with a unique fixed point at infinity. For instance the lamplighter group (semi-direct product of $\mathbb{Z}$ and $\mathbb{Z}_2[t, t^{-1}]$) fixes a unique point in the boundary of the tree of $GL_2(\mathbb{Z}_2[t^{-1}, t])$. In fact, the lamplighter group can be indentified with upper triangular matrices with one eigenvalue equal to 1 the other being t^n . As all these matrices have a common eigenspace, they fix one point in the boundary of the tree of GL_2 (the projective line on $\mathbb{Z}_2[t, t^{-1}]$); but this group contains the Jordan matrix and therefore cannot fix two points in the boundary of this tree.

In what follows, X denotes a complete Gromov hyperbolic geodesic space. We will assume that either X is proper (closed balls are compact) or that X is a complete $\mathbb{R}$ -tree, i.e., a complete 0-hyperbolic geodesic metric space. In the first case, $X \cup \partial X$ is a compact set (in the natural topology) and an unbounded sequence of points in X admits a subsequence which converges to a point in ∂X . Important examples of such spaces are Cayley graphs of hyperbolic groups (see [Gro87] for instance). Other examples are universal covers of compact manifolds of non positive curvature. Note that properness implies completeness for a metric space, but the converse is false. Recall also that a geodesic space is proper if and only if it is complete and locally compact [Gro99]. The Gromov hyperbolicity of a geodesic space can be defined in several ways (thinness of geodesic triangles, properties of the Gromov product etc.) which are equivalent (see [CDP90] Chap. 1); we will prefer the definition in terms of the Gromov product ([CDP90] Chap. 1, Def. 1.1).

For the rest of the statements in this section we will assume that the following condition on the pair (X, G) holds:

- (C) X is a complete geodesic space and X is either proper hyperbolic or is an $\mathbb{R}$ -tree. G is a group and $\varphi : G \rightarrow \text{Isom}(X)$ is an isometric action of G on X .

Note that such an action extends uniquely to a continuous action on $X \cup \partial X$.

Theorem 3. *Let G be a branch group acting isometrically on a hyperbolic space X . Suppose the pair (X, G) satisfies the condition (C). Then:*

- a) *The image of G in $\text{Isom}(X)$ is elementary.*
- b) *Suppose furthermore that G satisfies the property (FL), and X is a hyperbolic graph with uniformly bounded valence of vertices. Then $\varphi(G)$ is elliptic or parabolic.*
- c) *If X is $\text{CAT}(0)$ and if the group $\varphi(G)$ is elliptic, then it has a fixed point in X .*
- d) *If X is $\text{CAT}(-1)$, or is an $\mathbb{R}$ -tree, then $\varphi(G)$ fixes a point in X or in ∂X , or preserves a line in X .*
- e) *Let X be an $\mathbb{R}$ -tree. Suppose further that G is f.g.; then G cannot be parabolic.*

Corollary 1. *Let G be a f.g. branch group. G has fixed point property for actions on $\mathbb{R}$ -trees if and only if it has property (FL).*

Proof. A tree is $CAT(-1)$, so if G acts on a tree and does not fix a point, it must either preserve a line or a unique point on ∂X . The last possibility is excluded by e). $\square$

From d) we also deduce:

Corollary 2. *If X is $CAT(-1)$ and G acts on $X \cup \partial X$ by isometries, then G fixes a point or contains a subgroup of index 2 which fixes two points in ∂X .*

Recall (see [Ser80]) that a group G is an amalgam (resp. an HNN extension) if it can be written as a free product with amalgamation $G = A *_C B$, with $C \neq A, B$ (resp. $G = A *_t C t^{-1} = C'$). We say that this amalgam (resp. HNN extension) is *strict* if the index of C in A is at least 3 and the index of C in B is at least 2 (resp. the indexes of C and C' in A are at least 2). If G splits as an amalgam or HNN extension, then G acts on a simplicial tree T without edge inversion s.t. T/G has one edge and 2 vertices in the case of an amalgam, and one edge and one vertex in the case of an HNN extension. It is easy too see that if a group is a strict amalgam or HNN extension its action on Serre's tree is not elementary. If $G = A *_C B$ with C of index 2 in A and B Serre's tree is a line, and G permutes the two ends of this line. If $G = A *_t C t^{-1} = C'$ and $C = C' = A$, Serre's tree is a line and G fixes the two ends of this line. If $G = A *_t C t^{-1} = C'$ is a strictly ascending HNN extension ($C' = A$, but $C \neq A$), the group G contains a hyperbolic element (the letter t for instance) and fixes exactly one end of the tree. Therefore the property e) implies the following:

Corollary 3. *Let G be a f.g. branch group. Then G is neither a strict amalgam nor a strict HNN extension nor a strictly ascending HNN extension.*

Before proving Theorem 3 let us state and prove some statements that have independent interest and will be used later.

Recall that an isometry f of a hyperbolic space X is called elliptic (resp. parabolic, resp. hyperbolic) if the subgroup generated by f is elliptic (resp. parabolic, resp. loxodromic). It can be proved (see [CDP90], chap. 9) that an isometry is either elliptic, or parabolic or hyperbolic, and that if X is an $\mathbb{R}$ -tree an isometry cannot be parabolic. An elliptic group cannot contain a hyperbolic or a parabolic element, a loxodromic group cannot contain a parabolic element. In order to simplify the notation, if $\phi : G \rightarrow \text{Isom}(X)$ is an action of the group G , we denote by gx the image of x under the isometry $\phi(g)$.

Proposition 1. *Let the pair (X, G) satisfy (C). Assume that each element of G is either elliptic or parabolic. Then G is either elliptic or parabolic; if X is an $\mathbb{R}$ -tree, and G is finitely generated, then G is elliptic.*

The proof of this proposition is of dynamical nature and based on the following

Lemma 1. ([CDP90], Chap. 9, Lemma 2). *Let X be a δ -hyperbolic space. Let g, h be two elliptic or parabolic isometries of X . Suppose that $\min(d(gx, x), d(hx, x)) \geq 2\langle gx, hx \rangle_x + 6\delta$. Then $g^{-1}h$ is hyperbolic.*

Recall that the Gromov product $\langle x, y \rangle_z$ is defined as $1/2(d(x, z) + d(y, z) - d(x, y))$

Proof of Proposition 1. Let us first consider the case where X is an $\mathbb{R}$ -tree, which we denote by T , and G is finitely generated. Recall that projection of a point x in a $CAT(0)$ space onto a complete convex subset Y is the unique closest point to x in Y (see [BH99], page 176). We claim that in an $\mathbb{R}$ -tree T , if g is some elliptic isometry, and T^g the subtree of fixed points of g , then for every x the midpoint of the segment $[x, gx]$ is the projection of x on T^g : indeed let p be this projection, so that the image of the segment $[x, p]$ is $[gx, p]$; if the Gromov product $\langle x, gx \rangle_p = d$ is strictly positive, we can consider the point $q \in [p, x]$ s.t. $d(p, q) = d$; it is fixed by g as it belongs to $[p, x]$ and it is the unique point on this segment with $d(q, p) = d$, but q is closer than p to x , contradiction. Thus $\langle x, gx \rangle_p = 0$, and as the two segments $[x, p]$ and $[gx, p] = g[x, p]$ have the same length, p is the midpoint of $[x, gx]$. For every subset $\Sigma \subset G$, let T^Σ be the fixed subset of Σ . Let $\{g_1, \dots, g_n\}$ be a finite generating subset of G , and let us prove by induction that $T^{\{g_1, \dots, g_n\}}$ is not empty. For $n = 1$ this is the hypothesis. Suppose that $T^{\{g_1, \dots, g_{n-1}\}} \cap T^{\{g_n\}} = \emptyset$. The minimal distance between these two subtrees is achieved along a segment $[a, b]$, with $a \in T^{\{g_1, \dots, g_{n-1}\}}$, $b \in T^{\{g_n\}}$. Let x_0 be the midpoint of this segment: $x_0 \notin T^{\{g_n\}}$. Therefore $b \in [x_0, g_n x_0]$ is the midpoint. As $x_0 \notin T^{\{g_1, \dots, g_{n-1}\}}$, we have that $x_0 \notin T^{\{g_i\}}$, for some i . The intersection $T^{\{g_i\}} \cap [a, x_0]$ is a segment $[a, c]$; the right extremity c of this segment is the projection of x_0 on $T^{\{g_i\}}$, and therefore $c \in [x_0, g_i x_0]$ is the midpoint. Thus $x_0 \in [g_i x_0, g_n x_0]$ and, in other words, $\langle g_i x_0, g_n x_0 \rangle_{x_0} = 0$. Lemma 1 applies and proves that the isometry $g_i g_n$ is hyperbolic, a contradiction.

Suppose now that X is a proper geodesic hyperbolic space. Let G be as in the statement, and $x_0 \in X$ be some base-point. If the orbit Gx_0 is bounded, then it is a bounded G invariant set, and G is elliptic. Assume that Gx_0 is not bounded. We consider the set $\overline{Gx_0} \cap \partial X$.

1) Assume that this set has only one point a . It must be G invariant. Let us prove that G is parabolic. Suppose that G fixes another point b on the boundary. Then it acts on the union Y of geodesic lines between a and b . Let $L \subset Y$ be a geodesic between a and b , so that every point in Y is at distance $< 100\delta$ of L . Let $x_0 \in L$; as $\overline{Gx_0} \cap \partial X = \{a\}$, we can find two isometries g, h in G such that $d(x_0, gx_0) > 1000\delta$, $d(x_0, hx_0) > d(x_0, gx_0) + 1000\delta$ and the projections of gx_0 and hx_0 on L are on the right of x_0 .

Considering these projections of gx_0 and hx_0 on L , we see that $d(x_0, hx_0) \geq d(x_0, gx_0) + d(gx_0, hx_0) - 200\delta$, thus $\langle x_0, hx_0 \rangle_{g(x_0)} \leq 100\delta$. By isometry, we get $\langle g^{-1}x_0, g^{-1}hx_0 \rangle_{x_0} < 100\delta < 1/2(\min(d(x_0, g^{-1}x_0), d(x_0, g^{-1}hx_0)) - 3\delta)$, and h must be hyperbolic by Lemma 1.

2) Assume that $\overline{Gx_0} \cap \partial X$ has at least two points, $a, b \in \overline{Gx_0} \cap \partial X$. There exists two sequences g_n and h_n such that $g_n x_0 \rightarrow a$, and $h_n x_0 \rightarrow b$. Then $d(g_n x_0, x_0) \rightarrow \infty$ as well as $d(h_n x_0, x_0)$, but $\langle g_n x_0, h_n x_0 \rangle_{x_0} \rightarrow \langle a, b \rangle_{x_0}$ and remains bounded

(by the very definition of the Gromov boundary). Lemma 1 applies and we get a contradiction. $\square$

Corollary 4. *Let (G, X) satisfy (C). If G has a subgroup of finite index which is elliptic or parabolic, the G is also elliptic or parabolic.*

Proof. No element of G can be hyperbolic, as any power of a hyperbolic element is hyperbolic. $\square$

Proposition 2. *Let the pair (X, G) satisfy (C). If G is elliptic, then it has an orbit of diameter $\leq 100\delta$. If, furthermore, X is $CAT(0)$, then G has a fixed point.*

Proof. In a metric space, the radius of a bounded set Y is the infimum of r s.t. there exists a x with $Y \subset B(x, r)$. A center is a point c s.t. $Y \subset B(c, r')$ for every $r' > \text{radius}(Y)$. The proof of Proposition 2 is a direct consequence of the following generalization of Elie Cartan center's theorem [BH99], II.2.7. $\square$

Proposition 3. *In a proper geodesic δ -hyperbolic space, the diameter of the set of the centers of a bounded set is $\leq 100\delta$. In a complete $CAT(0)$ space, every bounded set admits a unique center.*

Proof. The second point is proved in [BH99], II.2.7. Let us prove the first assertion. Let a, b be two centers and suppose that $d(a, b) > 100\delta$. Let c be a midpoint of a, b . Let us prove that for every x in Y , $d(y, c) < r - 10\delta$ and in such way get a contradiction. By assumption $d(a, x)$ and $d(b, x)$ are less than $r + \delta$. By the 4 points definition of δ -hyperbolicity ([CDP90] Prop. 1.6) we know that $d(x, c) + d(a, b) \leq \max(d(x, a) + d(b, c), d(x, b) + d(a, c)) - 2\delta$. As $d(b, c) = d(a, c) = 1/2d(a, b) > 50\delta$ we get that $d(x, c) \leq \max(d(x, a), d(x, b)) - 48\delta \leq r - 48\delta$ and we are done. $\square$

Proposition 4. *Let the pair (X, G) satisfy (C). If the G -orbit of some point of ∂X is finite and has at least 3 elements, then G is elliptic.*

Proof. If the orbit is finite and has at least 3 elements $w_1, \dots, w_k$, let us construct a bounded orbit of G in X . For every triple of different points w_i, w_j, w_k in this orbit, let us consider the set C_{ijk} consisting of all points being at a distance less than 24δ from all geodesics between w_i, w_j , and w_k . By hyperbolicity this set is not empty and has diameter $\leq 100\delta$. This follows from [CDP90], Chap. 2, Prop. 2.2, p. 20. A finite union of bounded sets is bounded. Therefore, the union of the sets C_{ijk} is a bounded G invariant set. $\square$

Proposition 5. *Let X be a δ -hyperbolic graph of bounded valence. If $G \subset \text{Isom}(X)$ is loxodromic, then there exists an epimorphism $m : G \rightarrow \mathbb{Z}$ or $m : G \rightarrow \mathbb{D}_\infty$ such that $\ker m$ is elliptic.*

Proof. We will construct a combinatorial analogue of the Busemann cocycle (compare [RS95]). As G is loxodromic, the action of G fixes two points $w^\pm$ at infinity. It contains a subgroup of index at most two G^+ which preserves these two points, and contains some hyperbolic element h . Let U be the union of all geodesics between these two points at infinity, and choose a preferred oriented line L between

this two points. If $x \in U$, there exists a point in L such that $d(x, p(x)) < 24\delta$ ([CDP90], Chap. 2, Prop. 2.2, p. 20). Choose such a point and call it a projection of x . If $x \in U$, let $R(x) = \{y \in U \mid d(x, y) > 1000\delta, \text{ and the projection of } y \text{ to } L \text{ is on the right to that of } x\}$. Note that our hypothesis implies that for every pair x, y , $\{R(y)/R(x)\}$ is contained in the ball centered at y and of radius $d(x, y) + 2000\delta$ and is therefore finite: by definition, a point of $R(y)$ which is at distance $> d(x, y) + 2000\delta$ from y must project on L on a point which is at the distance $> 1000\delta$ of x . Note also that if h is hyperbolic, $R(h^n x)$ is strictly contained in $R(x)$ if n is $\gg 1$. Let $c(x, y) = \text{Card}\{R(y) \setminus R(x)\} - \text{Card}\{R(x) \setminus R(y)\}$. Note that $c(y, x) + c(x, y) = 0$, and that $c(x, y) + c(y, z) = c(x, z)$. Moreover, if g is in G^+ , then $R(gx) = gR(x)$. Choose some point $x_0 \in U$. The formula $m(g) = c(x_0, gx_0)$ defines a nontrivial morphism $G^+ \rightarrow \mathbb{Z}$. The orbit of x_0 under the action of the kernel of m is bounded, contained in $B(x_0, 2000\delta)$, and $\ker m$ is elliptic. If G/G^+ is not trivial, and $\varepsilon \in G \setminus G^+$, then $m - m(\varepsilon g \varepsilon^{-1}) = d(g)$ extends to a nontrivial epimorphism $G \rightarrow \mathbb{D}_\infty$. $\square$

Proof of Theorem 3. Let H_1 be the rigid stabilizer of the first level of G . It is a product of n subgroups of G , $H_1 = L_1 \times \dots \times L_n$ conjugate in G .

i) Suppose first that L_1 contains no hyperbolic element.

Then L_1 has either (1) a bounded orbit or (2) a unique fixed point w at infinity.

(1) In the first case, let $C_1 = \{x \mid \forall g \in L_1, d(gx, x) < 100\delta\}$ (by Proposition 2 this set is nonempty). As L_2 commutes with L_1 it preserves C_1 . Being conjugate to L_1 , every orbit of L_2 is bounded. If $x_0 \in C_1$ and $D = \text{diam}(L_2 x_0)$, we see that the diameter of $(L_1 \times L_2)x_0$ is $\leq D + 2 \cdot 100\delta$, hence $L_1 \times L_2$ is elliptic, and the set $C_2 = \{x \mid \forall g \in L_1 \times L_2, d(gx, x) \leq 100\delta\}$ is not empty (Proposition 2). By induction we prove that $C_k = \{x \mid \forall g \in L_1 \times L_2 \times \dots \times L_k, d(gx, x) < 100\delta\}$ is not empty; thus G admits a subgroup of finite index which is elliptic, and G is itself elliptic.

(2) In the second case, the unique fixed point w is stable under the action of the subgroup $L_2 \times \dots \times L_n$, and G has a subgroup of finite index which is parabolic, thus G is parabolic itself.

ii) Suppose L_1 contains some hyperbolic element h . Let $w^\pm$ be the two distinct fixed points of h at infinity. As $L_2 \times \dots \times L_n$ commutes with h this group fixes this set. Now L_2 contains a hyperbolic element h_2 , conjugated to h : thus h_2 has the same fixed points at infinity as h , and H_1 must also fix the set $\{w, w^-\}$. Thus the orbit of $w^\pm$ is finite and Proposition 4 applies. The orbit of G cannot have more than 2 elements unless G is elliptic: therefore it has exactly two elements, and G is loxodromic. This proves a).

To prove b) apply Proposition 5. Proposition 2 and Proposition 3 (the unique center for bounded sets) give rise to the desired fixed point for c). Claim d) follows from the fact that between two points at infinity in a CAT(-1) space there exists a unique geodesic (visibility property). For claim e), let w be an end of a tree fixed by the group G . Let $t \rightarrow r(t)$ be a geodesic ray converging to w . Note that

when the point x is fixed, the function $t \rightarrow d(x, r(t)) - t$ is constant for $t \gg 1$. The value of the constant $b_w(x)$ is called the Busemann function associated to w (see [BH99], Chap. II.8. for a study of Busemann functions in CAT(0) spaces). If the point w is fixed by some isometry g , then $g.r(t)$ is another ray converging to w . But two rays converging to the same point in a tree must coincide outside a compact set. Therefore $d(gr(t), r(t)) = b(g)$ is constant for $t \gg 1$, and this constant is $b_{g.w} - b_w$. By construction, $g \rightarrow b(g)$ is a homomorphism from G to $\mathbb{R}$, which is non-trivial unless every element of G is elliptic, and takes values in $\mathbb{Z}$ if X is a combinatorial tree. Suppose that the restriction of b to L_1 is trivial. Then L_1 consists of elliptic elements. Since G is finitely generated, L_1 is finitely generated as well. Thus L_1 is elliptic and i) applies. Otherwise, L_1 contains a hyperbolic element and ii) applies.

Theorem 3 is proved. $\square$

3. An indicable branch group

Let $\mathcal{G}$ be a branch group acting on a rooted tree $\mathcal{T}$. It is proved in [Gri00] that, if $N \triangleleft \mathcal{G}$ is a nontrivial normal subgroup, then the group N contains the commutator subgroup of the rigid stabilizer $\text{rist}_{\mathcal{G}}(n)'$, for some level n . As $\text{rist}_{\mathcal{G}}(n)$ is of finite index in G , $G/\text{rist}_{\mathcal{G}}(n)$ is finite, $G/\text{rist}_{\mathcal{G}}(n)'$ is virtually abelian and we have:

Proposition 6. *A proper quotient $\mathcal{G}/N$ of a branch group is a virtually abelian group.*

We construct in this section an example of a finitely generated branch group which surjects onto the infinite cyclic group. The construction starts from the finitely generated torsion 2-group firstly defined in [Gri80b] and later studied in [Gri84] and other papers (see also the Chapter VIII of the book [dlH00]).

We will list briefly some properties of G that will be used later.

Let $(\mathcal{T}, \emptyset)$ be the rooted binary tree whose vertices are the finite sequences of 0, 1 with its natural tree structure (see [dlH00], VIII.A for details), the empty sequence $\emptyset$ being the root. If v is a vertex of $\mathcal{T}$ we denote by $\mathcal{T}_v$ the subtree consisting of the sequences starting in v . In other words, the subtree $\mathcal{T}_v$ of $\mathcal{T}$ consists of vertices w that contain v as a prefix. Deleting the first $|v|$ letters of the sequences in $\mathcal{T}_v$ yields a bijection between $\mathcal{T}_v$ and $\mathcal{T}$, called the canonical identification of these trees.

The group G (see [dlH00], VIII.B.9 for details) acts faithfully on the binary rooted tree $(\mathcal{T}, \emptyset)$ and is generated by four automorphisms a, b, c, d of the tree where a is the rooted automorphism permuting the vertices of the first level, while b, c, d are given by the recursive rules

$$b = (a, c), c = (a, d), d = (1, b).$$

This means that b does not act on the first level of the tree, it acts on the left subtree $\mathcal{T}_0$ as a and acts on the right subtree $\mathcal{T}_1$ as c , with similarly meaning of the relations for c and d . Here we use the canonical identifications of $\mathcal{T}$ with

$\mathcal{T}_i, i = 0, 1$. An alternative description of G is that it is the group generated by the states of the automaton drawn on the figure 1.

The group G is 3-generated as we have the relations

$$a^2 = b^2 = c^2 = d^2 = bcd = 1$$

there are many other relations and G is not finitely presented [Gri84].

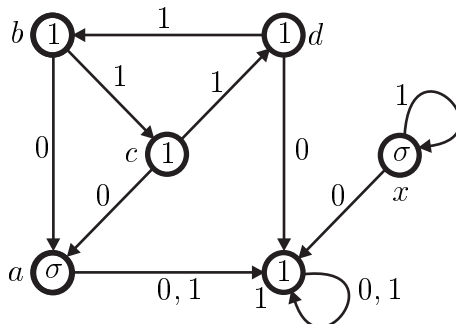


FIGURE 1. The automaton defining L

In order to study groups acting on the binary rooted tree $\mathcal{T}$, it is convenient to use the embedding

$$\begin{aligned} \psi : \text{Aut}(\mathcal{T}) &\longmapsto \text{Aut}(\mathcal{T}) \wr S_2, \\ g &\longmapsto (g_0, g_1)\alpha. \end{aligned}$$

In this description S_2 is a symmetric group of order 2, $\alpha \in S_2$ describes the action of g on the first level of the tree and the sections g_0, g_1 describe the action of g on the subtrees $\mathcal{T}_0, \mathcal{T}_1$. We will usually identify the element g and its image $(g_0, g_1)\alpha$. Relations of this type will be often used below.

Let x be the automorphism of $\mathcal{T}$ defined by the recursive relation $x = (1, x)a$. This automorphism is called the adding machine as it imitates the adding of a unit in the ring of diadic integers [GNS00]. An important property of x is that it acts transitively on each level of $\mathcal{T}$ and therefore has infinite order.

Let $L = \langle x, G \rangle$ be the subgroup of $\text{Aut}(\mathcal{T})$ generated by G and the adding machine x .

Theorem 4. *The group L is branch, amenable, and has infinite abelization.*

The next two lemmas are the first steps towards the proof of the fact that L is a branch group.

Lemma 2. *The following formulas hold in the group L :*

$$\begin{aligned} [x, a] &= (x^{-1}, x), \\ [x, d] &= (x^{-1}bx, b), \\ [[x, a], d] &= (1, [x, b]), \\ (1, [[x, b], c]) &= [[[x, a], d], b]. \end{aligned}$$

Proof. This follows by direct computation:

$$\begin{aligned} [x, a] &= x^{-1}axa = a(1, x^{-1})a(1, x)aa = (x^{-1}, 1)(1, x) = (x^{-1}, x). \\ [x, d] &= x^{-1}dxd = a(1, x^{-1})(1, b)(1, x)a(1, b) = (x^{-1}bx, b) \\ [[x, a], d] &= [(x^{-1}, x), (1, b)] = ([x^{-1}, 1], [x, b]) = (1, [x, b]). \\ [[[x, a], d], b] &= [(1, [x, b]), (a, c)] = (1, [[x, b], c]). \end{aligned}$$

□

Lemma 3. *The group L is self-replicating, and hence level transitive.*

Proof. Consider the elements $b = (a, c)$, $c = (a, d)$, $d = (1, b)$, $aba = (c, a)$, $xa = (1, x)$. They stabilize the two vertices of the first level of $\mathcal{T}$, and their projections on $\text{Aut}(\mathcal{T}_1) \simeq \text{Aut}(\mathcal{T})$ are c, d, b, a, x , i.e., the generators of L . Note that these elements generate L . Hence the projection of $st_L(1)$ on $\text{Aut}(\mathcal{T}_1)$ is L modulo the canonical identification of $\mathcal{T}$ and $\mathcal{T}_1$. The conjugation by a permutes the coordinates of elements in $st_L(1)$, hence the same holds for the first projection. The self-replicating property (Definition 4) follows by induction on the level. The level transitivity is an immediate consequence of the transitivity of L on the first level and the self-replicating property. □

Let

$$\begin{aligned} K &= \langle [a, b] \rangle^G, S = \langle [x, a] \rangle^L, \\ R &= \langle K, S, \gamma_2(L) \rangle^L = KS\gamma_2(L). \end{aligned}$$

These subgroups will play an important role in our further considerations.

Lemma 4. *We have the following inclusions: $\gamma_2(G) \supseteq \gamma_2(G) \times \gamma_2(G)$, $K \supseteq K \times K$, and $R \supseteq S \times S$.*

Proof. The first two inclusions are known [Gri89, Gri00].

Using the commutator relations and the fact that conjugation by a permutes the coordinates we have

$$\begin{aligned} (1, [c, x]) &= [(a, c), (1, x)] = [b, xa] = [b, a][b, x][[b, x], a] \in R, \\ (1, [x, b]) &= [(x^{-1}, x), (1, b)] = [[x, a], d] \in R, \end{aligned}$$

by Lemma 2,

$$(1, [a, x]) = a[(a, c), (x, 1)]a = a[b, (x, 1)]a = ab^{-1}(x^{-1}, 1)b(x, 1)a.$$

But $x = (1, x)a$ and $axa = (x, 1)a$ which leads to

$$(1, [a, x]) = ab^{-1}x^{-1}abaxa = ab^{-1}ab[b, axa] = [a, b][b, axa].$$

Now we have

$$[b, axa] = a[aba, x]a \in S,$$

and $[a, b] \in K$ which gives $(1, [a, x]) \in R$.

Finally

$$(1, [x, d]) = (1, [x, bc]) = (1, [x, c][x, b][[x, b], c]) = (1, [x, c])(1, [x, b])(1, [[x, b], c])$$

and

$$(1, [[x, b], c]) = [[[x, a], d], b] \in R$$

by Lemma 2. Therefore the elements $(1, [a, x])$, $(1, [b, x])$, $(1, [c, x])$, $(1, [d, x])$ belong to R and, as $S = \langle [a, x], [b, x], [c, x], [d, x] \rangle^L$, the lemma is proved. $\square$

Lemma 5. *We have the inclusion: $\gamma_2(L) \succeq \gamma_2(L) \times \gamma_2(L)$.*

Proof. Consider the subgroup $Q = \langle d, c, aca, xa \rangle \subset L$. As $d = (1, b)$, $c = (a, d)$, $aca = (d, a)$, $xa = (1, x)$, the group Q is a subdirect product in $D_4 \times L$ where $D_4 \simeq \langle a, d \rangle$ is a dihedral group of order 8. As $\gamma_2(D_4) = 1$ we get

$$\gamma_2(Q) = (1, \gamma_2(L)),$$

$$\gamma_2(aQa) = (\gamma_2(L), 1),$$

and therefore

$$\gamma_2(L) \succeq \gamma_2(L) \times \gamma_2(L). \quad \square$$

Lemma 6. *The group L is a weakly regular branch group over R .*

Proof. We know that $K \succeq K \times K$, $\gamma_2(G) \succeq \gamma_2(G) \times \gamma_2(G)$, $\gamma_2(L) \succeq \gamma_2(L) \times \gamma_2(L)$ and $R \succeq S \times S$. But R is generated by S , $\gamma_2(L)$ and K . This implies the statement. $\square$

In order to prove that L is a branch group, we consider its subgroup $P = \langle R, \langle x^4 \rangle \rangle^L$.

Lemma 7. *The group P has finite index in L .*

Proof. Every element $g \in L$ can be written as a product $g = x^i a^j c^k d^l h f x^{4t}$, where $h \in [G, G]$, $f \in S$, $i \in \{0, 1, 2, 3\}$, $j, k, l \in \{0, 1\}$, $t \in \mathbb{Z}$. This implies that the index of P in L is ≤ 128 . $\square$

Let $P_n \simeq P \times \cdots \times P \subset \text{Aut}(\mathcal{T})$ (2^n factors) be the subgroup of $\text{Aut}(\mathcal{T})$ that is the product of 2^n groups isomorphic to P that act on the corresponding 2^n subtrees rooted at the vertices on the n -th level.

Lemma 8. *The group L contains P_n for every n .*

Proof. For $n = 0$ the statement is obvious. For $n = 1$, let us consider $(xa)^4 = (1, x^4)$ which is an element of L . As L is self-replicating, for any given element $h \in L$ there exists an element k in L s.t. $k = (f, h)$. Conjugating $(1, x^4)$ by an element of L of the form (f, h) , we get that $(1, (x^4)^h) \in L$. But P is generated by conjugates of x^4 . This together with Lemma 6 proves the inclusion $1 \times P \succeq L$. The inclusion $P \times 1 \succeq L$ is obtained by conjugating L by a . Then we get that $P \times P = P_1 < L$.

In order to prove the lemma for $n = 2$ we observe that

$$L \ni [x, a] = (xa)^2 = (1, x^2) = (1, 1, x, x)_2$$

(the index 2 indicates that we rewrite the considered while considering its action on the second level; we will use such type of notations for further levels as well). Multiplying $(1, x^2)$ (which is in L) by

$$(1, [x, a]) = (1, 1, x^{-1}, x)_2,$$

we get $(1, 1, 1, x^2)_2 \in L$. Therefore $(1, 1, 1, x^4)_2 \in L$ and hence $P_2 < L$ (by level transitivity and the self-replicating property of L we see that $(x^4, 1, 1, 1)$, $(1, x^4, 1, 1)$ and $(1, 1, x^4, 1)$ also belong to L).

Let us prove the lemma by induction on $n \geq 2$. Suppose that, for every $k \leq n$, the inclusion $P_k < L$ holds and let us prove that $P_{n+1} < L$. Consider the element μ

$$L \ni \mu = (1, \dots, 1, x^4)_{n-2} = (1, \dots, 1, x^2, x^2)_{n-1},$$

As L is self-replicating, there exists an element $\rho \in St_G(u_{n-2})$, where u_{n-2} is the last vertex on the $(n-2)$ -th level, whose projection at this vertex is equal to b . We have

$$\begin{aligned} L \ni [\mu, \rho] &= (1, \dots, 1, [x^2, a], [x^2, c])_{n-1} \\ &= (1, \dots, 1, [x, a], [x, d])_n = (1, \dots, 1, x^{-1}, x, x^{-1}bx, b)_{n+1}, \end{aligned} \quad (3.1)$$

As $b^2 = 1$ we get the relation

$$[\mu, \rho]^2 = (1, \dots, 1, x^{-2}, x^2, 1, 1)_{n+1}$$

Now we have

$$\begin{aligned} L \ni \eta &= (1, \dots, 1, x^4, 1)_n = (1, \dots, 1, x^2, x^2, 1, 1)_{n+1}, \\ [\mu, \rho]^2 \eta &= (1, \dots, 1, x^4, 1, 1)_{n+1} \end{aligned} \quad (3.2)$$

and we come to the conclusion that $1 \times 1 \times \dots \times 1 \times P \times 1 \times 1 \succeq L$, hence $\underbrace{P \times \dots \times P}_{2^{n+1}} \succeq R$, and $P_{n+1} < L$, as L is level transitive. $\square$

We can now prove that L is a branch group. This group acts transitively on each level of the rooted tree $\mathcal{T}$, and contains P_n for every $n = 1, 2, \dots$. In order to prove that it is branch, as $P_n < \text{rist}_L(n)$, and L is level transitive, it is enough to check that P_n has finite index in L . We have the following diagram

$$\begin{array}{ccccccc} L & & & & & & \\ \uparrow & \psi_n & & & & & \\ st_L(n) & \twoheadrightarrow & \tilde{H} & < & L & \times \dots \times & L \\ \uparrow & & & & & & \\ \text{rist}_L(n) & & \uparrow & & \uparrow & \dots & \uparrow \\ \uparrow & \psi_n & & & & & \\ P_n & \twoheadrightarrow & \tilde{P}_n & = & P & \times \dots \times & P \end{array}$$

(the vertical arrows are inclusions, $\tilde{H}$ and $\tilde{P}_n$ are ψ_n images of $st_L(n)$ and P_n respectively, where ψ_n is the n -th iteration of ψ).

As the group P has finite index in L , we get that $\tilde{P}_n$ has finite index in $\tilde{H}$ and therefore P_n has finite index in $st_L(n)$ and hence in L . This establishes the first statement of Theorem 4.

The group L is the self-similar group generated by the states of the automaton in Figure 1. The diagram of this automaton satisfies the condition of Proposition 3.9.9 of [Nek05]: it is therefore a bounded automaton in the sense of Sidki [Sid00]. This proposition states that an automaton is bounded if and only if its Moore

diagram has the following property: every two nontrivial cycles are disjoint and are not connected by a directed path; a cycle is called trivial if all of its states represents the identity automorphism of the tree.

It is easy to see that automaton determining the group L satisfies this property.

By a theorem of Bondarenko and Nekrashevych (Theorem 3.9.12 in [Nek05]) every group generated by the states of a bounded automaton is contracting. Moreover, by a theorem of Bartholdi, Kaimanovich, Nekrashevych and Virag [BKNV06] such a group is amenable. This establishes the amenability of L , as well as its contracting property.

In order to compute the abelianization of L , we need to combine the contracting property of L with a rewriting process which corresponds to the embedding ψ . The combination of this rewriting process and the contraction property will produce an algorithm for solving the word problem in L : the branch algorithm. This type of algorithm appeared in [Gri84] for the first time: it is a general fact that the branch algorithm solves the word problem for contracting groups [Sav03].

The group

$$\Gamma = \langle a, b, c, d, x : a^2 = b^2 = c^2 = d^2 = bcd = 1 \rangle,$$

defined by generators and relations, naturally covers L . It is isomorphic to the free product

$$\mathbb{Z}/2\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}) * \mathbb{Z}.$$

Therefore, the elements in Γ are uniquely represented by words $w = w(a, b, c, d, x)$ in the reduced form (for this free product structure).

Similarly the group G is naturally covered by the group

$$\langle a, b, c, d : a^2 = b^2 = c^2 = d^2 = bcd = 1 \rangle \simeq \mathbb{Z}/2\mathbb{Z} * (\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}).$$

The elements in G can be represented by reduced words (with respect to this free product structure).

Let w be a word representing an element of Γ , $w = u_1 x^{i_1} u_2 x^{i_2} \dots u_k x^{i_k} u_{k+1}$, where u_i are reduced words in a, b, c, d , u_i is nonempty for $i \neq 1, k+1$, and $i_j \neq 0$, for $j = 1, \dots, k$.

Let us consider the following rewriting process:

- 1) In each word u_i replace b, c, d by the corresponding element of the wreath product $L \wr S_2$, using the defining relations $b = (a, c)$, $c = (a, d)$, $d = (1, b)$, $x = (1, x)a$.
- 2) Move all the letters a to the right using the relations $a(v_0, v_1) = (v_1, v_0)a$. Use the relation $a^2 = 1$ for simplification of words, and take the componentwise product of all involved pairs. One obtains in such a way a relation of the form $w = (w_0, w_1)a^\varepsilon$ with $\varepsilon \in \{0, 1\}$, which holds in L .
- 3) Reduce the words w_i in Γ , obtaining a pair (w_0, w_1) of reduced words.

Note that the length of w_i , $i = 0, 1$ is strictly shorter than of w if at least one letter a appears in the word w .

We can represent this rewriting process as a pair $\varphi = (\varphi_0, \varphi_1)$ (or a product $\varphi_0 \times \varphi_1$) of two rewritings $w \rightarrow w_0$ and $w \rightarrow w_1$. We will apply these maps to words with an even number of occurrences of a , i.e., words representing the elements in $st_L(1)$: in this case $\varepsilon = 0$. We can therefore iterate this rewriting procedure φ n times for words representing elements in $st_L(n)$, and get 2^n words $w_{i_1, \dots, i_n}$ with $i_j \in \{0, 1\}$. (For formal definition of φ_0, φ_1 in case of the group G see [Gri98a], for L the formal description is similar.)

Proposition 7. *The rewriting process is 3-step contracting with core $\mathcal{N} = \{1, b, c, d, x, x^{-1}, bx, cx, dx, x^{-1}b, x^{-1}c, x^{-1}d, x^{-1}bx, x^{-1}cx, x^{-1}dx\}$. In other words: for every word w representing an element in $stab_L(3)$, $\varphi^3(w)$ consists of 8 words $w_{i,j,k}$, $i, j, k \in \{0, 1\}$ of strictly shorter length than w .*

Proof. Let the word $w = w_1 x^{i_1} w_2 x^{i_2} \dots w_k x^{i_k} w_{k+1}$ be as above and represents an element in $stab_L(3)$. As we already have noted, if the letter a occurs in some of the w_i , then rewriting process is strictly shortening in one step. In order to study reduced words without the letter a , we will make use of the relations in Table 1.

Observe that w is a product of subwords in the form presented by the left side in the relations in Table 1, followed by an element of the set

$$\{1, b, c, d, xbx, cx, dx, xb, xc, xd, x^{-1}b, x^{-1}c, x^{-1}d, bx^{-1}, cx^{-1}, dx^{-1}\}.$$

In all relations marked by A or B the rewriting process gives shortening in one step (case A) or in two steps (case B); in the latter case note the presence of the letter a , which insures reduction of length in one more step.

If the word w is not shortened after applying twice the rewriting procedure, then either it belongs to N , or it is of the form $*x^{-1} * x \dots x^{-1} * x * t$, with $* \in \{b, c, d\}$ except for the first or last $*$ which may also represent the unit, and $t \in \{x^{-1}b, x^{-1}c, x^{-1}d, bx^{-1}, cx^{-1}, dx^{-1}\}$.

Let $x^{-1}bx = \tilde{b}$, $x^{-1}cx = \tilde{c}$, $x^{-1}dx = \tilde{d}$. These elements are of order two, and satisfy the relations $\tilde{b} = (\tilde{c}, a)$, $\tilde{c} = (\tilde{d}, a)$, $\tilde{d} = (\tilde{b}, 1)$. Since these relations are of the same form as the relations that hold for b, C and d , the group $\tilde{G}$ generated by $\langle a, \tilde{b}, \tilde{c}, \tilde{d} \rangle$ is isomorphic to G .

Let $A < L$ be the subgroup generated by $\langle b, c, d, \tilde{b}, \tilde{c}, \tilde{d} \rangle$. Note that A stabilizes the first level of the tree. Consider the embedding $\psi : A \rightarrow \tilde{G} \times G$ obtained by projecting the elements of A on the left and right subtrees (we use the same notation ψ for the embedding as before).

Lemma 9. *The group $\psi(A)$ is a subdirect product of finite index in $\tilde{G} \times G$.*

Proof. We have $\tilde{c} = (\tilde{d}, a)$, $c = (a, d)$, $\tilde{d} = (\tilde{b}, 1)$, $d = (1, b)$, $\tilde{b}\tilde{d} = \tilde{c}$ and $bd = c$. Therefore the projection of $\psi(A)$ on each of two factors is onto.

Let $B = \langle b \rangle^G$ and $\tilde{B} = \langle \tilde{b} \rangle^{\tilde{G}}$.

As $d = (1, b) \in \psi(A)$ and as for every $g \in G$ there exists some h s.t. $(g, h) \in \psi(A)$, we see that $(1, gb g^{-1}) \in \psi(M)$. Therefore the group $1 \times B$ is contained in $\psi(A)$ and, by a symmetric argument, $\tilde{B} \times 1$ is contained in $\psi(A)$. Thus $\tilde{B} \times B <$

$bx = (a, c)(1, x)a = (a, cx)a$	
$cx = (a, d)(1, x)a = (a, dx)a$	
$dx = (1, b)(1, x)a = (1, bx)a$	
$x^{-1}b = a(1, x^{-1})(a, c) = (x^{-1}c, a)a$	
$x^{-1}c = a(1, x^{-1})(a, d) = (x^{-1}d, a)a$	
$x^{-1}d = a(1, x^{-1})(1, b) = (x^{-1}b, 1)a$	
$xb = (1, x)a(a, c) = (c, xa)a$	<i>B</i>
$xc = (1, x)a(a, d) = (d, xa)a$	<i>B</i>
$xd = (1, x)a(1, b) = (b, x)a$	<i>A</i>
$bx^{-1} = (a, c)a(1, x^{-1}) = (ax^{-1}, c)$	<i>B</i>
$cx^{-1} = (a, d)a(1, x^{-1}) = (ax^{-1}, d)$	<i>B</i>
$dx^{-1} = (1, b)a(1, x^{-1}) = (x^{-1}, b)$	<i>A</i>
$xbx = (1, x)a(a, c)(1, x)a = (1, x)(c, a)(x, 1) = (cx, xa),$	<i>A</i>
$x^{-1}bx^{-1} = a(1, x^{-1})(a, c)a(1, x^{-1}) = (x^{-1}c, ax^{-1})$	<i>A</i>
$xbx^{-1} = (1, x)a(a, c)a(1, x^{-1}) = (1, x)(c, a)(1, x^{-1}) = (c, xax^{-1})$	<i>B</i>
$x^{-1}bx = a(1, x^{-1})(a, c)(1, x)a = (x^{-1}cx, a),$	<i>C</i>
$xcx = (1, x)a(a, d)(1, x)a = (1, x)(d, a)(x, 1) = (dx, xa)$	<i>A</i>
$x^{-1}cx^{-1} = a(1, x^{-1})(a, d)a(1, x^{-1}) = (x^{-1}, 1)(d, a)(1, x^{-1}) = (x^{-1}d, ax^{-1})$	<i>A</i>
$xcx^{-1} = (1, x)a(a, d)a(1, x^{-1}) = (1, x)(d, a)(x^{-1}, 1) = (dx^{-1}, xa)$	<i>A</i>
$x^{-1}cx = a(1, x^{-1})(a, d)(1, x)a = (x^{-1}dx, a)$	<i>C</i>
$xdx = (1, x)a(1, b)(1, x)a = (1, x)(b, 1)(x, 1) = (bx, x)$	<i>A</i>
$x^{-1}dx^{-1} = a(1, x^{-1})(1, b)a(1, x^{-1}) = (x^{-1}, 1)(b, 1)(1, x^{-1}) = (x^{-1}b, x^{-1})$	<i>A</i>
$xdx^{-1} = (1, x)a(1, b)a(1, x^{-1}) = (b, 1)$	<i>A</i>
$x^{-1}dx = a(1, x^{-1})(1, b)(1, x)a = (x^{-1}bx, 1)$	<i>C</i>
$x^2 = (x, x)$	<i>A</i>
$x^{-2} = (x^{-1}, x^{-1}).$	<i>A</i>

TABLE 1. Some relations in L

$\psi(A) < \tilde{G} \times G$. But the groups B and $\tilde{B}$ have finite index in G and $\tilde{G}$, respectively, and the lemma is proved. $\square$

We now finish the proof of Proposition 7. Consider a reduced word u which represents an element of L . Suppose that this element stabilizes the first level but is not shortened after applying twice the rewriting process. The word u has to be

of the form $u = wb$, where w represents an element of A and

$$t \in \{x^{-1}b, x^{-1}c, x^{-1}d, bx^{-1}, cc^{-1}, dx^{-1}\}.$$

Rewrite it as a word in the letters $\langle b, c, d, \tilde{b}, \tilde{c}, \tilde{d} \rangle$. Use the relations $\tilde{b} = (\tilde{c}, a)$, $\tilde{c} = (\tilde{d}, a)$, $\tilde{d} = (\tilde{b}, 1)$, $b = (a, c)$, $c = (a, d)$, $d = (1, b)$ to rewrite it as an element $(\tilde{w}_0, w_1)$ of $\tilde{G} \times G$.

Recall that, endowed with its natural system of generators, the group G is one step contracting with core $\mathcal{N}_0 = \{1, b, c, d\}$ (and contracting coefficient $\frac{1}{2}$ [Gri84]). In other words, applying the rewriting procedure to reduce a word v in a, b, c, d with an even number of occurrences of the letter a yields a couple a words of length $\leq 1/2|v|$ unless $v \in \{1, b, c, d\}$. More precisely, if $v \rightarrow (v_0, v_1)$ is obtained by rewriting in the group G , then $|v_i| \leq |v|/2 + 1$.

By isomorphism the same property is true for a reduced word in the alphabet $a, \tilde{b}, \tilde{c}, \tilde{d}$ determining an element in $\tilde{G}$ (and the core in this case is $\tilde{\mathcal{N}}_0 = \{1, \tilde{b}, \tilde{c}, \tilde{d}\}$).

Split the word w as a product of monads $*$ and triads $x^{-1} * x$. If there are at least two monads or at least two triads we get after rewriting shortening at each of coordinates. The remaining case is the case of a word of the form $x^{-1} * x *$ and $*x^{-1} * x$ for which one checks that reduction of length occurs in the second step.

This completes the proof of Proposition 7. $\square$

From this proposition we get an algorithm to solve the word problem: the branch algorithm for L . Let us describe it further.

Let w be a word in the letters a, b, c, d, x . The problem is to check if $w = 1$ in L . The notation $w \equiv_L w'$ means that the two elements of L defined by the words w and w' are equal.

1) Reduce w in Γ . If w is the empty word, then in L , $w \equiv_L 1$. If it is not the empty word, compute the exponent $\exp_a w$ (that is the sum of exponents of a in w). Check if this number is even. If NO, then $w \not\equiv_L 1$. If YES go to 2).

2) Rewrite w as a pair (w_0, w_1) using the rewriting map $\varphi = (\varphi_0, \varphi_1)$. Apply 1) successively to w_0, w_1 and follow steps 1) and 2) alternatively. Either, at some step one obtains a word with odd $\exp_a$ or (after n steps) one obtains that all 2^n words represent the identity element in Γ (observe that the word problem in Γ is solvable by using the normal form for elements).

Note that $w \equiv_L 1 \Leftrightarrow (w_0 \equiv_L 1 \text{ and } w_1 \equiv_L 1)$. Applying this procedure 3 times yields either the answer NO (the element is not the identity) or a set of 8 words $w_{i,j,k}$ with $i, j, k \in \{0, 1\}$ which - by Proposition 7- are strictly shorter than w . This algorithm solves the word problem.

Lemma 10. *Let w be a word in the generators. Let $w \equiv_L (w_0, w_1)\alpha$, $\alpha = a$ or $\alpha = 1$ depending on the parity of the exponent $\exp_a w$, and the triple $(w_0, w_1), \alpha$ is obtained from w by applying once the rewriting process described above. Then*

$$\exp_x(w) = \exp_x(w_0) + \exp_x(w_1).$$

Proof. The rewriting process uses the relations $b = (a, c)$, $c = (a, d)$, $d = (1, b)$ and $x = (1, x)a$, $x^{-1} = (x^{-1}, 1)a$ which do not change the total exponent of x . The reduction in group Γ also doesn't change the exponent. $\square$

Lemma 11. *The abelianization $L/[L, L]$ is infinite. The image of x in $L/[L, L]$ is of infinite order.*

Proof. Any element in the commutator group can be expressed as a product of commutators $[u, v]$. Choosing the words in a, b, c, d, x representing u and v , we get that any element in $[L, L]$ can be written as a word w with $\exp_x w = 0$. Suppose that for some $n \geq 1$, $x^n \in [L, L]$. We get a word $w = x^n \Pi[u_i, v_i]$ in the letters a, b, c, d, x with total exponent n for x which represents the identity element in L . Choose w of minimal length with this property. Applying the rewriting process at most 3 times to w , we get a set of 8 words w_{ijk} , $i, j, k \in 0, 1$ representing the identity element in L with the sum of exponents of the symbol x different from zero. Hence at least one of them has non zero $\exp_x$. The words w_{ijk} are shorter than w , a contradiction. $\square$

The proof of Lemma 11 completes the proof of Theorem 4.

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Thomas Delzant
 Département de mathématiques
 Université de Strasbourg
 7 rue Descartes
 F-67084 Strasbourg
 e-mail: delzant@math.u-strasbg.fr

Rostislav Grigorchuk
 Department of Mathematics
 Texas A & M University
 MS-3368, College Station, TX, 77843-3368
 USA
 e-mail: grigorch@math.tamu.edu