A finiteness property on monodromies of holomorphic families.

Thomas Delzant
IRMA, Université de Strasbourg et CNRS

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Abstract. Let $X$ be a compact Kähler manifold, and $g$ a fixed genus. Due
to the work of Parshin, Arakelov and Caporaso, it is known that there are only
a finite number of non isotrivial holomorphic families of Riemann surfaces of
genus $g \geq 2$ over $X$. We prove that this number only depends on the funda-
mental group of $X$. Our approach uses asymptotic group theory (limit groups,
$\mathbb{R}$–trees), and Gromov-Shoen theory.

1 Introduction

Let $X$ be a compact, connected Kähler manifold. A holomorphic family of Rie-
mann surfaces of genus $g$ over $X$ is a pair $(Y, \pi)$ where $Y$ is a complex manifold
and $\pi : Y \to X$ a holomorphic submersion whose fibers are Riemann surfaces of
genus $g$ (we assume $g \geq 2$). It is called non isotrivial if the family of Riemann
surfaces $Y_s = \pi^{-1}(s)$ is not constant in the moduli space of Riemann surfaces.
A holomorphic family of Riemann surfaces determines a monodromy, which is a
homomorphism $\varphi$ from the fundamental group $\pi_1(X, s_0)$ to the mapping class
group $M(S)$ of the topological surface underlying $Y_{s_0}$, the fiber at the point $s_0$.

A fundamental result due to Parshin and Arakelov ([Ar], [Par]) and answer-
ing a question of Shafarevich asserts that given a Riemann surface $B$ the set
of families of given genus over $B$ is finite (for a proof based on the study of
Teichmüller spaces, see [Im-Sh], or [McM]). Another consequence of the study
of [Ar] is that the number of non isotrivial families over a projective manifold
$X$ can be bounded in terms of this manifold.

A uniform result has even been described by L. Caporaso [Ca] who proved
that the Hilbert polynomial of a complex surface which is a non singular bundle
of genus $g \geq 2$ over a base of genus $p \geq 2$ can only take a finite number of
values. One consequence is the following. Given a surface $\Sigma_p$ of genus $p$ and a
topological surface $S$ (of genus $g$), the cardinality of the set of homomorphisms
from the fundamental group of $\Sigma_p$ to the mapping class group of $S$ which
can be realized as a monodromy is finite up to conjugacy at the target and automorphism at the source.

In this article, we want to give a bound in terms of the fundamental group $\Gamma$ of the base $X$, independent of the manifold $X$. Before stating our main result let us recall some definitions.

A finitely presented group $\Gamma$ is Kähler if it can be realized as the fundamental group of a compact Kähler manifold. The group $\Gamma$ fibers if there exist a topological 2-orbifold $\Sigma$ of hyperbolic type together with a surjective homomorphism $\pi : \Gamma \to \pi_1^{orb}(\Sigma)$ whose kernel is finitely generated. This is equivalent to the fact that every compact Kähler manifold $X$ with fundamental group $\Gamma$ admits a holomorphic map with connected fibers on a complex hyperbolic 1-orbifold whose underlying topological orbifold is $\Sigma$ (see paragraph 2.1). Analogously, one says that the family $Y$ over $X$ factorizes through a curve $B$ if there exist a Riemann orbifold $B$ and a map $q : X \to B$ so that $Y$ is the pull-back of a family over $B$. We shall see that this property only depends on the monodromy of the family and not on the manifold $X$. The main result of this article is:

**Theorem 1** Let $\Gamma$ be a Kähler group and $S$ a topological surface. There exists only finitely many conjugacy classes of homomorphisms $\varphi : \Gamma \to M(S)$ which can be realized as the monodromy of a holomorphic family of Riemann surfaces on some Kähler manifold with fundamental group $\Gamma$, but do not factorize through a curve.

Combining this result with the case of curves ([Ca]) one obtains that the number of non isotrivial families over a Kähler manifold $X$ can be bounded in terms of its fundamental group (see Corollary 3 in Paragraph 4.4).

In fact, Theorem 1 appears as a special case of a general factorization theorem for actions of Kähler groups on Gromov-hyperbolic spaces (see Corollary 1 in Paragraph 3.3).

Let $H$ be a hyperbolic space in the sense of Gromov ([Gr]), and $G$ a subgroup of the group of isometries of $H$. If $\Gamma$ is a finitely generated group, one can study infinite sequences of non elementary homomorphisms from $\Gamma$ to $G$ with an asymptotic method (sometimes called the Bestvina-Paulin method). Let $\Sigma$ be a fixed set of generators of $\Gamma$. The energy of the homomorphism $h$ is $e(h) = \min_{x \in H} \max_{s \in \Sigma} d(x, h(s)x)$. An infinite sequence of pairwise non conjugate homomorphisms is diverging if $\limsup e(h_n) = +\infty$. After choosing some ultrafilter, infinite sequences of diverging energy converge to an action of $\Gamma$ on some $\mathbb{R}$-tree : the asymptotic cone of $H$ (see Paragraph 3). Due to the fundamental work of Gromov-Shoen [Gr-Sh], one deduces that the Kähler group “fibers”, i.e. admits an epimorphism $\pi$ to a 2-orbifold group with finitely generated kernel. In many important cases, one further proves that for infinitely many integers $n$, $h_n$ factorizes through $\pi$; this is the case in particular if the action of $G$ on $H$ is acylindrical in the sense of Bowditch [Bo], and even weakly acylindrical (see Paragraph 3.3).
Unfortunately this simple method cannot be applied directly to the mapping class group acting on the complex of curves (which is a hyperbolic space due to the work of Masur and Minsky [Ma-Mi]) as there is no reason that an infinite sequence of homomorphisms to the mapping class group acting on the complex of curves has diverging energy. The whole machinery of Bestvina, Bromberg, and Fujiwara [Be-Br-Fu] which creates a proper action of the mapping class group on a finite product of hyperbolic spaces will be needed together with the work of Drutu Berhstock and Sapir [Be-Dr-Sa] on the asymptotic geometry of this group (see paragraph 3.4).

In the first paragraph we review some basic results concerning Kähler groups acting on \( \mathbb{R} \)-trees, in the second paragraph we recall the Bestvina-Paulin method to construct actions on \( \mathbb{R} \)-trees and we introduce the notion of a limit group associated to an infinite sequence. This theory is applied to the study of Kähler groups. In the last paragraph these results are combined with the description of the asymptotic geometry of the mapping class group to study monodromies.

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2 Kähler groups and \( \mathbb{R} \)-trees

2.1 2-Orbifolds and their fundamental groups

Let us recall some basic facts about 2 – orbifolds, their fundamental groups and geometric structures as introduced by W. Thurston ([Th] Chap. 13).

An (oriented) 2-orbifold is a topological (oriented) compact surface \( S \) endowed with a finite set of marked points \( \{(p_1, m_1),\ldots,(p_k, m_k)\} \), where \( m_i \) is an integer greater than 2. We shall denote it by \( \Sigma = \{S; (p_1, m_1),\ldots,(p_k, m_k)\} \).

For each point \( p_i \), let \( \gamma_i \) be a small simple loop which is the boundary of a small embedded disc \( D_i \) centered at \( p_i \) and chosen so that the disks \( (D_i)_{1 \leq i \leq k} \) are disjoints.

The fundamental group of \( \Sigma \) is defined as a quotient :

\[
\pi_1^{\text{orb}} = \pi_1(S\setminus\{p_1,\ldots,p_k\})/ \langle \gamma_1^{m_1},\ldots,\gamma_k^{m_k} \rangle.
\]

The Euler characteristic of \( \Sigma \) is \( \chi(\Sigma) = 2 - 2g - k + \sum_{1 \leq i \leq k} \frac{1}{m_i} \). A 2-orbifold is called of hyperbolic type if its Euler characteristic \( \chi(X) \) is strictly negative.

If \( \Lambda \) is a co-compact lattice in \( \text{PSL}(2, \mathbb{R}) \), the quotient of the unit disk \( D \) by \( \Lambda \) has naturally a structure of a 2-orbifold : \( D/\Lambda \) is a topological oriented surface \( S \) and modulo \( \Lambda \), only finitely many points \( (p_i)_{1 \leq i \leq k} \) have a non trivial isotropy group, which is a finite cyclic subgroup of order \( (m_i)_{1 \leq i \leq k} \) of \( \Lambda \). If \( \Sigma = \{S; (p_i, m_i)\} \) is the underlying orbifold, one proves that \( \Lambda = \pi_1^{\text{orb}}(\Sigma) \). A hyperbolic structure on \( \Sigma \) is a realization of its fundamental group as a co-compact lattice in \( \text{PSL}(2, \mathbb{R}) \).
A complex structure on $\Sigma$ is a complex structure on $S$, marked at the points $p_i$, and the uniformization theorem for 2-orbifolds implies that there is a one-to-one correspondence between hyperbolic and complex structures on general 2-orbifolds.

Let $Y$ be a complex structure on $\Sigma$. In order to define holomorphic maps with values in $Y$, let us choose for every $i$ a small closed disk $D_i$ around $p_i$. Let $G_i : (\tilde{D}_m, p_i) \to (D_i, p_i)$ be the $m_i$-th cover ramified at the origin. If $X$ is a complex manifold, a map $f : X \to \Sigma$ is called holomorphic if it is holomorphic in the usual sense in $X - f^{-1}\{1 \leq i \leq n\}$, and if for every $i$ and every point $x$ such that $f(x) = p_i$, the map $f$ admits a local holomorphic lift through the map $G_i$. In other words, $f - p_i$ is locally the $m_i$-th power of a holomorphic map.

This enables us to endow $Y$ with the Kobayashi metric which coincides with the hyperbolic structure.

The following result is due to F. Catanese ([Cat] Thm 5.14). We propose below a different proof based on the work of Carleson and D. Toledo [Ca-To].

**Theorem 2** Let $\Gamma$ be the fundamental group of a compact Kähler manifold $X$ and $\Lambda$ the fundamental group of a compact 2-orbifold $\Sigma$ of hyperbolic type. The following are equivalent:

i. There exists a surjective homomorphism $\psi : \Gamma \to \Lambda$ with finitely generated kernel.

ii. There exists a complex structure $Y$ on $\Sigma$ together with a holomorphic map $X \to Y$ with connected fibers.

**Proof** Let us choose some discrete co-compact action of $\Lambda$ on the unit disk, with orbifold quotient $\Sigma_{aux}$ (an auxiliary hyperbolic structure on $\Sigma$). The theory of harmonic maps (developed by J. Carlson and D. Toledo [Ca-To]) proves that there exists an equivariant harmonic map from the universal cover of $X$ to $D$ which leads to a differentiable map $f$ from $X$ to $\Sigma_{aux}$. Using a Bochner formula, one proves that the map $f$ is pluriharmonic and that the connected components of the fibers of $f$ are complex hypersurfaces. The set of connected components of fibers of $f$ is a complex orbifold $Y$ whose singular points are the multiple fibers, with their multiplicity. There exists therefore a continuous map $q : Y \to \Sigma$ that induces a surjective homomorphism on fundamental groups. In order to prove that this homomorphism is an isomorphism, we adapt the argument of F. Catanese. Let us consider the image of $\ker \psi$ in $\pi_{1i}^{\text{orb}}(Y)$. As $\ker \psi$ is f.g. and $\psi' : \Gamma \to \pi_{1i}^{\text{orb}}(Y)$ is onto, the image of $\ker \psi$ in $\pi_{1i}^{\text{orb}}(Y)$ must be trivial or have finite index (finitely generated normal subgroups in 2-orbifold groups have finite index). This group, which is the kernel of $q_*$, cannot have finite index as $q_* : \pi_1(\Sigma') \to \pi_1(\Sigma)$ is onto, thus it is trivial and $q$ is an isomorphism.

Using the fact that holomorphic maps are 1-Lipschitz for the Kobayashi metric (which is the hyperbolic metric on a hyperbolic 2-orbifold), one proves the following proposition (see [De] Thm. 2 or [Co-Si] for a purely algebraic proof).
Proposition 1 Let $X$ be a compact complex manifold. There exist only finitely many pairs $(Y_i, F_i)$ where $Y_i$ is a hyperbolic/complex 2-orbifold and $F_i$ a holomorphic map from $X$ to $Y_i$. □

This suggests the following definition (see [ABCKT] Chapter 2, Paragraph 3).

Definition 1 A Kähler group $\Gamma$ fibers if there exist a 2-orbifold of hyperbolic type $\Sigma$ and a surjective homomorphism $\Gamma \to \pi_1(\Sigma)$ whose kernel is finitely generated.

We wish to emphasize that this definition only depends on $\Gamma$ and not on the choice of a Kähler manifold with $\Gamma$ as a fundamental group. Combining the previous two theorems, one gets the following :

Proposition 2 Let $\Gamma$ be a Kähler group. There exist a finite family of pairs $(\Sigma_i, \pi_i)_{1 \leq i \leq p}$ of 2-orbifolds of hyperbolic type and surjective homomorphisms $\pi_i : \Gamma \to \pi_1(\Sigma_i)$ with finitely generated kernel, such that for every Kähler manifold $X$ with fundamental group $\Gamma$, every hyperbolic orbifold $Y$ and every holomorphic map $F : X \to Y$ with connected fibers, there exists an integer $i$ such that the underlying orbifold of $Y$ is isomorphic to $\Sigma_i$ by an isomorphism inducing $\pi_i$ on the fundamental groups. □

2.2 Trees

Recall that an $\mathbb{R}$–tree is a connected geodesic metric space which is 0–hyperbolic (see [Be2] for an introduction to this subject).

An $\mathbb{R}$–tree, endowed with an action of a group $\Gamma$ is called minimal if it does not contain an invariant subtree. If the group $\Gamma$ is finitely generated, such a minimal subtree exists and is unique. The action is called non-elementary if it is neither elliptic (fixing a point) nor axial (fixing a line but no point on this line) nor parabolic (fixing a unique point at infinity).

The main example of a Kähler group acting on an $\mathbb{R}$–tree is the fundamental group of a Riemann surface (or 2-orbifold) endowed with a holomorphic quadratic differential $\omega$: the $\mathbb{R}$–tree is the set of leaves of the real part of $\omega$.

After the fundamental work of Gromov-Schoen [Gr-Sh] several authors ([Ko-Sh], [Su]) studied actions of Kähler groups on $\mathbb{R}$–trees. The following theorem summarizes the situation.

Theorem 3 Let $\Gamma$ be a Kähler group acting on an $\mathbb{R}$–tree $T$. Assume that $T$ is minimal and is not a line. Then $\Gamma$ fibers. Moreover, there exist a 2-orbifold $\Sigma$ of hyperbolic type, a surjective map $\pi : \Gamma \to \pi_1(\Sigma)$ with finitely generated kernel, an action of the fundamental group of $\Sigma$ on an $\mathbb{R}$–tree $T'$, and a $\pi$–equivariant map $T' \to T$. If the action of $\Gamma$ on $T$ is faithful, then $\pi$ is an isomorphism.
If \( T \) is a simplicial locally finite tree, the proof is explained in [Gr-Sh]; the general case is sketched in Paragraph 9.1 of the same article. Let us recall the main steps of the proof.

Let \( X \) be a compact Kähler manifold with fundamental group \( \Gamma \). One constructs a \( \Gamma \)-equivariant harmonic map \( h \) from the universal cover of \( X \) to the tree \( T \) with finite \( \Gamma \)-energy ([Gr-Sh], [Ko-Sh]). The regularity of this harmonic map ([Gr-Sh]) has been detailed by [Su]; in particular, its set of singular points \( X_{\text{sing}} \) has codimension 2 and the image of a connected fundamental domain in \( X \) appears to be a finite tree (the convex hull of a finite number of points). At this point one can copy the argument of [Gr-Sh]: using the Kähler structure, one proves that the map \( h \) is pluriharmonic: outside from \( X_{\text{sing}} \) it is locally the real part of a holomorphic function, and there exists a holomorphic quadratic differential \( \omega \) on \( X \) such that locally \( \pm dh = \omega \). Using the fact that the tree is not a line one proves that one leaf of the foliation defined by \( \omega \) on \( X \) is singular (entirely contained in the set \( \omega = 0 \)) hence compact. One deduces that all leaves are compact and \( X \) fibers on some hyperbolic 2-orbifold \( \Sigma \) with fibers the leaves of \( \omega \). Thus \( \omega \) comes from \( \Sigma \) and if \( T_{\omega} \) stands for the \( \mathbb{R} \)-tree dual to the leaves of \( \omega \), the harmonic map \( h \) factorizes through the \( \pi_1(\Sigma) \)-equivariant map from \( D \) to \( T_{\omega} \). □

The case of axial actions of Kähler groups is also quite useful (see [De2]).

Theorem 4 Let \( \Gamma \) be a Kähler group with an isometric action on a line (isometric to \( \mathbb{R} \)). If the kernel of this action is not finitely generated, the group \( \Gamma \) fibers; more precisely, there exist a 2-orbifold of hyperbolic type and a surjective morphism with finitely generated kernel \( \pi : \Gamma \to \pi_1^{\text{orb}}(\Sigma) \) such that the action factorizes through \( \pi \).

Let \( X \) be a compact manifold with fundamental group \( \Gamma \). In the case of an oriented line, such an action is defined by a holomorphic form and [De2] applies. In the general case, the action defines a holomorphic quadratic differential \( \omega \) which becomes a differential on a (perhaps) ramified double cover. All the leaves of this form are compact, hence all the leaves of \( \omega \) are compact and the same argument as in Theorem 3 completes the proof. □

3 Limit groups and \( \mathbb{R} \)-trees as limit spaces

In numerous cases, actions of groups on \( \mathbb{R} \)-trees are obtained by a limit process of actions on hyperbolic spaces, as observed by Bestvina [Be1] and Paulin [Pa].

3.1 Limit spaces and limit groups.

Let \( \Gamma \) be a finitely generated group, \( (H_n, x_n, d_n) \) a sequence of pointed metric spaces, and \( \varphi_n : \Gamma \to \text{Isom}(H_n) \) a sequence of isometric actions of \( \Gamma \) on \( H_n \).

In order to study the asymptotic behavior of this sequence of actions, let us fix a non principal ultrafilter \( \omega \) on \( \mathbb{N} \). Recall that \( \omega \) is a subset of the set of
infinite subsets of $\mathbb{N}$ such that, if $A, B \in \omega$ then $A \cap B \in \omega$, the complementary $F^c$ of every finite set $F$ is in $\omega$, and $\omega$ is maximal for these properties. One says that a family of propositions $(P_n)_{n \in \mathbb{N}}$ is true $\omega$-almost surely if the subset of $\mathbb{N}$ such that $P_n$ is true belongs to $\omega$. If $(u_n)_{n \in \mathbb{N}}$ is a sequence in a Hausdorff topological space, one says that $l$ is an $\omega$–limit of $(u_n)$ and writes $l = \lim_\omega u_n$, if for every neighborhood $U$ of $l$, the set $\{n \in \mathbb{N} : u_n \in U\}$ is an element of $\omega$.

Recall that in a compact Hausdorff space every sequence has a unique $\omega$–limit. In particular every bounded sequence of real numbers has a limit in $\mathbb{R}$ and every sequence of positive numbers a $\omega$–limit in $\mathbb{R} \cup \{+\infty\}$. The following definitions are useful to describe the asymptotic behavior of a family of actions.

**Definition 2** The space $\Pi^{\text{bounded}}_H H_n$ is the subset of the usual product $\Pi_{n \in \mathbb{N}} H_n$ made with sequences $(y_n)_{n \in \mathbb{N}}$ such that $\lim_\omega d(x_n, y_n) < \infty$.

The limit space $H_\omega = \lim_\omega (H_n, d_n, x_n)$ is the quotient of the set $\Pi^{\text{bounded}}_H H_n$ by the equivalence relation $\lim_\omega d_n(x_n, z_n) = 0$.

The space $\lim_\omega H_n$ has a natural base-point namely the equivalence class of $(x_n)$ and a natural distance $d_\omega$ defined by $d_\omega(y, z) = \lim_\omega d(y_n, z_n)$. It is a complete metric space. If for every element $g$ in a generating set of $\Gamma$, the sequence $d_n(x_n, \varphi_n(g)x_n)$ is bounded, the group $\Gamma$ acts isometrically on $H_\omega$ by the obvious action, as $\Gamma$ acts on the product and on $\Pi^{\text{bounded}}_H H_n$.

**Definition 3** The stable kernel of $(\varphi_n)_{n \in \mathbb{N}}$ is the subgroup $K_\omega$ of $\Gamma$ defined as $K_\omega = \{g \in \Gamma / \{n / \varphi_n(g) = e\} \in \omega\}$ and the limit group $\Gamma_\omega$ is the quotient $\Gamma / K_\omega$.

By definition, the limit group $\Gamma_\omega$ acts on the limit space $H_\omega$. The following proposition emphasizes the importance of finiteness properties of kernels of actions.

**Proposition 3** Let $K$ be a finitely generated subgroup contained in $K_\omega$. Then for $\omega$–almost every integer $n$, $K \subseteq \ker(\varphi_n)$ the kernel of the action of $\varphi_n$.

**Proof** In fact, if $\sigma = \{a_1, \ldots, a_n\}$ is a finite set of generators of $K$, for every $i$ the set $A_i = \{n \in \mathbb{N} / \varphi_n(a_i) = e\}$ is an element of $\omega$. The finite intersection $\cap A_i$ is therefore an element of $\omega$. □

**Remark.** This definition of a “limit group” extends the definition of Sela [Se], which deals with limit groups for sequences of homomorphisms of a given group $\Gamma$ to a fixed free (or hyperbolic) group. The explicit use of an ultrafilter simplifies the definition of [Se]; with our terminology if $(\Gamma, \sigma)$ is a finitely generated group and $\varphi_n : \Gamma \to F_r$ is a sequence of pairwise non conjugate homomorphisms to the free group of rank $r$, the limit group associated to $\varphi_n$ is the limit group associated to the sequence $(\varphi_n, a_n, \frac{1}{\varepsilon_n} T)$, where $T \supset F_r$ is the Cayley graph of the free group, $e_n = \min_{a \in F} \sup_{s \in \sigma} d(a, \varphi_n(s)a)$, $a_n$ is a point where this minimum is achieved, and $\frac{1}{\varepsilon_n} T$ is the tree $T$ with the renormalized distance $d/e_n$. 

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3.2 Hyperbolic metric spaces and $\mathbb{R}$-trees

Let $H$ denote a $\delta$-hyperbolic space (in the sense of Gromov [Gr]), $\Gamma$ a finitely
generated group, $\sigma$ a fixed set of generators of $\Gamma$, and $\varphi : \Gamma \to \text{Isom}(H)$ an
isometric action.

The energy of a point $x$ is defined by $e(x) = \sup_{y \in \sigma} d(sx, x)$, and the energy
of the action is the minimum $e(\varphi) = \min_{x \in H} e(x)$. If the action of $\Gamma$ on $H$ is not
elementary (neither elliptic, nor parabolic, nor loxodromic), then the function
$e(x)$ is metrically proper (it goes to infinity with the distance of $x$ to a fixed
base-point). Then, the set $\{x \in H : e(x) \leq e(h) + 1\}$ is not empty and has
bounded diameter. If $H$ is furthermore a $\text{CAT}(-1)$ complete metric space, the
minimum is in fact achieved, as $e$ is a convex (along geodesics) metrically proper
function. (See [Be2]).

The main result concerning limits of actions of a group acting on a hyperbolic
space is due to M. Bestvina and F. Paulin see ([Be1], [Be2], [Ko-Sh], [Pa]).

**Theorem 5** Let $\Gamma$ be a finitely generated group, $(H,d)$ a hyperbolic space and
$\varphi_n : \Gamma \to \text{Isom}(H)$ be a sequence of actions whose energy $e_n$ goes to infinity.
Let $x_n$ be chosen so that $e_n(x_n) \leq e_n + 1$, and $d_n$ be the renormalized distance:
$$d_n = \frac{d}{e_n}.$$ Then the limit space $\limomega(H,d_n,x_n)$ is a complete $\mathbb{R}$-tree $H_\omega$, the
action of $\Gamma$ on $H_\omega$ has energy 1, and the minimum of energy is achieved at the
origin $x_\omega$ of $H_\omega$.

Proof. (Compare [Be2], [Ko-Sh], [Gr2] 2.B.b). Recall that a complete metric
space with a base point $x$ is hyperbolic (resp. geodesic) if $\forall y,z,t$, $<y,z> = 1/<y,t>$,
and $d(y,z) = 1/2d(y,z)$ (resp. $\forall y,z \exists m/d(y,m) = d(m,z) = 1/2d(y,z)$). These two properties are
formulated in the first order language of metric spaces: a finite number of
quantifiers and inequalities only involving distances. Therefore, they behave
nicely through ultra-limits. As $\limomega d_n = 0$, the limit space is a 0–hyperbolic
geodesic space, hence an $\mathbb{R}$-tree. Using the finiteness of $\sigma$, one proves that
$e(x_\omega) = 1$. Let $x'_\omega$ be the minimum of energy of the limit action and assume that
$e(x'_\omega) = \alpha < e(x_\omega) = 1$. The point $x'_\omega$ is defined by a sequence of points $x_n'$, with
$\sup_{x_n \in \Sigma} d_n(sx_n', x_n') = f_n$. As $\lim f_n = \alpha < 1$, for almost all $n$, $f_n < 1+\alpha e_n < e_n$, contradicting the definition of $e_n$. \boxdot

As the energy of the limit action is 1, this action cannot be elliptic: an
elliptic action has a fixed point. The limit tree contains therefore a unique
minimal invariant subtree which will be denoted by $T_\omega \subset H_\omega$.

The following two definitions, acylindricality (due to Bowditch [Bo]) and weak
acylindricality are useful to study the kernel of limit action of $\Gamma_\omega$ on the tree $H_\omega$.

**Definition 4** The action of $G$ on a $\delta$-hyperbolic space $H$ is acylindrical, if
there exists an integer $N$ and a real $K$ such that for every pair of points $a,b$
with $d(a,b) \geq K$ the set $\{g \in G : d(ga,a) + d(gb,b) \leq 1000\delta\}$ is finite with
cardinality $\leq N$. 

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Let \( G \) be a group acting on a \( \delta \)-hyperbolic space \( H \). The translation length \([g]\) of an element \( g \) is \([g] = \text{Min}_{x \in H} d(x,gx)\). Let \( g \) be some element with translation length \( \geq 100\delta \). Then \( g \) is hyperbolic: if \( x_0 \) realizes the minimum \( \text{Min}_{x \in H} d(x,gx) \), the union of geodesic segments \( L_g = \bigcup_{n \in \mathbb{Z}} g^n[x_0, gx_0] \) is a \([g]\)-local geodesic and respectively converges as \( n \) tends to \(+\infty\) and \(-\infty\), to the fixed points \( g^+ \) and \( g^- \) of \( g \) at infinity, (see for instance [Co-De-Pa], Chap. 9 with \( 6\delta \) instead of \( 100\delta \)). We consider \( L_g \) as a \([g]\)-local geodesic oriented by the action of \( g \). The set \( L_g \) is called a quasi-axis of \( g \). Note that a different choice of minimizing point or a different choice of geodesic segment \([x_1, gx_1]\) will give another quasi-axis \( L'_g \). Using standard properties of Gromov hyperbolic spaces, one sees that the Hausdorff distance between \( L_g \) and \( L'_g \) is bounded above by \( C\delta \) where \( C \) is a universal constant (for instance 10).

Let \( E(h) \) be the elementary group generated by \( h \), i.e. the subgroup \( E(h) = \{g \in G / g \{h^-,h^+\} = \{h^-,h^+\}\} \), where \( \{h^-,h^+\} \) are the two fixed points of \( h \) in the boundary of \( H \).

**Definition 5** The action of \( G \) on a \( \delta \)-hyperbolic space is weakly acylindrical if, there exists a real \( D > 0 \) such that for every hyperbolic element \( g \), and every \( h \notin E(g) \), the diameter of the projection of \( hL_g \) to \( L_g \) is bounded by \( D(1 + [g]) \).

This definition of "weak acylindricity" is a uniform version of the WWPD property introduced by [Be-Br-Fu] to study the action of the mapping class group on some hyperbolic spaces: the WWPD property requires that the same set \( \{x \in L_g / d(x,hx) \leq 100\delta \} \) is bounded by a constant depending on \( g \). Note that this definition assumes hyperbolicity of the space, whereas acylindricity does not.

**Proposition 4** If the action is acylindrical, then it is weakly acylindrical.

Let \((K,N)\) be the two constants involved in the definition of acylindricity, and \( \Delta \) be the maximum of the diameter of the sets \( \{a \in L_g / \exists a \in hL_g / d(a,a') \leq 10\delta \} \) for \( h \notin E(g) \). Assume that \( \Delta > N([g] + 1000\delta) + K \). Then \( L_g \) contains a segment \( I = [a,b] \) of length \( \Delta \) such that \( \exists a', b' \subset hL_g \) with \( d(a,a') \leq 10\delta, d(b,b') \leq 10\delta \). Note that for every \( c \in [a,b] \) there exists \( c' \in [a',b'] \) with \( d(c,c') \leq 12\delta \). Let \( I_1 = [a_1,b] \subset [a,b] \) with \( d(a,a_1) = [g] \). As \( g \) acts as a translation of length \([g]\) on \( L_g \), its conjugate \( hgh^{-1} \) acts as a translation of the same length on \( hL_g \). Therefore, if \( h_1 = h^{-1}ghg^{-1}, \) for every \( x \in I_1, d(h_1x,x) \leq 20\delta \). Replacing \( g \) by \( g^k, k \leq K \), and defining \( a_k \) as the point of \([a,b]\) with \( d(a_k,a) = k[g] \), and \( h_k = h^{-1}g^khg^k \), we see that if \( \Delta - N[g] - 100\delta > 0 \), then \( d(a_k,b) > K \), and for every \( k \), we have \( d(h_ka_k,a_k) \leq 20\delta \) and \( d(h_kb_k,b_k) \leq 20\delta \). Therefore, for two different indexes, \( k, l, h^{-1}g^khg^{-k} = h^{-1}g^lhg^{-l} \), or \( [h,g^{k-l}] = 1 \), hence \( h \in E(g) \).

**Example 1** Let \( G \) be a group acting discretely on some simply connected manifold with pinched negative curvature \( X \), \( \delta \) its hyperbolicity constant, and \( \mu \) its
Proposition 5. Let \( (H,d) \) be a hyperbolic space, \( G \) a group of isometries of \( H \), \( \Gamma \) a finitely generated group and \( \varphi_n : \Gamma \to G \) a sequence of actions whose energy \( e_n \) goes to infinity. Let \( x_n \) be chosen so that \( e(x_n) \leq e_n + 1 \), let \( H_\omega = \lim_\omega (H_n, d_n, x_n) \) be the limit tree and \( T_\omega \subset H_\omega \) the minimal invariant subtree. Let \( N^\omega \) the stable kernel and \( \Gamma^\omega \) the limit group of this sequence.

1. Assume that the action of \( G \) on \( X \) is weakly acylindrical. Let \( K \subset N^\omega \subset G \) be a finitely generated subgroup. Then for almost every \( n \), \( \varphi_n(K) \) is contained in \( E(\varphi_n(h)) \), the maximal elementary subgroup containing some hyperbolic element \( \varphi_n(h) \) of \( G \).

2. If \( T_\omega \) is not a line, for infinitely many \( n \), the group \( \varphi_n(K) \) is an elliptic subgroup of \( G \).

3. Assume that the action of \( G \) on \( X \) is acylindrical. Then the kernel of the action of \( \Gamma_\omega \) on \( T_\omega \) is finite if \( T_\omega \) is not a line, or virtually abelian if \( T_\omega \) is a line.

The proof is very similar to the one of [Be2], Prop. 3.10.

As \( \Gamma \) is finitely generated, and as its action on \( T_\omega \) is not elliptic, it contains an element \( g \) whose image in \( \Gamma_\omega \) is an hyperbolic isometry of \( T_\omega \). Let \( L \subset T_\omega \) be the \( g \)-invariant line, and \( [g]_\omega \) its translation length. Let us first note that \([g]_\omega = \lim_\omega [\varphi_n(g)]_n \). Indeed, if \( m_n \in H_n \) is a midpoint of a segment \([x_n, \varphi_n(g)x_n]\), it is known that \([\varphi_n(g)] \leq d_n(m_n, \varphi_n(g)m_n) \leq [\varphi_n(g)] + 10\delta_n \). Note that \( d_n(x_n, \varphi_n(g)x_n) \) is bounded, therefore \( m_n \) converges to a point which is a midpoint of \([x_\omega, g, x_\omega] \), hence on \( L \). If \( y_n \in L_{\varphi_n(g)} \) is such that \( d_n(x_n, y_n) \) is bounded, the same argument proves that \( \lim_\omega y_n \in L \). As \( \lim_\omega [\varphi_n(g)]_n = [g]_\omega \), for \( \omega \)-almost every \( n \), \([\varphi_n(g)]_n \gg \frac{1}{\delta_n} = \delta_n \), the hyperbolicity constant of \( H_n \). Note that \( g \) acts on its axis \( L_n \) as a translation of length \([\varphi_n(g)]_n \). Let \( h \in \Gamma \), and assume that \( \varphi_\omega(h) = 1_{\Gamma_\omega} \). Let us prove that for \( \omega \)-almost \( n \), \( \varphi_n(h) \in E(\varphi_n(g)) \). Let \( \alpha, \beta \) be two points on \( L \) at a distance \( > D([g]_\omega + 1) + 1 \). These points are \( \omega \)-limits of points \( a_n, b_n \) or of \( \varphi_n(g)a_n, \varphi_n(g)b_n \). In particular, \( d(a_n, \varphi_n(g)a_n) = \alpha(e_n), d(b_n, \varphi_n(g)b_n) = \alpha(e_n), \) but \( d(a_n, b_n) > D([\varphi_n(g)]_n) + 1 + e_n \). Let \( a'_n \) and \( b'_n \in [a_n, b_n] \) be the points at distance \( e_n \) from \( a_n \) and \( b_n \) respectively. Then for \( \omega \)-almost every \( n \), the distance between the point \( a'_n \) (respectively \( b'_n \)) and \( \varphi_n(g)[a_n, b_n] \) is bounded above by \( 10\delta_n \), and \( \varphi_n(h) \in E(\varphi_n(g)) \). This proves Point 1.

If the action is acylindrical, the subgroup generated by the commutators...
\(\varphi_n([h, g])\) is finite for \(n \gg 1\) and the kernel of the action \(\varphi_n\) is therefore virtually abelian.

If \(T_\omega\) is not a line, for some \(u\), \(\Lambda_{uhu^{-1}} = h_\omega(u)\Lambda_g \neq \Lambda_g\), and for the same reason, if \(H\) is a f.g. subgroup of the kernel of the action, \(\varphi_n(H) \subset E(\varphi_n(uhu^{-1}))\).
The group \(\varphi_n(H)\) is contained in two elementary subgroups generated by \(\varphi_n(h)\) and \(\varphi_n(uhu^{-1})\) and thus is elliptic, in particular finite in the acylindrical case.

\[\square\]

### 3.3 Kähler groups.

Let us keep the notations of Section 3.2 unchanged, but assume now that \(\Gamma\) is a Kähler group. Applying Theorem 3 (or 4 in the axial case) and the result of the previous section one gets a first factorization result.

**Corollary 1** Let \(\Gamma\) be a Kähler group; let \(G\) be a group of isometries of a hyperbolic space \((H, d)\), \(h_n : \Gamma \to G\) a sequence of actions whose energy \(\epsilon_n\) tends to infinity, and \(x_n\) be chosen so that \(\epsilon_n(x_n) \leq \epsilon_n + 1\). Let \(X_\omega = \lim\omega(\frac{1}{\epsilon_n}X_n, x_n)\), and \(T_\omega \subset X_\omega\) be the minimal invariant subtree.

If \(T_\omega\) is not a line, then \(\Gamma\) fibers. There exist a 2-orbifold \(\Sigma\) and a surjective homomorphism \(\pi : \Gamma \to \pi_{orb}^1(\Sigma)\) with finitely generated kernel \(N\). The same is true if \(T_\omega\) is a line, but the kernel of the action of \(\Gamma\) on this line is not finitely generated.

If furthermore, the action of \(G\) on \(H\) is acylindrical, then for infinitely many \(n\), \(h_n(\ker \pi)\) is a finite group if \(T_\omega\) is not a line or is a line but the kernel is infinitely generated. Therefore \(h_n\) factorizes, via \(\pi\), through a finite extension of \(\pi_{orb}^1(\Sigma)\).

If the action is weakly acylindrical, then for infinitely many \(n\), the group \(h_n(N)\) is contained in the elementary subgroup of a hyperbolic element of \(G\).

**Example 2** This corollary implies a compactness result on non elementary discrete actions on a simply connected manifold with pinched non positive curvature, for instance the complex or quaternionic hyperbolic space. Let \(G = SU(n, 1)\) or \(Sp(n, 1)\). As \(\Gamma\) is finitely generated the set \(\text{Hom}_{\text{dense, disc}}(\Gamma, G)/\text{conj}\) of conjugacy classes of Zariski dense representations of \(\Gamma\) to \(G\) is endowed with a natural topology, such that the set of representations of bounded energy is compact. Our corollary applies. Thus, the subset of representations which do not factorize through the fundamental group of a 2-orbifold is a compact subset of \(\text{Hom}_{\text{disc}}(\Gamma, G)\). Note that, in the case of \(SO(n, 1)\) this set is even known to be empty (see [Ca-To]).

### 3.4 Projection complexes.

Let us review the construction of M. Bestvina, K. Bromberg and K. Fujiwara [Be-Br-Fu] of a projection complex of \(\delta\)-hyperbolic metric spaces.
Let \( Y \) be a set of \( \delta \)-hyperbolic geodesic metric spaces; note that \( \delta \) is uniform for \( Y \in Y \). For each metric space \( Y \in Y \), a function \( \pi_Y \) is given, called the projection, from \( Y - \{ Y \} \) to the set of subsets of \( Y \) of diameter \( \leq \theta \). Here, \( \theta, \delta \) are two fixed constants. One extends this function \( \pi_Y \) on \( Y \) by setting \( \pi_Y(x) = x \) and on the union of \( X \in Y - \{ Y \} \), by setting \( \pi_Y(x) = \pi_Y(X) \). We assume that the function \( d^p_Y(X, Z) = \text{diam}\{\pi_Y(X) \cup \pi_Y(Z)\} \) endows \( Y \) with the structure of a projection complex in the sense of [Be-Br-Fu], i.e., satisfies two further axioms:

- for every \( A, B, C \) a projection, from \( Y \in Y \) and on the union of \( X \in Y - \{ Y \} \), by setting \( \pi_Y(x) = \pi_Y(X) \). We assume that the function \( d^p_Y(X, Z) = \text{diam}\{\pi_Y(X) \cup \pi_Y(Z)\} \) endows \( Y \) with the structure of a projection complex in the sense of [Be-Br-Fu], i.e., satisfies two further axioms:

If \( K \) is a given positive constant, let \( Y_K(X, Z) = \{ Y/d_Y(X, Z) > K \} \), and let \( P_K(Y) \) be the graph whose vertex set is \( Y \) and where two vertices are joined by an edge if \( Y_K(X, Z) = \emptyset \).

The first important result of [Be-Br-Fu] (Thm 3.16) is that for sufficiently large \( K \), this metric space is a quasi-tree, i.e., a metric space that is quasi-isometric to a tree. In particular, it is a Gromov-hyperbolic space. It appears to be \( \beta \)-hyperbolic for a uniform constant \( \beta \), twice the “bottleneck constant”: for every pair of points \( a, b \) and \( c \in [a, b] \), every continuous path between \( a \) and \( b \) contains a point \( c' \) with \( d(c, c') \leq 2 \). The exact value of the hyperbolicity constant \( \beta \) is not meaningful, but it is important that it remains constant as \( K \) increases; as the bottleneck constant of \( P_K(Y) \) is 2, we can choose \( \beta = 10 \).

Then [Be-Br-Fu] constructs a quasi-tree of metric spaces \( C(Y) \): this metric space is obtained by taking the disjoint union of the metric spaces \( C(Y) \), for \( Y \in Y \), and adding an edge of length \( L \) from every point of \( \pi_X(Z) \) to every point in \( \pi_Y(Z) \), if \( [X, Z] \) is an edge of \( P_K(Y) \). In this new metric space, the subsets \( C(Y) \) are totally geodesically embedded subspaces. As all metric spaces \( Y \in Y \) are uniformly \( \delta \)-hyperbolic, if \( K \) is large enough and \( L \) is correctly chosen between \( K \) and \( 2K \), there exists a constant \( \Delta \) such that the metric space \( C(Y) \) is \( \Delta \)-hyperbolic [Be-Br-Fu] (Thm 4.17). Moreover this number \( \Delta \) is independent of the choice of \( K \). If necessary, we will write \( C_K(Y) \) to remember the dependence on the parameter \( K \).

Assume that \( G \) is a group acting on \( Y \), preserving the maps \( \pi \), and such that for every \( Y \in Y \) and \( g \in G \), there is map \( F_g : \cup C(Y) \to \cup C(Y) \), which for each \( Y \) restricts to an isometry \( F_g : C(Y) \to C(g(Y)) \), and such that for all \( g, h \) \( F_{g,h} = F_g F_h \). Then, the group \( G \) acts isometrically on \( C(Y) \).

The following definition, the \( \nu \)-invariant, is useful to describe elementary subgroups of certain hyperbolic groups and study iterated small cancelation groups or Burnside-like groups (compare [Co]). The finiteness of \( \nu \) can substitute the existence of the Margulis constant when there is no lower bound on the curvature (as in Example 3.11).

**Definition 6** Let \( G \) act on a set \( Y \). One says that \( \nu(G, Y) \leq \nu \) if the following property holds. Let \( h, g \in G \). Suppose that \( h, ghg^{-1}, g^2hg^{-2}, \ldots, g^n hg^{-n} > \)
Proposition 6 Assume that there exists a constant $D$ such that for every $Y \in \mathcal{Y}$, the action of the stabilizer $G^Y$ on $C(Y)$ is $D$–weakly acylindrical and assume that $\nu(G,Y) \equiv \nu$ is finite. Then, for $K$ large enough, the action of $G$ on $C_K(Y)$ is weakly-acylindrical.

In order to apply the results of [Be-Br-Fu], we will study the action of $G$ on the metric space $C_K^∗(Y)$, where $K^∗$ will be defined later. Recall that $\Delta$ is a fixed common hyperbolicity constant for the metric spaces $C_K^∗(Y)$ provided $K^∗ \geq K$. Let $g \in G$ be an element acting on $C_K^∗(Y)$ and assume that $g$ is a hyperbolic isometry. Up to replacing $g$ by some power, we may assume that the translation length $[g]$ of $g$ is large compared to the hyperbolicity constant: $\min_{x \in C_K^∗(Y)} d(x, gx) = [g]_{C_K^∗(Y)} \geq 10^4 \Delta$.

In order to prove this proposition, we will distinguish two very different cases in the next two sections (3.4.1 and 3.4.2 respectively). If the isometry $g$ is hyperbolic for its action on $P_K^∗(Y)$, we will see that it has an axis when acting on $P_K$. If it is elliptic or parabolic, then it fixes a point in $Y$. Note that both cases have been already discussed in [Be-Br-Fu], and our discussion very closely follows this work.

3.4.1 Elliptic or parabolic case.

Recall that for all $K$ large enough, the graph $P_K(Y)$ is $10$–hyperbolic.

Lemma 1 If $g$ is elliptic element for its action on $P_K^∗(Y)$, then $g$ fixes a point in $Y$.

As $P_K^∗(Y)$ is $10$-hyperbolic, for every $N_0$, one can find a point in this graph (i.e an element $Y_0 \in Y$), such that $d(Y_0, g^n Y_0) \leq 100$ for all $1 \leq n \leq N_0$ (see [Co-De-Pa] Chap. 9). Let $n$ be a fixed integer ($n$ will be chosen later) and consider a geodesic path of length $\leq 100$ between $Y_0$ and $g^n Y_0$. This path can be written as a sequence of adjacent vertices $Y_0 = X_0, X_1, \ldots, X_k = g^n X_0$ with $k \leq 100$. For every $i \in \{0, 1, \ldots, k-1\}$, one chooses a point $x_i^+$ in $\pi_{X_i}(X_{i+1})$ (the diameter of this set is $\leq \theta$) and for $i \in \{1, \ldots, k\}$, a point $x_i^−$ in $\pi_{X_{i-1}}(X_i)$. One sets $x_0^+ = g^n x_0$, and for every $i \in \{1, \ldots, k\}$, one considers a geodesic segment $[x_{i−1}, x_i^+]$ in $X_i$. Note that, for every $i$, $d(x_i^+, x_{i+1}^−) \leq L^*$, and that $[x_i^+, x_{i+1}^−]$ is an edge of $P_K^∗(Y)$. We consider the path $\gamma = [x_0^+, x_1^−] \cup [x_1^+, x_2^+] \cup \ldots \cup [x_k^−, x_{k+1}^+]$. We remark that if $g^n Y_0 = Y_0$, this path is entirely contained in $Y_0$.

Applying [Co-De-Pa] 3.1.5 page 25, one gets that in $C_K^∗(Y)$ any geodesic $[x_0^+, g^n x_0]$ remains in the $(\ln 2 k)\Delta$-neighborhood (or $10\Delta$ as $k \leq 100$) of this path $\gamma$.

Let $y_0$ be a point where the function $d(y, gy)$ is minimal, equivalently $[g] = d(y_0, g y_0)$. By minimality, the $g$–invariant set $L_g = \cup_{k \in \mathbb{Z}} g^k [y_0, g y_0]$ is a $[g]$–local geodesic and is therefore $100\Delta−\text{quasi-convex}$ ([Co-De-Pa] Prop. 10.3.1).
Let $p$ be a projection of $x_0$ on this set ($p$ realizes the minimal distance of $x$ to $L_g$). Then $q = g^np$ is at the distance $\geq n([g] - 100\Delta)$ of $p$. Therefore the segment $[q, g^aq]$ of $L_g$ contains a subsegment $[a, b]$ of length $\geq \frac{n([g] - 100\Delta) - L^*}{K}$ which is $10\Delta$-close to a segment $[x^-, x^+]$. Let $A$ be some positive number. Choosing $n$ large enough, one may assume that $d(a, b) \geq [g] + A$ and find an arbitrary long segment (of size $A$) in some metric space $X_i$ whose image by $g$ is $10\Delta$-close to $X_i$. Choosing $A > \theta + 1000\Delta$ (and $n$ consequently), we see that the projection of $gX_i$ to $X_i$ has an arbitrary large diameter; by definition of a projection complex, the point $X_i$ is fixed by $g$. This proves the lemma 1. □

Now, in order to study the action of $g$ on $C_K(Y)$ we can copy the argument of [Be-Br-Fu], Prop. 4.20. The metric space $X_i$ is a totally geodesic subspace of $C_K(Y)$ fixed by $g$, and $L_g \subset X_i$ is a $g$-invariant $[g]$-local geodesic. If the diameter of the projection of $h.L_g$ on $L_g$ is greater than $\theta$, then $h$ must fix $X_i$. This implies that $E(g) \subset G_{X_i}$ and the $D$-acylindricity of the action of $G_{X_i}$ on $X_i$ implies that if $h \notin E(g)$ but $h \in G_{X_i}$, the diameter of the projection of $h.L_g$ to $L_g$ is bounded by $D(1 + [g])$.

### 3.4.2 Hyperbolic case.

Let us now assume that $g$ is a hyperbolic isometry for its action on $P_K$, and let $E(g)$ be the elementary subgroup containing $g$. Recall that an axis of $g$ is a $g$-invariant subset of a bi-infinite geodesic (joining the two fixed points of $g$ at infinity). It is contained in a bi-infinite geodesic, but might be a proper subset.

If a point $X$ belongs to every axis between the two fixed points at infinity, we call it stable ([Be-Br-Fu]). The set of stable points (if not empty) is an axis, invariant by $E(g)$. We call it the stable axis of $g$ (if it is not empty); if $X_0$ belongs to the stable axis of $g$, if $X_n = g^nX_0$ and $[X_0, X_1]$ is a fixed geodesic segment, the union $\cup_{n \in \mathbb{Z}} g^n[X_0, X_1]$ is a $g$-invariant axis. The following lemma is very similar to Lemma 3.22 from [Be-Br-Fu]: an element acting on $P_K$, which is “hyperbolic enough” has an axis for its action on $P_K$.

**Lemma 2**

1. In the metric space $P_K$, the isometry $g$ has a stable axis $\alpha$.

2. If furthermore $h \in G$ is such that the diameter of the projection of $h.\alpha$ on $X$ is greater than $(k + 1)[g] + 1000$, then this stable axis contains a point $X'_k$ such that for every $p \in \{0, \ldots, k\}$, the commutator $h' = hg^{-1}h^{-1}g$ fixes $g^pX'_k \in \alpha$.

3. If furthermore $k \geq \nu(G, Y)$, then $h \in E(g)$.

As $g$ is hyperbolic for its action on $P_K$, up to replacing $g$ by some power, one can assume that its translation length $d = \text{Min}_Y d(Y, gY)$ is greater than $10^5$ ($10^4$ times the hyperbolicity constant). Let $X_0$ be chosen so that in the graph $P_K$, $d(X_0, gX_0)$ is minimal and let $[X_0, gX_0]$ be a geodesic segment between these points. Let $[X_0, X_1] = \{X_0 = Y_0, Y_1, \ldots, Y_d = X_1\}$. By isometry, if $Y_{-1} = g^{-1}Y_{-1}$, $\{X_{-1} = Y_{-d}, Y_{-d+1}, \ldots, Y_{-1}, Y_0 = X_0\}$ is also a geodesic segment
between $X_{-1} = g^{-1}X_0$ and $X_0$. Let $X_n = g^nX_0$, and $Y_{n,d+k} = g^uY_k$. Note that by minimality the sequence $(Y_j)_{j \in \mathbb{Z}}$ is a $d$-local geodesic path in our graph. Let $k \leq \frac{n}{2}$. As $d(Y_{-k}, Y_k) = 2k$, the set $Y_{k^*}(Y_{-k}, Y_k) = \{Z/d_Z(Y_{-k}, Y_k) \geq K^*\} \cup \{Y_{-k}, Y_k\}$ is another (injective) path of length $\geq 2k$ ([Be-Br-Fu] proof of Prop. 3.7).

By the 10-bottleneck property ([Be-Br-Fu] Theorem 3.16), this path contains a point $X'_0$ which is 10-close to $X_0$. Now $d(X'_0, Y_i) \geq l - 10 \geq 3$ if $|l| \geq 13$. Therefore, applying ([Be-Br-Fu] Prop. 3.14) to the path $\{Y_1, Y_{l+1}, \ldots, Y_n\}$ and the vertex $X'_0$, one obtains that if $13 \leq l \leq n$, then $dX'_0(Y_i, Y_n)$ as well as $dX'_0(Y_{l-1}, Y_{n})$ is bounded by a constant $c_0$ depending only on $\theta$. By the triangle inequality, if $l, m \geq 13$, then $dX'_0(Y_{l-1}, Y_{m}) \geq K^* - 2c_0$. In particular if $K^*$ is sufficiently large, then for every $p, q \geq 1$, $dX'_0(g^{-p}X_0, g^qX_0) > K'$, where $K'$ is the constant introduced in Lemma 3.18 of [Be-Br-Fu]. Similarly, let $l, q > 200$ and consider a pair of points $\{A_{-p}, A_q\}$ such that $d(A_{-p}, Y_p) \leq 100$ and $d(A_q, Y_q) \leq 100$. Let $[A_q, Y_q]$ and $[A_{-p}, A_q]$ be two geodesic paths. Then the paths $\{Y_{-i}, Y_{l+1}, \ldots, Y_p\} \cup \{A_{-p}, Y_{-p}\}$ and $\{Y_1, Y_{l+1}, \ldots, Y_q\} \cup \{A_q, Y_q\}$ remain at a distance at least 3 from $X'_0$. Moreover if $K^*$ is sufficiently large then $dX'_0(A_{-p}, A_q) > K'$.

Lemma 3.18 of [Be-Br-Fu] shows that in $P_K(Y)$ for every $p, q > 0$, the point $X'_0$ belongs to every geodesic between $g^{-p}X_0$ and $g^qX_0$, and even between two points $A_{-p}, A_q$ as before.

By equivariance, for every integer $n$, the point $g^nX'_0$ belongs to every geodesic between $g^{-l}X_0$ and $g^mX_0$ if $l, m > 2n$, and every geodesic segment $[A_{-l}, A_m]$ between two points $A_{-l}, A_m$ such that $d(A_{-l}, g^mX_0) \leq 100$. All the points $g^nX'_0$ are therefore on a $g$-invariant geodesic $\alpha$. In fact, the same argument shows that for any geodesic $\beta$ parallel to $\alpha$, the point $X'_0$ belongs to $Y_{K^*}^\nu(\beta)$. Thus Cor. 3.19 from [Be-Br-Fu] proves that $X'_0$ belongs to every geodesic $\beta$ which is parallel to $\alpha$: the isometry $g$ has a stable axis, and Point 1. is established.

For the second assertion, note that the hypothesis proves that one can find two points $P, Q$ on $\alpha$ such that the distance between $P$ or $Q$ and the geodesic $ho\alpha$ is bounded by 100, but $d(P, Q) > (\nu + 1)|g| + 200$. Let $k \leq \nu$. The length of the segment $[P, Q]$ being $>(k + 1)|g| + 200$, $[P, Q]$ contains at least $k + 1$ consecutive translates of $X'_0$. Reordering them, we may assume that these points are $X'_0, gX'_0, \ldots, g^kX'_0$. By hyperbolicity, these points are $10$-close to some points $Z_0, Z_1, Z_k$ on $h\alpha$, chosen in such a way that $d(Z_i, Z_i+1) = |g|$. Thus $h^{-1}Z_i \in \alpha$, and $gh^{-1}Z_i = h^{-1}Z_i+1$, as $d(h^{-1}Z_i, h^{-1}Z_i+1) = |g|$. Since $d(Z_0, X'_0) \leq 100$ and $d(Z_k, g^2X'_0) \leq 100$, every geodesic between $Z_0$ and $Z_k$ must go through $g^kX'_0$. Replacing $Z_0$ by a point $Z'_0 \in ho\alpha$ such that $d(Z'_0, gX'_0) = |g|$, and setting $Z'_0 = hg^{-1}h^{-1}Z_0$ we now have $Z'_i = g^iX'_0$ for $i \in \{1, \ldots, k\}$. Let $h' = hg^{-1}h^{-1}g$. For every $i \in \{0, \ldots, k-1\}$, $h'g^iX'_0 = hg^{-1}h^{-1}g^{k+i+1}X'_0 = hg^{-1}h^{-1}Z'_{i+1} = Z'_i = g^iX'_0$. The points $(g^iX'_0)_{0 \leq i \leq k-1}$ are fixed by $h'$. The second assertion is proved.

By definition of the $\nu$-invariant, if $k \geq \nu$, the isometry $h'$ fixes $g^nX'_0$ for all $n$. Therefore $h' = hgh^{-1}g^{-1} \in E(g)$, hence $hgh^{-1} \in E(g)$, and $h \in E(g)$. The
apply the results of Paragraph 3, and in particular Corollary 1. Br-Fu] gives a new geometrization of the mapping class group which enables to does not diverge. The work of M. Bestvina, M. Bromberg, and K. Fujiwara [Be-

study monodromy groups we will need a deeper result and consider sequences in Hom(Γ, M(S))/conj which are infinite but whose energy when acting in X(S) does not diverge. The work of M. Bestvina, M. Bromberg, and K. Fujiwara [Be-

pairs of curves that can be made disjoint by a homotopy.

4.1 Asymptotic geometry of the mapping class group

4 Mapping class group, complex of curves and holomorphic families of Riemann surfaces

Let S be a cionpact topological surface (possibly with a boundary); the complex of curves (and arcs) of S, denoted by X(S) (or simply X if only one surface is involved) is the graph whose vertices are the homotopy classes of simple closed loops on S which are non parallel to the boundary and arcs from the boundary to the boundary which are not homotopic to the boundary and whose edges are pairs of curves that can be made disjoint by a homotopy.

A fundamental result due to H. Masur and Y. Minsky asserts that X is hyperbolic [Ma-Mi]; this statement has been improved by B. Bowditch [Bo] who proved that the action of M(S) on X is acylindrical. Therefore, in principle, the results in paragraph 3 can be applied to this example. However, in order to study monodromy groups we will need a deeper result and consider sequences in Hom(Γ, M(S))/conj which are infinite but whose energy when acting in X(S) does not diverge. The work of M. Bestvina, M. Bronnberg, and K. Fujiwara [Be-

third assertion and the lemma are proved. □
Theorem 6 (Be-Br-Fu) The mapping class group \( M(S) \) of a compact surface \( S \) contains a subgroup of finite index \( M_1(S) \) which admits a product action on a finite product of hyperbolic spaces \( X = \Pi_{i \in I} X_i \). Furthermore the orbit map \( M(S) \to X \) is a quasi-isometric embedding.

The space \( X \) is not hyperbolic, however any of its asymptotic cones is a finite product of \( \mathbb{R} \)-trees.

The spaces \( X_i \) constructed by [Be-Br-Fu] are projection complexes of complexes of curves. In order to apply the factorisation theorem of Paragraph 3 (Cor. 1) we have to check that these actions are weakly acylindrical. In fact, the metric spaces \( X_i \) are defined as \( C(Y_i) \), where \( Y_i \) is a family of connected subsurfaces of \( S \) under the action of a finite index subgroup \( M_1(S) \) of \( M(S) \), and the metric space attached to \( Y \) is the complex of curves and arcs \( C(Y) \) of the surface \( Y \). As Bowditch proved that the action of the mapping class group of \( Y \) on \( C(Y) \) is acylindrical [Bo], in order to check the weak acylindricity of the action of \( M_1(S) \) on this space, it is enough to prove that the \( \nu \) invariant for the action of \( M(S) \) on the set \( Y \) of isotopy classes of connected subsurfaces of \( S \) if finite.

Lemma 3 The action of the mapping class group on the set of isotopy classes of connected subsurfaces of \( S \) has a finite \( \nu \)-invariant.

Let \( g, h \) be two elements in \( M(S) \) such that \( h \) fixes the non empty connected surfaces \( Y_0, g^{-1}Y_0, g^{-2}Y_0, \ldots, g^{-k}Y_0 \). Let us consider \( Y_1, \ldots, Y_c \) the pieces of the Nielsen decomposition of \( S \) associated to \( h \) (the orbits under \( h \) of connected components of the Nielsen decomposition) and \( I \subset \{ 1, \ldots, c \} \) the components on which \( h \) is homotopic to the identity. For every \( i \), the surface \( g^{-i}Y_0 \) is contained in a unique \( Y_i \), say in \( Y_{f(i)} \). As \( h \) fixes \( g^{-i}Y_0 \), this subsurface \( Y_{f(i)} \), must be connected. Then either the restriction of \( h \) to \( Y_{f(i)} \) is the identity (\( f(i) \in I \)), or it is pseudo-Anosov and in this case, \( g^{-k}Y_0 = Y_c \) is fixed by \( h \). Let us assume that \( k > c \). Then either for two different indices \( i, j \leq k \), \( g^{-i}Y_0 = g^{-j}Y_0 = Y_c \) and \( g^{-i}Y_0 = Y_0 \), or if \( k \) is sufficiently large (greater than \( \Sigma_{1 \leq i \leq c} - \xi(Y_i) + 1 \), where \( \xi \) the Euler characteristic), \( g^{-k}Y_0 \) must be contained in \( \bigcup_{0 \leq i < k, f(i) \in I} g^{-i}Y_0 \). Therefore \( g^{-(k+1)}Y_0 \) is contained in a set on which \( h \) induces the identity. □

4.2 Infinite sequences of homomorphisms.

We want to explain how to modify the argument of 3.2 to create actions on \( \mathbb{R} \)-trees from an infinite sequence of pairwise non-conjugate homomorphisms of a finitely generated group \( (\Gamma, \Sigma) \) to the mapping class group \( M(S) \). At this point one could directly apply the work of J. Behrstock, C. Drutu and M. Sapir [Be-Dr-Sa] Cor.6 from the appendix, however as we need to get more information on the kernel of the action on the limit tree, we prefer to deduce our result from [Be-Br-Fu].
In order to apply our Corollary 1 to this action, we recall a fundamental result of Ivanov [Iv] concerning the mapping class group: a finitely generated subgroup of $M(S)$ which is not reducible must contain a pseudo-Anosov element. The following lemma follows.

**Lemma 4** Let $G \subset M(S)$ be a f.g. irreducible group and $N \subset G$ an infinite f.g. normal subgroup. Then $N$ contains a pseudo-Anosov.

Proof. If $N$ contains no pseudo-Anosov and is finitely generated, then its action on the complex of curves is elliptic and the set of curves preserved by $N$ is bounded (for instance because of the acylindrity of the action). As $N$ is normal, this set is preserved by $G$ which is therefore elliptic acting on this complex. Applying Ivanov’s theorem, we get that $G$ must be reducible. □

Now, let $\Gamma$ be a group and $h_n : \Gamma \to M(S)$ be an infinite sequence of pairwise non conjugate homomorphisms. As the set $I$ defined in Thm. 6 is finite, $\Gamma$ admits a finite index subgroup $\Gamma_1$ such that for every $n$, $h_n(\Gamma_1) \subset M_1(S)$. Thus, the restriction of $h_n$ to $\Gamma_1$ does not permute the factors $X_i$. Let us fix a generating system $\Sigma$ of $\Gamma$. Simultaneously, we fix a generating system of $M(S)$ and denote by $|\gamma|$ the word length of an element $\gamma$.

Let $\alpha_0 \in X$ be a fixed base-point in $X$. It is known ([Be-Br-Fu]) that the orbit map $\psi : M(S) \to X$ defined by $\psi(\gamma) = \gamma.\alpha_0$ is a quasi-isometric embedding. In other words, there exists a constant $K$ such that: $K^{-1}|\gamma| - K \leq d(\alpha_0, \gamma.\alpha_0) \leq K|\gamma|$. On the other hand, the family $(h_n)$ is pairwise non conjugate, therefore the sequence $E_n = \min_{\gamma \in M(S)} \max_{s \in \Sigma} |\gamma^{-1}h_n(s)\gamma|$ is unbounded, as well as the sequence $e_n = \min_{\gamma \in M(S)} \max_s d(h_n(s).\gamma.\alpha_0, \gamma.\alpha_0)$. Indeed we have $K^{-1}E_n - K \leq e_n \leq KE_n$. Up to conjugating $h_n$, we may assume that this minimum is achieved at $\gamma = 1$.

Let $C = \lim_n \frac{1}{\omega} Ca(M(S)), e$ be the asymptotic cone of the mapping class group associated to this diverging sequence. The orbit map (for some fixed origin in $\Pi_{i \in I} X_i$) induces an equivariant bilipshitz embedding of $C$ in a product of trees $\Pi_{i \in I} T_i$, with $T_i = \lim_n \frac{1}{\omega} X_i$.

**Proposition 7** Assume that for all $n$, the group $h_n(\Gamma)$ is neither reducible nor virtually abelian. Let $N$ be a f.g. normal subgroup contained in the kernel of the action of $\Gamma$ on $T_i$; then for infinitely many $n$, $h_n(N)$ is finite. Moreover $\Gamma$ has a finite index subgroup $\Gamma_1$ such that the restriction of $h_n$ to $\Gamma_1$ factorizes through $\Gamma_1/N$.

Proof. The first fact (for some $i$, the action of $\Gamma$ on $T_i$ is not elliptic) directly follows from [Be-Dr-Sa], Section 6, Thm. 6.2, who proved that the orbit of the limit group $\Gamma_\omega$ on $C$ is unbounded. As the embedding of $C$ in this product is bilipshitz (thus metrically proper) it cannot project onto bounded orbits. As $\Gamma$ is finitely generated, we can find an element $h \in \Gamma_1$ and an index $i$ such $\lim_\omega(h_n(h))$ is hyperbolic and acting on $X_i$, $h_n(h)$ is hyperbolic for $\omega$-almost
all $n$. Due to the weak acylindricity property, $h_n(N) \subset E(h_n(h))$. If $h_n(h)$ is reducible, then so is $E(h_n(h))$. Applying Lemma 1, we get a contradiction. Therefore $h_n(h)$ is a pseudo-Anosov. In addition, $E(h_n(h))$ is virtually $\mathbb{Z}$ and cannot contain an infinite normal subgroup of $h_n(G)$, unless $h_n(G) \subset E(h_n(h))$ is virtually abelian. Thus $h_n(N)$ is finite. □

4.3 Holomorphic family of Riemann surfaces

Let $X$ be a compact Kähler manifold. A holomorphic family of Riemann surfaces of genus $g$ is a pair $(Z, \pi)$ where $Z$ is a compact complex surface and $\pi : Z \rightarrow X$ a holomorphic fibration such that the fibers of $\pi$ are Riemann surfaces of genus $g$. Let us review some facts about these families (see for instance [H-M], [We]).

Let $x_0$ be a base point in $X$, $Z_{x_0} = \pi^{-1}(x_0)$ its fiber and $S$ the underlying topological surface. Let $T = T_0$ be the Teichmüller space of $S$. Recall that $T$ is defined as the set of isotopy classes of complex structures on $S$, i.e. the quotient of the set of complex structures on $S$ by the group of diffeomorphisms of $S$ that are isotopic to the identity. The mapping class group $M(S)$, being defined as the group of isotopy classes of diffeomorphisms of $S$, naturally acts on $T$.

A differentiable trivialization $\Psi : U \times S \rightarrow Z \subset Y$ over a connected open neighborhood $U$ of $x_0$ gives a map $\Phi_U$ sending $x$ to the complex structure on $S$ obtained by pulling back by $\Psi(x, .) : S \rightarrow Y$ the complex structure of $Y$. A fundamental result (due to Teichmüller and Bers) endows $T$ with a structure of a complex manifold so that the map $\Phi_U$ is holomorphic (see for instance the beautiful text of Weil [We]). One proves that this map is uniquely defined (i.e it does not depend on the trivialization over $U$) modulo the action of the finite group $\text{Aut}(S_{x_0})$. The action of $M(S)$ on $T$ is holomorphic and properly discontinuous.

Let $(\tilde{X}, x_0)$ be the universal cover of $X$, computed at the base point $x_0$. Using analytic continuation, one extends the map $\Phi_U$ to a holomorphic map $\Phi : (\tilde{X}, x_0) \rightarrow (T, y_0)$ from the universal cover of $X$ to the Teichmüller space, called the classifying map, constructed along the following lines.

Let $(U_\alpha)_{\alpha \in A}$ be a covering of $X$ by open sets which is an atlas for the complex structure and over which the family has a trivialization. We may assume that the intersections $U_\alpha \cap U_\beta$ are empty or contractible. Let us consider holomorphic maps $\Phi_\alpha : U_\alpha \rightarrow T$ over each of these sets such that the complex structure over a point $p$ is isomorphic to $\Phi_\alpha(p)$. The family $(\Phi_\alpha)$ is well defined up to the action of $M(S_0)$. For every non empty intersection $U_\alpha \cap U_\beta$ there exists a mapping class $g_{\alpha, \beta}$ such that $\Phi_\alpha = g_{\alpha, \beta} \Phi_\beta$. In the langage of étale groupoids ([Br-Ha] Chapter III.G.2), the set of map $(\Phi_\alpha)_{\alpha \in A}$ and isometries $g_{\alpha, \beta}$ is a continuous homomorphism from the étale groupoid of holomorphic changes of charts defining $X$ to the (mapping class) group of isometries of the Teichmüller space endowed with its Weil-Peterson Kähler structure. From the developing lemma ([Br-Ha] Prop. 3.17, p. 611), one deduces that there exists a unique holomorphic map $\Phi$ from the universal cover of $X$ computed at the point $x_0$ to
the Teichmüller space which extends the map $\Phi_0$ defined on $U_o \ni x_0$. Moreover this map is equivariant for the induced morphism $\varphi : \pi_1(X, x_0) \to M(S)$ which is called the monodromy.

This (rather abstract) point of view allows us to immediately extend the definitions to the case where $X$ is a complex orbifold, viewed as an étale groupoid (see [Br-Ha]). We will use this definition when the base is a hyperbolic (hence complex) 2-orbifold. Such an orbifold is developable, its fundamental group is a lattice in $\text{PSL}(2, \mathbb{R}) = \text{Aut}(U)$ the group of automorphisms of the unit disc. Conversely, let $\Gamma \subset \text{PSL}(2, \mathbb{R})$ be a Fuchsian group, then $X = U/\Gamma$ is a complex 2-orbifold. A family of Riemann surfaces over $X$ is given by a representation $h : \Gamma \to M(S)$ and a holomorphic $\Gamma$-equivariant map from $U \to \mathcal{T}$.

Summing up, a holomorphic family $\pi : Y \to X$ of Riemann surfaces of genus $g$ over a complex manifold (orbifold) $X$ determines two objects:

1. The monodromy $\varphi$, which is a homomorphism from the fundamental group of $X$ to the mapping class group of its fiber.

2. The classifying map which is a holomorphic map from the universal cover of $X$ to the Teichmüller space of $S_{x_0}$ marked at the complex structure on this fiber, which is $\varphi$-equivariant.

One says that the family is isotrivial if $\Phi$ is constant (equivalently, the image of $\varphi$ is finite).

**Definition 7** Let $(Z, \pi, X)$ be a family of Riemann surfaces over $X$. One says that $\pi$ factorizes through a curve if there exits a hyperbolic 2-orbifold $Y$, a holomorphic map with connected fibers $F : X \to Y$ and a family $(Z', \pi', Y)$ over $Y$ such that $Z = F^*Z'$.

Let us recall that the Teichmüller space is contractible (it is homeomorphic to a ball) and let $\Omega_{\text{WP}}$ be the imaginary part of the Weil-Peterson Kähler metric (defined in [We]). As $\mathcal{T}$ is contractible and the action of $M(S)$ is proper with finite stabilizers, this space serves as a classifying space of $M(S)$ over $\mathbb{R}$, and the form $\Omega_{\text{WP}}$ determines a cohomology class $[\Omega_{\text{WP}}] \in H^*(M(S), \mathbb{R})$, the Weil-Peterson class.

**Proposition 8** The following assertions are equivalent:

i. The family factorizes through a curve

ii. $\varphi^*([\Omega_{\text{WP}}])^2 = 0$ (in $H^4(\Gamma, \mathbb{R})$)

iii. The complex rank of the (holomorphic) classifying map $\Phi$ is 1.

Proof. $i \Rightarrow ii$ and $iii \Rightarrow i$ are obvious. Assume that the complex rank of $\Phi$ is $r \geq 2$. As $\Phi$ is holomorphic and $\Omega_{\text{WP}}$ is a Kähler form on the Teichmüller space, $\Phi^*(\Omega_{\text{WP}})$ is a non zero harmonic $(r, r)$-form on $\tilde{X}$, which is $\Gamma$-equivariant
and defines a non zero harmonic form of degree $2r$ on $X$, providing a non zero class in $H^{2r}(X, \mathbb{R})$. As the Teichmüller space is contractible, this class vanishes on $\pi_k(X)$ for every $k \in \{1, \ldots, 2r\}$ and belongs to the image of the cohomology of the fundamental group $c^* : H^*(\Gamma, \mathbb{R}) \to H^*(X, \mathbb{R})$. □

**Corollary 2** The fact that a family of Riemann surfaces factorizes only depends on its monodromy (the morphism $\varphi : \Gamma \to M(S)$), not on the manifold.

If there exists a finite cover $X_1$ of $X$ such that the pullback of the bundle $Z$ factorizes through a curve, then $Z$ itself factorizes.

The following theorem, due to Imayoshi and Shiga [Im-Sh] (see also [McM]) is a key point in their proof of Parshin’s finiteness theorem.

**Theorem 7** Let $\varphi$ be the monodromy of a family of Riemann surfaces; then the image of $\varphi$ cannot be reducible nor virtually cyclic.

The first point is just a reformulation of Paragraph 4, Case 2 from [Im-Sh], or the “Irreducibility” Part in the proof of [McM]. In order to check that the image cannot be virtually $\mathbb{Z}$, note that by contradiction this would imply that $\varphi^* \Omega = 0$, as $H^2(\mathbb{Z}, \mathbb{R}) = 0$. Thus $\Phi$ would be constant.

### 4.4 Finiteness of monodromies.

Let us say that a morphism $\Gamma \to M(S)$ from a Kähler group to the mapping class group of a topological surface $S$ of genus at least 2 is a monodromy if it can be realized as the monodromy of a family of Riemann surfaces over a Kähler manifold whose fundamental group is $\Gamma$.

**Theorem 8** Let $\Gamma$ be a Kähler group. Then there exist only finitely many conjugacy classes of monodromies $\varphi : \Gamma \to M$ which do not factorize through a Riemann surface.

**Proof.** Let $\varphi_n$ be an infinite sequence of pairwise non conjugate monodromies. Applying the results of Paragraph 2 and 3, one constructs a finite index subgroup of $\Gamma$ which fibers over a Riemann surface group $S$ such that if $N$ is the kernel of this fibration, $\varphi_n|_N$ is finite. Thus, $\Gamma$ admits a finite index subgroup such that $\varphi_n$ restricted to this subgroup factorizes. By Cor. 2, $\varphi_n$ itself factorizes through a Riemann surface, a contradiction. □

**Corollary 3** The number of non isotrivial families over a compact Kähler manifold $X$ can be bounded in terms of its fundamental group.

The case of a Riemann surface is a theorem of L. Caporaso [Ca]. Applying simultaneously Proposition 2 and Corollary 2, we are reduced to study families which do not factorize through a curve. According to Theorem 8, we know that these families can only have finitely many possible monodromies. In order to conclude, we need to prove that on a given compact Kähler manifold, a non
isotrivial family is determined by its monodromy. The case of curves is the
Rigidity Theorem of [Im-Sh], see also [McM]. The case where $X$ is a projective
manifold follows by induction on the dimension, while considering the restriction
of the family to hyperplane sections. The general case follows from the
existence of a algebraic reduction of $X$, namely a projective manifold $X'$ with
a holomorphic map $\pi : X \to X'$ such that every map from $X$ to a projective
manifold must factorize through $\pi$ (see [Ue], p. 24).

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