

CODIMENSION ONE SUBGROUPS AND BOUNDARIES OF HYPERBOLIC GROUPS

THOMAS DELZANT AND PANOS PAPASOGLU

ABSTRACT. We construct hyperbolic groups with the following properties: The boundary of the group has big dimension, it is separated by a Cantor set and the group does not split. This shows that Bowditch's theorem that characterizes splittings of hyperbolic groups over 2-ended groups in terms of the boundary can not be extended to splittings over more complicated subgroups.

1. INTRODUCTION

Let G be a finitely generated group and let H be a subgroup of G . We say that H is a co-dimension 1 subgroup if C_G/H has more than 1 end, where C_G is the Cayley graph of G . If G splits over H then one easily sees that H is co-dimension 1. The opposite is not true, for example any closed geodesic on a surface group gives a cyclic codimension 1 subgroup of the fundamental group of the surface. On the other hand only simple closed geodesics correspond to splittings.

The surface example can be generalized to $CAT(0)$ complexes to produce examples of codimension 1 subgroups: If X is a finite $CAT(0)$ complex of (say) dimension 2 and if R is a locally geodesic track on X then the subgroup of $G = \pi_1(X)$ corresponding to R is a codimension 1 free subgroup of G . Wise ([11]) has exploited this idea producing codimension 1 subgroups for small cancellation groups. In the setting of small cancellation groups of course one needs some combinatorial analog for the convexity property of geodesics (or tracks) and Wise develops such a notion. Pride ([6]) has shown that there are small cancellation groups that have property FA, so such groups have codimension 1 subgroups but do not split.

Stallings showed that if a compact set separates the Cayley graph of a finitely generated group G , into at least two unbounded components, then G splits over a finite group. Bowditch ([1]) showed something similar for hyperbolic groups: if the boundary ∂G of a 1-ended hyperbolic group G has a local cut point, then the group splits over a 2-ended

1991 *Mathematics Subject Classification.*

We acknowledge support from the French-Greek grant Plato.

group, unless it is a triangle group. There have been other generalizations of Stallings theorem similar in spirit. The general idea is that if a ‘small’ set (coarsely) separates the Cayley graph of a group then the group splits over a subgroup quasi-isometric to the ‘small set’. For a precise conjecture see [10].

The main purpose of this paper is to show the limitations of this ‘philosophy’. Given any $n > 0$, we produce an example of a hyperbolic group G , such that $\dim(\partial G) > n$, ∂G is separated by a set of dimension 0 (a Cantor set) and G has property FA (so it does not split over any subgroup). Our example is based on Wise’s construction which we generalize to the setting of small cancellation theory over free products.

2. PRELIMINARIES

Definition . A diagram is a finite connected planar graph. The faces of a diagram D are the closures of the bounded components of $\mathbb{R}^2 - D$.

In what follows we assume always that each interior vertex (i.e. not on ∂D) of a diagram has degree at least 3. We can always achieve this by erasing all interior vertices of degree 2.

We will need some small cancellation results about diagrams shown by McCammond and Wise in [2]. For the reader’s convenience and also because our setting is slightly different we include these results here. These results strengthen classical small cancellation results (see e.g. [4]).

We need some notation: If D is a diagram we denote by ∂D the boundary of the unbounded component of $\mathbb{R}^2 - D$ (so if U is the unbounded component of $\mathbb{R}^2 - D$, $\partial D = \bar{U} - U$). We say that the diagram is *non singular* if ∂D is homeomorphic to S^1 . We say that an edge of D is interior edge if it does not lie on ∂D .

If D is a diagram we denote by E, F, V respectively the total number of edges, faces and vertices of the diagram.

We denote by $E^\bullet, E^\circ$ respectively the number of edges of the diagram that lie (do not lie) on ∂D . We denote by V^+ the number of vertices on ∂D that lie in exactly one face and by V^- the number of vertices on ∂D that lie on more than one face. We denote by V° the number of vertices of D that do not lie on ∂D . We say that a diagram verifies the $C(6)$ condition if every face of the diagram has at least 6 sides. We have the following version of Greedlinger’s lemma (see [4]):

Lemma 2.1. *Let D be a non singular diagram which verifies the condition $C(6)$. Then $V^+ \geq V^- + 6$.*

Proof. We have the following inequalities:

$$6F \leq 2E^\circ + E^\bullet$$

This is because each face has at least 6 edges and each interior edge lies in at most 2 faces while boundary edges lie in one face.

$$2E \geq 3V^\circ + 3V^- + 2V^+$$

This is because each edge has at most 2 vertices and each interior edge has degree at least 3.

Using Euler's formula and the first inequality we obtain:

$$V = E - F + 1 \geq E - \frac{E^\circ}{3} - \frac{E^\bullet}{6} + 1 = \frac{2E}{3} + \frac{E^\bullet}{6} + 1$$

We remark now that

$$E^\bullet = V^- + V^+$$

Substituting $E^\bullet$ above and using the second inequality for E we obtain

$$V = V^- + V^+ + V^\circ \geq \frac{2}{3}(\frac{3}{2}V^\circ + \frac{3}{2}V^- + V^+) + \frac{V^- + V^+}{6} + 1 \Rightarrow$$

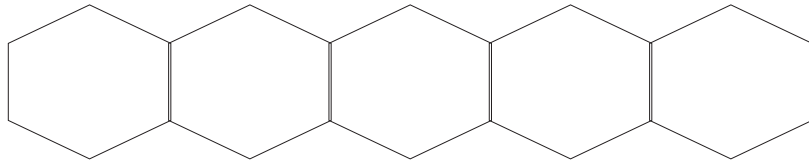
$$V^+ \geq V^- + 6$$

□

We recall some definitions from [2]:

Definition . Let D be a non singular diagram. A face F of the diagram is called an i -spur if the intersection of F with the boundary of D is connected and exactly i -edges of F are interior edges of D .

Definition . We say that a diagram D is a ladder if there are at most two faces F_1, F_2 of D such that $D - F_1, D - F_2$ are connected while for every other face F of D , $D - F$ has exactly 2 components.



A ladder

We have the following corollary from lemma 2.1:

Corollary 2.2. *Let D be a non singular disc diagram which is $C(6)$ and contains no 3-spurs and at most two i -spurs for $i \leq 2$. Then either D has a single region or it contains exactly two i -spurs with $i \leq 2$ and it is a ladder.*

Proof. We modify D as follows: if a face F of D has more than 6 edges and it intersects the boundary we erase successively vertices of F that do not lie on any other face till F has 6 edges (or there are no more vertices to erase). Let's call D_1 the new diagram. D_1 is still a $C(6)$ diagram. D_1 contains also no 3-spurs and at most two i -spurs for $i \leq 2$. We consider now a face F of D_1 that intersects the boundary and we see how it contributes to V^+, V^- . If F is not an i -spur then 2 vertices of F contribute to V^+ while at least 4 vertices of F contribute to V^- . So the total contribution of all such faces to the difference $V^+ - V^-$ is at most 0 (note that the contribution is not necessarily negative as we count twice the V^- vertices as they lie in at least 2 faces). The contribution of an i -spur to the difference $V^+ - V^-$ is $4 - i$.

Since D contains no 3-spurs and at most 2 i -spurs for $i \leq 2$, the inequality

$$V^+ - V^- \geq 6$$

implies that if D_1 has more than one face then D_1 has exactly two 1-spurs, say F_1, F_2 . If we erase F_1 we obtain a diagram D_2 which is again $C(6)$. We note that F_1 intersects exactly one face of D_1 so after erasing it the diagram D_2 has still the other 1-spur of D_1 and at most one new i -spur for some $i \leq 3$. By the inequality $V^+ \geq V^- + 6$ again we conclude as before that either D_2 has only one face or it has exactly two 1-spurs F_2, F_3 . Inductively we see that D_1 is a ladder hence D is also a ladder. $\square$

We will need a more technical result. If v is a vertex in a diagram we denote by d_v the valency of v . The result below will be used to show that small cancellation products of word hyperbolic groups are word hyperbolic.

Lemma 2.3. *Let D be a non singular diagram that verifies the condition $C(7)$. Then*

$$\frac{1}{3} \sum_{v \in D^0} \frac{d_v}{2} - \frac{2E^\circ}{7} \leq V^\bullet + \frac{E^\bullet}{7}$$

In particular

$$F \leq 3E^\bullet + 3V^\bullet$$

i.e. D satisfies a linear isoperimetric inequality.

Proof. We denote by D^0 the set of vertices of D (the 0-skeleton).

Clearly

$$\sum_{v \in D^0} \frac{d_v}{2} = E \quad (1)$$

We have also the following inequality:

$$7F \leq 2E^\circ + E^\bullet$$

This is because each face has at least 7 edges and each interior edge lies in at most 2 faces while boundary edges lie in one face.

Using Euler's formula and the inequality above we obtain:

$$E + 1 = V + F \leq V^\bullet + \frac{E^\bullet}{7} + V^\circ + \frac{2E^\circ}{7} \quad (2)$$

Since $d_v \geq 3$ for every v in the interior of D :

$$\sum_{v \in D^0} \frac{d_v}{2} - V^\circ \geq \frac{1}{3} \sum_{v \in D^0} \frac{d_v}{2}$$

By (1) and (2) we have

$$\frac{1}{3} \sum_{v \in D^0} \frac{d_v}{2} - \frac{2E^\circ}{7} \leq V^\bullet + \frac{E^\bullet}{7} \quad (3)$$

Since

$$\frac{1}{3} \sum_{v \in D^0} \frac{d_v}{2} \geq \frac{E^\circ}{3}$$

we have

$$\frac{1}{3} \sum_{v \in D^0} \frac{d_v}{2} - \frac{2E^\circ}{7} \geq \frac{E^\circ}{42} \geq \frac{3F}{7} - \frac{E^\bullet}{84}$$

and using (3)

$$V^\bullet + \frac{2E^\bullet}{7} \geq \frac{3F}{7} - \frac{E^\bullet}{84} \Rightarrow F \leq 3E^\bullet + 3V^\bullet$$

□

3. SMALL CANCELLATION THEORY OVER FREE PRODUCTS

Small cancellation theory can be developed over free products (see [4]). We show in this section that small cancellation products have codimension 1 subgroups. This generalizes a result of Wise ([11]). We recall that the free product factors embed in small cancellation products ([4] cor. 9.4, p.278). Osin ([5], lemma 4.4) showed that free product factors embed quasi-isometrically in small cancellation products (this also follows from [3]). For the reader's convenience we include a proof of this below.

Definition . Let $\langle S|R \rangle$ be a presentation of a group G . We say that $\langle S|R \rangle$ is symmetrized if for any $r = y_1 \dots y_n \in R$ all n cyclic permutations of r are also in R and r^{-1} is in R too. We assume that all elements of R are reduced words. If $r_1 = cb, r_2 = ca$ and the words cb, ca are reduced we call c a *piece* of the presentation.

Let now $\langle S|R \rangle$ be a symmetrized presentation. We have then the following small cancellation conditions:

- *Condition $C'(\lambda)$:* If $r \in R$ and $r = cb$ with cb reduced word and c a piece then $|c| < \lambda|r|$.
- *Condition $C(p)$:* No element of R is a product of fewer than p pieces.
- *Condition $B(2p)$:* If $r = ab$ and a is a product of p pieces then $|a| \leq |r|/2$.

Wise showed in [11] that groups that admit a presentation in which all relators have even length and condition $B(6)$ is satisfied, have codimension 1 subgroups. Clearly condition $C'(1/6)$ is stronger than condition $B(6)$ so Wise's result holds for these groups too.

We recall now how that the small cancellation conditions can be given for free products too ([4] ch V, sec. 9). Let G be the free product of the groups A_i .

We say that a word $a_1 \dots a_n$ is reduced if each a_j represents an element of one of the A_i and a_j, a_{j+1} belong to different factors for any j . Any element $g \in G$ can be represented in a unique way as a reduced word (normal form of g). If $g = a_1 \dots a_n$ is the normal form of g we define $\|g\| = n$. If $u = a_1 \dots a_n, v = b_1 \dots b_k$ are reduced words we say that the word $uv = a_1 \dots a_n b_1 \dots b_k$ is *semi-reduced* if $a_n b_1 \neq e$. Note however that a_n, b_1 might lie in the same factor. We say that a word $w = a_1 \dots a_n$ is *weakly cyclically reduced* if it is reduced and $a_n^{-1} a_1 \neq e$. We say that a sequence of words R is symmetrized if whenever $r \in R$ all weakly cyclically reduced conjugates of r and r^{-1} are in R . We say that c is a

piece if there are distinct $r_1, r_2 \in R$ such that $r_1 = ca, r_2 = cb$ and the words ca, cb are semi-reduced. As before we have the condition $C'(\lambda)$:

Condition $C'(\lambda)$: If $r \in R$ and $r = cb$ with cb semi-reduced word and c a piece then $\|c\| < \lambda\|r\|$.

Let now F be a free product $F = *A_i$ and let R be a symmetrized subset of F . The group G defined by the free product presentation $\langle F | R \rangle$ is the quotient

$$G = F / \langle\langle R \rangle\rangle$$

where $\langle\langle R \rangle\rangle$ is the normal closure of R in F .

We show now that if a group G has a free product presentation $\langle F | R \rangle$ that satisfies the $C'(1/6)$ condition then G has a codimension 1 subgroup. We start first by considering van-Kampen diagrams over G . We consider a usual presentation of G with a set of generators S given by the generators of A_i 's and a set of relators consisting of relators of the A_i 's together with a set R' such that R is obtained by taking all weak cyclic conjugates of elements in R' and their inverses. If R' is finite we say that G has a *finite free product presentation*. Let now w be a word in S representing the identity in G . Let D be a reduced Van-Kampen diagram for w for the presentation given above. We remark that if p is a simple closed path in the 1-skeleton of D such that all edges of p lie in a single factor A_i then the word corresponding to p represents the identity in A_i (see [4], cor. 9.4). Call such a simple closed path maximal if there is no other such simple closed path q in the interior of p . We modify now the diagram D as follows: For each maximal simple closed path p we erase all edges of p and all edges of D inside p and we introduce a new vertex v_p which we join with all vertices of p . Now each edge e of p has been replaced by two edges e_1, e_2 . We label e_1, e_2 by elements of A_i so that the product of their labels is equal to the label of e . This is clearly possible since p corresponds to the trivial element in A_i . We are allowed here to label an edge by the identity. After this operation some of the edges of D are 'subdivided'. We subdivide the rest of the edges of D so that the labels of the new edges lie in the same factor as the old ones and the product of their labels is equal to the label of the old edge. We call this diagram van-Kampen diagram over the free product.

We remark now that the $C'(\lambda)$ condition holds for this new diagram, i.e. if R_1, R_2 are adjacent regions of the diagram then

$$\text{length}(R_1 \cap R_2) \leq \lambda \min(\text{length}(\partial R_1), \text{length}(\partial R_2))$$

Theorem 3.1. *Let G be a finitely generated group with a finite free product $C'(1/6)$ presentation $\langle F | R' \rangle$. Assume further that all $r \in R'$*

are cyclically reduced words and $\|r\|$ has even length. Then G has a codimension 1 subgroup.

Proof. We construct a complex for G as usual. If $F = *A_i$ and K_i are complexes with a single vertex x_i such that $\pi_1(K_i, x_i) = A_i$ we take the wedge product of the K_i 's identifying all x_i 's. For each $r \in R'$ we glue a 2-cell to $\vee K_i$ in the obvious way to obtain a complex K such that $\pi_1(K, x) = G$. We argue now in a way similar to Wise ([11]). We slightly change approach and we consider bouquets of circles that go through x rather than tracks. We explain now how we construct a bouquet of circles Γ which will give the codimension 1 subgroup.

Let $r = a_1 \dots a_{2n}$ be the normal form in F of $r \in R'$. We represent the 2-cell $c(r)$ corresponding to r as a polygon where the a_i 's are the labels of the sides of this polygon. The bouquet of circles has a single vertex x and a set of edges corresponding to 'diagonals' of these polygons. We fix $r \in R'$ as above we pick the diagonal joining the beginning of the a_1 -edge to the vertex opposite to it, i.e. the end of the a_n edge.

We remark now that since $c(r)$ has an even number of sides each vertex has a vertex opposite to it, so we associate to this vertex the diagonal joining it with the opposite vertex. We remark that any vertex is determined by the edges adjacent to it. For example the beginning of the a_1 edge is the vertex corresponding to the consecutive edges a_{2n}, a_1 . We consider now the equivalence relation on vertices of the $c(r)$'s generated by the following relation: The vertex b_i, b_{i+1} of $c(r_1)$ is equivalent to the vertex c_j, c_{j+1} of $c(r_2)$ if b_i, b_{i+1} lie in the same free factors as c_j, c_{j+1} . We remark that r_1 might be equal to r_2 in this definition.

Now for each r we consider all vertices equivalent to the vertices of the chosen diagonal. We add to the bouquet of circles all diagonals corresponding to these vertices and we call the graph obtained in this way Γ . We remark that Γ is a bouquet of circles if we see it as an abstract graph but if we see it as immersed in K its edges are likely to intersect each other in the middle point of the polygons.

Γ corresponds to a subgroup of G . Indeed each diagonal gives a generator, for example the diagonal joining a_1, a_n gives the generator $a_1 a_2 \dots a_n$. Let's call this subgroup H . We will show that H is a codimension 1 subgroup of G .

Lemma 3.2. *There is a tree $\tilde{\Gamma} \subset \tilde{K}$ which has as edges diagonals of 2-cells which is invariant under H . $\tilde{\Gamma}$ separates $\tilde{K}$ in at least 2 components.*

Proof. Let $v \in \tilde{K}$ be a vertex. We define now a connected graph in $\tilde{K}$ as follows: We say that two vertices are related if they are opposite.

We take the equivalence relation generated by this relation and we consider the equivalence class of v . Let $\tilde{\Gamma}$ be the graph obtained by joining opposite vertices in this equivalence class by diagonals. We claim that $\tilde{\Gamma}$ is a tree. If it is not a tree then there is a path p in $\tilde{\Gamma}$ such that both endpoints of p lie on the same 2-cell of R' and p is not a single edge (a diagonal). Let's say a, b are the endpoints of p and they lie on a 2-cell σ . Let q be a path on $\partial\sigma$ joining a, b . We may assume q to have minimal normal form length in the free product among the 2 possible paths. Now $p \cup q$ is a closed loop. We change now p by replacing each diagonal by the corresponding path on the boundary on which the diagonal lie. We note that we have two choices and we replace the diagonals so that the path we obtain by replacing all of them corresponds to a reduced word of F . Let p' be the path we obtain in this way. We may arrange also that $p' \cup q$ is reduced at the vertex a (unless $a = b$). We consider now the van-Kampen diagram over the free product for $p' \cup q$ and we remark that if it has an i -spur for $i \leq 2$ then the boundary of this i -spur contains a neighborhood of the vertex b . But this contradicts Corollary 2.2. It follows that $\tilde{\Gamma}$ is a tree. By construction $\tilde{\Gamma}$ is invariant under H and separates locally (hence also globally) $\tilde{K}$. □

The first part of the next lemma follows also from work of Osin ([5], see also [3]). We include a proof here for the sake of completeness.

Lemma 3.3. *The vertex groups A_i and H embed quasi-isometrically in G . H is a codimension 1 subgroup of G .*

Proof. Let a be a geodesic word in the Cayley graph of A_i . We will show that a is a quasi-geodesic in $\tilde{K}$. Let S be the generating set of G and let $|w|$ be the length of a word in S . Let

$$M = \max\{|r| : r \in R'\}$$

We define a new length function L for words of S :

$$L(w) = M\|w\| + |w|$$

It is clear that an L -geodesic is a quasi-geodesic.

Indeed let p be a geodesic in the 1-skeleton of $\tilde{K}$ with respect to the length function L with the same endpoints as a . We consider the van-Kampen diagram over the free product for $a \cup p$. We may assume that $a \cap p$ is equal to the endpoints of a, p since along the intersection of a, p , a is quasi-geodesic.

We remark now that this diagram has at most 2 i -spurs (for $i \leq 2$), corresponding to the endpoints of p so by corollary 2.2 this diagram is

a ladder. By considering now the usual van-Kampen diagram for $a \cup p$ we have that $|a| \leq M|p|$ so a is quasi-geodesic.

We prove now that H is quasi-isometrically embedded. Since H acts freely on $\tilde{\Gamma}$ it is enough to show that $\tilde{\Gamma}$ is quasi-isometrically embedded. Let p be a geodesic path in $\tilde{\Gamma}$ joining two vertices v, u of $\tilde{\Gamma}$.

We change p by replacing each diagonal by the corresponding path on the boundary of the cell on which lies the diagonal. We pick this path so that the word of F corresponding to the path is reduced. Let p' be the path we obtain in this way. Let q be an L -geodesic path joining v, u . As before we may assume that p', q intersect only at their endpoints. Again by the definition of p', q if we consider the van-Kampen diagram over the free product for $p' \cup q$ we remark that it has at most 2 i -spurs for $i \leq 2$. Hence this diagram is a ladder. By considering the usual van-Kampen diagram we have that $\text{length}(p') \leq M \text{length}(q)$. Since q is a quasi-geodesic we have that p' is a quasi-geodesic, so H is quasi-isometrically embedded.

Finally we show that H is a codimension 1 subgroup. It suffices to show that $\tilde{\Gamma}$ separates $\tilde{K}$ in at least 2 components which are not contained in a finite neighborhood of $\tilde{\Gamma}$. By its definition $\tilde{\Gamma}$ separates locally $\tilde{K}$. Since $\tilde{K}$ is simply connected, $\tilde{\Gamma}$ separates $\tilde{K}$.

We introduce now some useful terminology. Let $r \in R'$ and let $c_1 c_2 \dots c_n$ be the normal form of r in F . Recall that r is cyclically reduced. Let k be smallest such that

$$\|c_1 \dots c_k\| \geq \frac{n}{6}$$

We say then that $c_1 \dots c_k$ is a *piece* of r . Similarly we define pieces of all cyclic permutations of $c_1 c_2 \dots c_n$.

Let R be a 2-cell in $\tilde{K}$ which intersects $\tilde{\Gamma}$ on an edge e . Let v, u be the vertices of e .

Let $c_1 c_2 \dots c_n$ be the label of R starting from v and written in free product normal form. Let s be the vertex corresponding to the endpoint of a piece $p_1 = c_1 \dots c_k$ of R starting at v . We construct a path starting from v and lying in the same component of $\tilde{X} - \tilde{\Gamma}$ as s . The path starts by p_1 . At s we continue p_1 by a piece p_2 of another 2-cell R_2 corresponding to $r_2 \in R'$. We pick $R_2 \neq R$ and so that $p_1 p_2$ is reduced in F . We continue inductively in the same way picking each time a new 2-cell and a piece so that the word we obtain is reduced in F . Let $\beta = p_1 \dots p_n$ be the path we obtain after n steps. If s_n is the endpoint of $p_1 \dots p_n$ we claim that $d(s_n, \tilde{\Gamma}) \rightarrow \infty$ as $n \rightarrow \infty$. Indeed let q be a geodesic joining s_n to a closest vertex $t \in \tilde{\Gamma}$. We consider a geodesic γ in $\tilde{\Gamma}$ joining v to t .

We distinguish now two cases. Assume first that u does not lie on γ . We change γ by replacing each diagonal by the corresponding path on the boundary on which the diagonal lie to obtain a path γ' . We make these replacements so that the word of F corresponding to the new path is reduced and $p_1^{-1}\gamma'$ corresponds also to a reduced word in F . Clearly this is possible since we have two choices for replacing each diagonal and the normal form of each starts from a different free factor. We consider now the loop

$$\beta \cup \gamma' \cup q$$

Since q is geodesic the van-Kampen diagram for free products for this loop has at most 2 i -spurs with $i \leq 2$ which appear around the end-points of q . It follows that this diagram is a ladder (see corollary 2.2) hence the lengths of q and $\beta \cup \gamma'$ are comparable so the length of q goes to infinity as $n \rightarrow \infty$.

We deal now with the second case, i.e. we assume that u lies on γ . We modify then $p_1 \dots p_n$ as follows. We replace p_1 by the path q_1 on the boundary of R joining s to u . We note that the new path $\beta' = q_1 p_2 \dots p_n$ might not be reduced at the endpoint of q_1 . We replace γ by a path γ' in the 1-skeleton of $\tilde{K}$ as before so that $q_1^{-1}\gamma'$ is reduced in the free product F . We remark that the van-Kampen diagram over free products for the loop

$$\beta' \cup \gamma' \cup q$$

is a ladder in this case too hence the length of q goes to infinity as $n \rightarrow \infty$.

Similarly we construct we see that the component of $R - \tilde{\Gamma}$ that does not contain v is not contained in a finite neighborhood of $\tilde{\Gamma}$. It follows that H is co-dimension 1. □

□

4. THE EXAMPLE

Theorem 4.1. *Given any $n > 0$ there is a one-ended hyperbolic group G such that*

- $\dim \partial G \geq n$
- ∂G is separated by a Cantor set.
- G does not split.

Proof. Let A be a torsion free 1-ended hyperbolic group with property T and such that $\dim(\partial A) \geq n$ (eg a lattice in $Sp(n, 1)$). Let's say $A = \langle a_1, \dots, a_k \rangle$. We may assume that $a_i^m \neq a_j^r$ for any $i \neq j$ and $m, r > 0$. We take now another copy of A . For notational convenience

we denote the second copy by B and its generators by $\langle b_1, \dots, b_k \rangle$. We consider now the free product $A * B$ and we define G to be the small cancellation quotient of $A * B$ given by the relations:

$$r_{i,j} = (a_i b_j)(a_i b_j^2)(a_i b_j^3)(a_i b_j^4) \quad 1 \leq i, j \leq k$$

By theorem 3.1 of the previous section G has a free codimension 1 subgroup H . As we showed in the proof of the theorem H is quasi-isometrically embedded so a Cantor set separates ∂G . We show now that G has property FA (i.e. it does not split). Clearly G is not an HNN extension since the abelianization of A is trivial, so the abelianization of G is trivial. We show now that G does not split as an amalgamated product. Let's say $G = X *_C Y$. Without loss of generality we may assume that $A \subset X$ and $B \subset gXg^{-1}$ or $B \subset gYg^{-1}$. Let $g = x_1 \dots x_n$ be the normal form of G in the free product decomposition. By replacing A, B by conjugates we may assume that either $g = 1$ and $B \subset Y$ or $x_1 \notin X$. However we see then that the word

$$r_{i,j} = (a_i b_j)(a_i b_j^2)(a_i b_j^3)(a_i b_j^4)$$

is reduced in $X *_C Y$ unless a_i or b_j is in C . As all $r_{i,j}$ are equal to the identity this implies that $A = C$ and B contained in Y or $B = C$ and A contained in X but in both cases, the splitting would be trivial.

We claim finally that G is hyperbolic. Indeed this follows by lemma 4.4 of [5], and [3]. For the reader's convenience we sketch a proof here using lemma 2.3. It is enough to show that G satisfies a linear isoperimetric inequality. Let w be a word on the generators of G and let D be a reduced van-Kampen diagram for G . As we describe in section 3 one obtains from D a new diagram, let's say D_1 , which is called the diagram for w over the free product. Since A, B are hyperbolic they satisfy some isoperimetric inequality of the form

$$A(p) \leq Kl(p)$$

for any simple closed path p in the Cayley graph of A or of B .

It follows that if p is a simple closed path of D such that all edges of p lie in A (or in B) and if v is the vertex of D_1 that we obtain by collapsing p to a point then

$$d_v = l(p) \geq \frac{1}{K} A(p)$$

where d_v is the degree of v . It follows that

$$A(D) \leq A(D_1) + K \sum_{v \in D_1^0} d_v$$

From lemma 2.3 we have the following inequality for the diagram D_1 :

$$\frac{1}{3} \sum_{v \in D_1^0} \frac{d_v}{2} - \frac{2E^\circ}{7} \leq V^\bullet + \frac{E^\bullet}{7}$$

We have

$$\sum_{v \in D_1^0} \frac{d_v}{2} \geq E^\circ \Rightarrow \frac{2E^\circ}{7} \leq \frac{2}{7} \sum_{v \in D_1^0} \frac{d_v}{2}$$

so

$$\frac{1}{3} \sum_{v \in D_1^0} \frac{d_v}{2} - \frac{2E^\circ}{7} \geq \frac{1}{42} \sum_{v \in D_1^0} \frac{d_v}{2}$$

We have also $l(\partial D_1) \leq l(\partial D)$ and $V^\bullet, E^\bullet \leq l(\partial D)$. So from lemma 2.3 we have

$$A(D_1) \leq 6l(\partial D)$$

and

$$\sum_{v \in D_1^0} d_v \leq 42V^\bullet + 7E^\bullet$$

so

$$A(D) \leq (6 + 49K)l(\partial D)$$

In other words G verifies a linear isoperimetric inequality, so it is hyperbolic. □

Remark 1. The above example shows also that for any n there is a finitely presented group G with $asdim G > n$ which is separated coarsely by a uniformly embedded set H of $asdim H = 1$ and which does not split. This answers a question in [10].

REFERENCES

- [1] B.H. Bowditch *Cut points and canonical splittings of hyperbolic groups* Acta Math. 180, No.2, pp.145-186 (1998)
- [2] J. McCammond, D.Wise, *Fans and ladders in small cancellation theory*, Proc. London Math. Soc. (3), 84 (3), p. 599-644 (2002).
- [3] T. Delzant, *Sous-groupes distingues et quotients des groupes hyperboliques*, Duke Math. J. Volume 83, Number 3 (1996), p. 661-682.
- [4] R. Lyndon, P. Schupp, *Combinatorial Group Theory*, Springer Verlag (1977)
- [5] D. Osin, *Small cancellations over relatively hyperbolic groups and embedding theorems*, arXiv:math/0411039
- [6] S. Pride *Some finitely presented groups of cohomological dimension two with property (FA)*, J. Pure Appl. Algebra 29 (2), p. 167-68, (1983)
- [7] D.Wise, *Cubulating small cancellation groups*, Geom. Funct. Anal. 14 (2004), no. 1, 150-214.

- [8] M. Gromov, *Hyperbolic groups*, Essays in group theory (S. M. Gersten, ed.), MSRI Publ. 8, Springer-Verlag, 1987 pp. 75-263.
- [9] M. Gromov, *Asymptotic invariants of infinite groups* in 'Geometric group theory', (G. Niblo, M. Roller, Eds.), LMS Lecture Notes, vol. 182, Cambridge Univ. Press, 1993
- [10] P. Papasoglu, *Group splittings and asymptotic topology*, J. Reine Angew Math. v.602, p. 1-16 (2007).
- [11] D. Wise *Cubulating small cancellation groups* GAFA, Volume 14, Number 1 / February, 2004.

E-mail address, Thomas Delzant: `delzant@math.u-strasbg.fr`

E-mail address, Panos Papasoglu: `panos@math.uoa.gr`

(Thomas Delzant) INSTITUT DE RECHERCHE MATHÉMATIQUE AVANCÉE, UNIVERSITÉ LOUIS PASTEUR ET CNRS, 7 RUE RENÉ DESCARTES, 67084 STRASBOURG CEDEX, FRANCE

(Panos Papasoglu) MATHEMATICS DEPARTMENT, UNIVERSITY OF ATHENS, ATHENS 157 84, GREECE