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TREES, VALUATIONS AND THE GREEN–LAZARSFELD SET

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Abstract. We study the Green–Lazarsfeld set from the point of view of geometric group theory and compare it with the Bieri–Neumann–Strebel invariant. Applications to the study of fundamental groups of Kähler manifolds are given.

1 Introduction

The aim of this paper is the study of the relationship between two objects, the Green–Lazarsfeld set and the Bieri–Neumann–Strebel invariant, which appeared simultaneously in 1987 ([GL1], [BiNS]). Let us recall some basic definitions.

Let Γ be a finitely generated group, and K be a field. A 1-character is an homomorphism from Γ to K^*; in this article we will only consider 1-characters, and call them characters. A character χ is called exceptional if $H^1(\Gamma,\chi) \neq 0$, or more geometrically if χ can be realized as the linear part of a fixed-point-free affine action of Γ on a K-line.

The set of exceptional characters, $E^1(\Gamma, K)$ is a subset of the abelian group $\mathrm{Hom}(\Gamma, K^*)$, and our aim is to understand its geometry, in particular if Γ is the fundamental group of a compact Kähler manifold.

Motivated by the pioneering work of M. Green and R. Lazarsfeld [GL1], algebraic geometers studied the case where $K = \mathbb{C}$ is the field of complex numbers, and $\Gamma = \pi_1(X)$ is the fundamental group of a projective or more generally a compact Kähler manifold. In this case, the geometry of $\mathrm{Hom}(\Gamma, K^*)$ is well understood: it is the union of a finite set, made up of torsion characters, and a finite set of translates of subtori. This result has been proved in some special cases by A. Beauville [Be] and conjectured

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by A. Beauville and F. Catanese, [Be]. The fact that the components are translates of tori was proved by Green and Lazarsfeld [GL2], and the result about torsion by C. Simpson [Si2] for projective manifolds, and extended by F. Campana to the Kähler case (see [C] for a detailed introduction), and D. Arapura for the quasi-projective case [Ar]. The main tools used by Simpson were the flat hyper-Kähler structure of $\mathrm{Hom}(\Gamma, \mathbb{C}^*)$ and the Schneider–Lang theorem in transcendence theory. Another proof, model theoretic and using Deligne–Illusie Hodge theory, has been proposed by R. Pink and D. Roessler [PR]; it concerns the geometry of line bundles over manifolds in characteristic $p \neq 0$. See also the recent survey by A. Dimca [Di] for other results and applications.

The definition of an exceptional class in the sense of Bieri–Neumann–Strebel is easier to explain in the case of an integral cohomology class (an element of $H^1(\Gamma, \mathbb{Z})$). Such a class is *exceptional* if it can be realized as the translation class of a parabolic, non-loxodromic action of Γ in some tree.

The link between these two notions, explained in the next section, can be sketched as follows. Let χ be an exceptional character of Γ. Suppose that $\chi(\Gamma)$ is not contained in the ring of algebraic integers of K. There exists a discrete non-archimedean valuation on the subfield of K generated by $\chi(\Gamma)$ such that $v \circ \chi$ is a non-trivial homomorphism to $\mathbb{Z}$. It appears that $v \circ \chi$ is an exceptional class in the sense of Bieri–Neumann–Strebel. More precisely, one can find a parabolic action of Γ on the Bruhat–Tits tree of the v completion of K, say K_v, with translation length $v \circ \chi$.

Due to the work of Simpson [Si3], M. Gromov and R. Schoen [GrS], exceptional cohomology classes on Kähler manifold are well understood (see also [D] for a detailed study of the BNS invariant of a Kähler group). Let X be a Kähler manifold, and ω an exceptional class; there exists a holomorphic map F from X to a hyperbolic Riemann orbifold Σ such that ω belongs to $F^*H^1(\Sigma, \mathbb{Z})$. Recall that a complex 2-orbifold Σ is a Riemann surface S marked by a finite set of marked points $\{(q_1, m_1), \dots, (q_n, m_n)\}$, where the m_i''s are integers $\geqslant 2$. A map $F : X \to \Sigma$ is called holomorphic if it is holomorphic in the usual sense, and for every q_i the multiplicity of the fiber $F^{-1}(q_i)$ is divisible by m_i. The main result of this paper is a description of the (generalized) Green–Lazarsfeld set of $\pi_1(X)$ in terms of the finite list of its fibrations on hyperbolic 2-orbifolds.

Theorem. *Let Γ be the fundamental group of a compact Kähler manifold X, $(F_i, \Sigma_i)_{1 \leqslant i \leqslant n}$ the family of fibration of X over hyperbolic 2-orbifolds. Let K be a field of characteristic p (if $p = 0$, $K = \mathbb{C}$), $\bar{F}_p \subset K$ the* *algebraic closure of F_p in K. Then $E^1(\Gamma, K)$ is the union of a finite set of torsion characters (contained in $E^1(\Gamma, \bar{F}_p)$ if $p > 0$) and the union $\bigcup_{1\leqslant i\leqslant n} F_i^* E^1(\pi_1^{\mathrm{orb}}(\Sigma_i), K^*)$.*

REMARKS. a) The study of $\mathrm{Hom}(G, K^*)$, with $\operatorname{char} K > 0$, seems to be a new idea. This set is abstractly $(K^*)^{b_1(G)} \times \Phi$, Φ a finite abelian group which depends on K. It is easy to construct a group G such that the set $E^1(G, K)$ strongly depends on the field K. For instance if $G = \mathbb{Z} \ltimes F_p[t, t^{-1}]$, with $\mathbb{Z}$ acting on $F_p[t, t^{-1}]$ by multiplication by t^n, $\mathrm{Hom}(G, K^*) = K^*$ for every field K. If $\operatorname{char} K \neq p$ the Green–Lazarsfeld set is reduced to the trivial character, but if $\operatorname{char} K = p$ it is the entire group $\mathrm{Hom}(G, K^*)$.

b) Let $\Sigma = (S; (q_i, m_i)_{1\leqslant i\leqslant n})$ be a hyperbolic 2-orbifold, and $\Gamma = \pi_1^{\mathrm{orb}}(\Sigma)$ its fundamental group. By a simple computation (see Proposition 4), one checks that $E^1(\pi_1^{\mathrm{orb}}(\Sigma), K^*) = \mathrm{Hom}(\pi_1^{\mathrm{orb}}(\Sigma, K^*)) = (K^*)^{2g} \times \Phi$, where Φ is a finite abelian group, unless $g = 1$ and for all i, $m_i \not\equiv 0(\operatorname{char} K)$. If $g = 1$ and for all i $m_i \not\equiv 0(\operatorname{char} K)$, $E^1(\pi_1^{\mathrm{orb}}(\Sigma), K^*)$ is *finite*, made up of torsion characters. This example shows the interest of considering fields of various characteristics.

In every case, the Green–Lazarsfeld set is the union of a finite set of torsion characters and a finite set of abelian groups which are translates of tori; this is our generalization of Simpson's theorem. As remarked by the referee, the theorem gives a geometric interpretation of the translates of subtori which, due to Beauville [Be], are the components of positive dimension of the Green–Lazarsfeld set: unless the curious exception $g = 1$, and for all i, $m_i \neq 0(\operatorname{char} K)$, these are the image by F_i^* of the abelian groups $\mathrm{Hom}(\pi_1^{\mathrm{orb}}(\Sigma, K^*)) = (K^*)^{b_1(\Sigma)} \times \Phi$; see also the preprint by Dimca [Di] who goes deeper into this new point of view and gives some applications.

c) The main tool used by Simpson to prove his theorem [Si2] was the study of algebraic triple tori; if $\operatorname{char} K \neq 0$ no such structure is available. Our proof furnishes a geometric (i.e. non-arithmetic) alternative to Simpson's proof in the case of characteristic 0. In fact, in this case ($\operatorname{char} K = 0$) our method proves that $E^1(\Gamma, K)$ is made with a finite set of *integral* characters (in the sense of Bass [B]), and the union $\bigcup_{1\leqslant i\leqslant n} F_i^* \mathrm{Hom}(\pi_1^{\mathrm{orb}}(\Sigma_i), K^*)$ the conclusion follows from the study of the the absolute value $|\chi|$ of exceptional characters, which was already done by Beauville [Be].

In a recent preprint [CoS], C. Simpson and K. Corlette study the variety of characters of a Kähler group Γ, $\mathrm{Hom}^{\mathrm{ss}}(\Gamma, \mathrm{PSl}(2, \mathbb{C})/\,\mathrm{PSL}(2, \mathbb{C})$ from a very similar point of view; they prove in particular that a Zariski dense representation of a Kähler group which is not *integral* in the sense of Bass

[B] factorizes through a fibration over a hyperbolic 2-orbifold. Their proof is based on the same idea as ours: if a representation ρ is not integral, there exists a valuation on the field generated by $\rho(\Gamma)$ such that the action of Γ on the Bruhat–Tits building is non-elementary. The conclusion follows by applying the theory of Gromov–Schoen on harmonic maps with value in a tree. Note that the same method applies for a field of positive characteristic. Using Simpson's work on Higgs bundles they prove further that a rigid representation comes from a complex variation of Hodge structure; this last part of the argument being meaningful only in the characteristic 0 case.

In section 2, we explain the relationship between the Green–Lazarsfeld and Bieri–Neumann–Strebel invariants; in section 3 we study the Green–Lazarsfeld set of a metabelian group: a finiteness result on this set is established. These two sections are purely group theoretic, and no Kähler structure is mentioned. In the section 4 we prove the main result.

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2 From an Affine Action on a Line to a Parabolic Action on a Tree

2.1 Affine action on the line: the Green–Lazarsfeld set. Let K be a field. The affine group of transformation of a K-line, $\mathrm{Aff}_1(K)$, is isomorphic to $K^* \ltimes K$. We identify this group with the set of upper triangular $(2,2)$matrices $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ with values in K.

Let Γ be a finitely generated group. An affine action of Γ on the line is a morphism $\rho : \Gamma \to \mathrm{Aff}_1(K)$. One can write $\rho(g) = \left(\begin{smallmatrix} \chi(g) & \theta(g) \\ 0 & 1 \end{smallmatrix}\right)$. The linear part of ρ is an homomorphism $\chi : \Gamma \to K^*$. Its translation part $\theta : \Gamma \to K$ is a 1-cocycle of Γ with value in χ, i.e. a function which satisfies $\theta(gh) = \theta(g) + \chi(g)\theta(h)$. The representation ρ is conjugate to a diagonal representation if and only if $\rho(\Gamma)$ fixes a point $\mu \in K$, or equivalently if and only if there exists a $\mu \in K$ such that $\theta(g) = \mu(-1+\chi(g))$ is a coboundary.

DEFINITION 1. *A character $\chi \in \mathrm{Hom}(\Gamma, K^*)$is exceptional if it can be realized as the linear part of a fixed-point-free affine action of Γ on the line, i.e. if $H^1(\Gamma,\chi) \neq 0$. The set of exceptional characters $E^1(\Gamma, K)$ is called the Green–Lazarsfeld set of Γ.*

2.2 Parabolic action on a tree: the Bieri–Neumann–Strebel invariant. Let T be a simplicial tree. We endow T with its natural simplicial metric, and think of T as a complete geodesic space. Let us recall the definitions of the boundary of T, and of the Busemann cocycle associated to a point in this boundary.

A ray in T is an isometric map $r : [a, +\infty[\to T$. Two rays $r : [a, +\infty[\to T$, $s : [b, +\infty[\to T$ are equivalent (or asymptotic) if they coincide after a certain time: there exists a', b' s.t. for all $t \geqslant 0$, $r(a' + t) = s(b' + t)$. The boundary of T, denoted ∂T, is the set of equivalence classes of rays. If $\alpha \in \partial T$ and $r : [a, +\infty[\to T$ represents α, for every point x, the function $t \to d(x, r(t)) - t$ is eventually constant. Its limit $b_r(x)$ is called the Busemann function of r. If s is equivalent to r, the difference $b_r - b_s$ is a constant.

Definition 2 (Busemann cocycle). *Let Γ be a group acting on T, and $\alpha \in \partial T$. If Γ fixes α, one define an homomorphism, the Busemann cocycle, by the formula*

$$\omega_\alpha : \Gamma \to \mathbb{Z} \,,$$

$\omega_\alpha(g) = b_r \circ g - b_r$.

Definition 3 (Exceptional classes). *The action of Γ is called parabolic if it fixes some point at infinity. It is called exceptional if it fixes a unique point at infinity, and if the associated Busemann cocycle is not trivial. A class $\omega \in H^1(\Gamma, \mathbb{Z})$ is exceptional if it can be realized as the Busemann cocycle of an exceptional action of Γ in some tree. The set of exceptional classes is denoted $\mathcal{E}^1(\Gamma, \mathbb{Z})$.*

Remark 1. A *topological* definition of a exceptional class can also be given, in the case where Γ is finitely presented. Let $\Gamma = \pi_1(X)$, where X is a compact manifold, and let ω be some class in $H^1(\Gamma, \mathbb{Z})$. One represents ω by a closed 1-form w on X and consider a primitive $F : \tilde{X} \to \mathbb{R}$ of the lift of w to the universal cover of X. Then ω is exceptional iff $F \geqslant 0$ has several components on which F is unbounded (see [BiS], [L], [Bro] for a study of this important notion).

Remark 2. The notion of an exceptional class, defined by Bieri, Neumann and Strebel, and studied by several authors, in particular [Bro], [L], is more general: it concerns a homomorphism with value in $\mathbb{R}$ and can be defined along the same lines, using $\mathbb{R}$-trees instead of combinatorial trees. Our point of view is that of Brown; it is interesting to remark that [Bro], [BiNS] and [GL1] are published in the same issue of the same journal, but apparently nobody remarked that [Bro] and [GL1] studied the same object from a different point of view.

2.3 Discrete valuations and Bruhat–Tits trees. In this section we fix a field K. Let $v : K^* \to \mathbb{Z}$ be a discrete non-archimedean valuation on K. Bruhat and Tits [BruhT] constructed a tree T_v with an action of $\mathrm{PGL}(2,K)$. One should think of the action of $\mathrm{PGL}(2,K)$ of T_v as an analogue of the action of $\mathrm{PGL}(2,\mathbb{C})$ on the hyperbolic space of dimension 3; we recall below some basic facts about this action (see [S] for a detailed study).

Let $O_v \subset K$ denote the valuation ring $v \geqslant 0$. The *vertices* of T_v are the homothety classes of O_v-lattices, i.e. free O_v-modules of rank 2, in K^2. The *boundary* of this tree is the projective line $P^1(\bar{K}_v)$ over the v-completion of K.

By the general theory of lattices, if Λ, Λ' are two lattices, one can find a O_v-base of Λ such that, in this base, Λ' is generated by $(t^a,0)$ and $(0,t^b)$ for some t with $v(t)=1$; hence up to homothety by $(1,0)$ and $(0,t^n)$, for $n = b-a$. Then the distance between Λ, and Λ' is $|n|$, and the segment between Λ and Λ' is the set of lattices generated by $(1,0)$ and $(0,t^k)$, $k=1,n$. More generally if l,l' are two different lines in K^2, considered as points in ∂T_v, the geodesic from l to l' is the set of product of lattices in l and l'.

The matrix $g_u = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right)$ fixes the lattice Λ_n generated by $(1,0)$ and $(0,t^n)$ for $n \leqslant v(u)$. The matrix $g_u = \left(\begin{smallmatrix} t^n & u \\ 0 & 1 \end{smallmatrix}\right)$ transforms Λ_m to Λ_{m+n} if $m+n \leqslant v(u)$.

Acting on T_v the Borel subgroup $\left(\begin{smallmatrix} * & * \\ 0 & 1 \end{smallmatrix}\right)$ is *parabolic*: it fixes an end of T_v (namely the line generated by the first basis vector), but neither a point of T_v nor a pair of points of ∂T_v.

The Busemann cocycle of this parabolic subgroup is $b\left(\left(\begin{smallmatrix} \alpha & \beta \\ 0 & 1 \end{smallmatrix}\right)\right) = v(\alpha)$.

The relationship between the Green–Lazarsfeld set and the Bieri–Neumann–Strebel invariant is now simple to explain.

Proposition 1. *Let $\chi \in H^1(\Gamma, K^*)$. Suppose that $\chi \in E^1(\Gamma, K^*)$ and let $\theta \in H^1(\Gamma, \chi) \neq 0$. Let $\rho : \Gamma \to \mathrm{Gl}(2,K)$ be defined by $\rho(g) = \begin{pmatrix} \chi(g) & \theta(g) \\ 0 & 1 \end{pmatrix}$. If $v \circ \chi \in H^1(\Gamma, \mathbb{Z})$ is not 0, ρ is an exceptional action on T_v.*

Proof. By construction the action of Γ on T_v fixes a point at infinity. It contains an hyperbolic element as $v \circ \chi \neq 0$, but the action cannot fix a line: the other point in the boundary $P^1(\bar{K}_v)$ would be fixed by the group Γ, and ρ would be conjugate to diagonalizable action. The orbit of any point of Γ is therefore a minimal tree which is not a line. □

3 Metabelian Groups

If Γ is a group, let $\Gamma' = [\Gamma, \Gamma]$ its derived group. Recall that a group is *metabelian* if Γ' is abelian, or $\Gamma^2 = (\Gamma')'$ is trivial. If Γ is a f.g. group, Γ/Γ^2 is metabelian.

3.1 The Green–Lazarsfeld set of a metabelian group. If K is a field, the Green–Lazarsfeld set $E^1(\Gamma, K)$ of the group Γ only depends on its metabelianized Γ/Γ^2 as it only depends of the set of representations of Γ in the metabelian group $\mathrm{Aff}_1(K) = K^* \ltimes K$.

Let Γ be a metabelian group. We write $1 \to [\Gamma, \Gamma] \to \Gamma \to Q \to 1$, where $Q = \Gamma/[\Gamma, \Gamma]$ is the abelianized group, and $[\Gamma, \Gamma]$ is abelian. As an abelian group, $M = [\Gamma, \Gamma]$ is not necessarily f.g., however we can let Q act on $[\Gamma, \Gamma]$ by conjugation, so that M can be promoted to a $\mathbb{Z}Q$ module. The following fact is basic and well-known.

Lemma 1. *The module M is finitely generated as a $\mathbb{Z}Q$ module.*

If $g_1, \ldots, g_r$ are generators of Γ, the commutators $h_{ij} = [g_i, g_j]$ generate $[\Gamma, \Gamma]$ as a $\mathbb{Z}Q$ module: if $[g, h]$ if $h = ab$ we have $[g, h] = [g, ab] = gag^{-1}a^{-1}agbg^{-1}b^{-1}a^{-1} = [g, a]a[g, b]a^{-1} = [g, a]a_*[g, b]$, and the result follows by induction. □

Theorem 1. *Let Γ be a finitely generated group. Given a prime number p (p might be 0), there exists a finite number of fields K_ν of characteristic p and finitely generated over F_p (if $p = 0$, set $F_p = \mathbb{Q}$) and characters $\xi_\nu : \Gamma \to K_\nu^*$ such that*

1. *$H^1(\Gamma, \xi_\nu) \neq 0$, i.e. $\xi_\nu \in E^1(\Gamma, K_\nu)$;*
2. *If K is a field of characteristic p and $\chi \in E^1(\Gamma, K)$ a Green–Lazarsfeld character, then there exists an index ν s.t. $\ker \chi \supset \ker \xi_\nu$.*

Proof. Let F_p be the field with p elements (or $\mathbb{Q}$ if $p = 0$) and let $F_p[Q]$ the group ring of Q with F_p coefficients. Let $M_p = [\Gamma, \Gamma] \otimes F_p$, $\mathcal{J} \subset F_p[Q]$ the annihilator of M_p, and $A = F_p[Q]/\mathcal{J}$. As Q is a finitely generated abelian group, isomorphic to $\mathbb{Z}^r \times \Phi$, with Φ finite abelian, A is a noetherian ring. Thus A admits a finite number of *minimal* prime ideals $(\mathfrak{p}_\nu)_{1 \leqslant \nu \leqslant \nu_0}$. Let k_i be the field of fraction of $A/\mathfrak{p}_i$, and ξ_i be the natural character $\Gamma \to Q \to A/\mathfrak{p}_i \to k_i$. Up to re-ordering the list of these ideals, we may assume that, for $1 \leqslant i \leqslant \nu_1$, $H^1(\Gamma, \xi_i) \neq 0$.

Note that, by very construction the fields K_ν are finitely generated over F_p. Therefore, the Theorem 1 is a consequence of the following:

Lemma 2. *Let $\chi \in E^1(\Gamma, K)$ be an exceptional character, $\chi \neq 1$. This character extends to an homomorphism $\chi : F_p[Q] \to K$.*

1. *The kernel of χ is a prime ideal in $F_p[Q]$ which contains $\mathcal{J}$, and defines a ring homomorphism $\bar{\chi} : A \to K$.*
2. *Let $\mathfrak{p}$ be a minimal prime ideal contained in* $\ker \bar{\chi}$, *and $K_{\mathfrak{p}}$ the field $A/\mathfrak{p}$. Then, the character $\xi_{\mathfrak{p}} : \Gamma \to k_{\mathfrak{p}}$ belongs to $E^1(\Gamma, k_{\mathfrak{p}}^*)$, i.e. $H^1(\Gamma, \xi) \neq 0$.*

Let χ as in the lemma and $\theta \in H^1(\Gamma, \chi)$ a non-trivial cocycle: θ defines a non-trivial morphism $M_p = [\Gamma, \Gamma] \otimes F_p \to K$. Let m_o with $\theta(m_0) \neq 0$. Let us extend χ to a ring homomorphism $\chi : F_p[Q] \to K$. If $j \in \mathcal{J}$, as $\chi(m_0) = 0$, we have $0 = \theta(j \cdot m_0) = \chi(j)\theta(m_0)$, hence $\chi(j) = 0$. Thus, the kernel of χ contains $\mathcal{J}$, and χ descends to A. This proves 1.

Let $M_{\mathfrak{p}} = M \otimes A_{\mathfrak{p}}$, and $M_0 = M \otimes_A K = M_{\mathfrak{p}}/\mathfrak{p}M_p$. Note that M_0 is a finitely generated $k_{\mathfrak{p}}$ vector space, on which Γ acts by homotheties: the action of g is the homothety of ratio $\xi(g)$. Let $\pi : [\Gamma, \Gamma] \to M_0$ the canonical map. We shall prove that $H^1(\Gamma, M) \neq 0$.

For some $g_0 \in \Gamma, \xi(g_0)$ is not 1 (as an element of $k_{\mathfrak{p}}$): if not $\Gamma = \ker \xi_p$ so $\chi = 1$.

The map $\Gamma \to M_0$ defined by $c(g) = \pi(g_0 g g_0^{-1} g^{-1})$ satisfies $c(gh) = \pi(g_0 g h g_0^{-1} h^{-1} g^{-1}) = \pi(g_0 g g_0^{-1} g^{-1}) + \pi(g g_0 h g_0^{-1} h^{-1} g^{-1}) = c(g) + \xi(g)\pi(g_0 h g_0^{-1} h^{-1}) = c(g) + \xi(g)c(h)$. Therefore c is a 1-cocycle of Γ with value in M.

Let us prove, by contradiction, that the cohomology class of c is not 0.

For every $m \in M_0$, $c(m) = (\xi(g_0)m - m) = (\xi(g_0) - 1)m$. If $c = 0$, as $\xi(g_0) \neq 1$, then $M_0 = 0$. But if $M_0 = 0$, $M_{\mathfrak{p}}/\mathfrak{p}M_p = 0$, i.e. $\mathfrak{p}M_{\mathfrak{p}} = M_{\mathfrak{p}}$, and $M_{\mathfrak{p}} = 0$ by the Nakayama lemma ($\mathfrak{p}$ is the unique maximal ideal of $A_{\mathfrak{p}}$), i.e. $M = \mathfrak{p}M$. But $\mathfrak{p} \subset \ker \bar{\chi}$, so this would implies that $M \otimes_A K = 0$ and $H^1(\Gamma, \chi) = 0$.

If this cocycle is a coboundary we could find some $m \in M_0$ s.t. $c(g) = (1 - \xi(g))m$, but $c(g_0) = 0$, and $\xi(g_0) \neq 1$, so c would be 0.

In order to prove Lemma 2, we see that, for every linear map $l = M_0 \to K$, $l \circ c$ is a non-trivial 1-cocycle.

This proves Theorem 1. □

Remark 3. The previous proof is a combination of arguments by [BiG] and [Br]. In their remarkable paper R. Bieri and J. Groves describe the BNS invariant of a metabelian group in terms of a finite set of field k_ν and characters ξ_ν: the fields k_ν are the fields of fractions of the minimal prime ideals $\mathfrak{p}_\nu$ of the noetherian ring $\mathbb{Z}Q/\operatorname{Ann}_{\mathbb{Z}Q} M$, and ξ_ν are the tautological characters. For every such a field and every valuation v on it, $v \circ \xi_\nu$ is exceptional. This provide a map from the cone of valuations on the family of fields k_ν to the BNS set. This set turns out to be the union of the images of these cones. In [Br], Breuillard proves along the same lines, that a metabelian not virtually polycyclic group admits a non-trivial affine action.

4 Fundamental Groups of Kähler Manifolds

4.1 Fibering a Kähler manifold. For the general study of orbifolds and their fundamental groups, we refer to Thurston's book [T, Ch. 13]. Complex 2-orbifolds are 2-orbifolds with singularities modeled on the quotient of the unit disk by the action of $\mathbb{Z}/n\mathbb{Z}$. The usefulness of this notion in our context of (fibering complex manifolds to Riemann surfaces) has been pointed out by Simpson [Si1].

Definition 4 (Complex 2-orbifold, and holomorphic maps). *A complex 2-orbifold Σ is a Riemann surface S marked by a finite set of marked points $\{(q_1, m_1) \dots, (q_n, m_n)\}$, where the m_i's are integers $\geqslant 2$.*

Let X be a complex manifold, $f : X \to \Sigma$ a map. Let $x \in X, q = f(x)$. Let $m \in \mathbb{N}^*$ be the multiplicity of q, so that there exists an holomorphic map $u : D(0, r) \subset \mathbb{C} \to (\Sigma, q)$ which is a ramified cover of order m of a neighborhood of q. Then, f is called holomorphic at x, if there exists a neighborhood U of q and a lift $\tilde{f} : U \to D$, holomorphic at x such that $f = u \circ \tilde{f}$.

Definition 5 (Fundamental group). *Let $\Sigma = (S; \{(q_1, m_1) \dots, (q_n, m_n)\})$ be a 2-orbifold. Let $q \in S \backslash \{(q_1, m_1), \dots, (q_n, m_n)\}$. The fundamental group – in the sense of orbifolds – of Σ at the point p is the quotient $\pi_1^{\mathrm{orb}}(\Sigma, p) = \pi_1(S \backslash \{q_1, \dots q_n\}) / \ll\gamma_i^{m_i}\gg$, where γ_i is the class of homotopy (well defined up to conjugacy) of a small circle turning once around q_i, and $\ll\gamma_i^{m_i}\gg$ is the normal subgroup generated by all the conjugates of $\gamma_i^{m_i}$.*

Example 1 (This is the main example, see [T, Ch. 13]). Let $\Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ be a uniform (discrete co-compact) lattice. The quotient $S = D/\Gamma$ of the unit disk by the action of Γ is a Riemann surface. If $p \in D$, its stabilizer is a finite hence cyclic subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Modulo the action of Γ there is only a finite set of points $\{q_1, \dots, q_n\}$ with non-trivial stabilizers of order m_i. The quotient orbifold is $\Sigma = (S; \{(q_1, m_1), \dots, (q_n, m_n)\})$. One proves that $\Gamma = \pi_1^{\mathrm{orb}}(\Sigma)$. An orbifold is called hyperbolic if it is obtained in this way; an orbifold is hyperbolic if and only if its Euler characteristic $\chi^{\mathrm{orb}}(\Sigma) = \chi(S) - \Sigma_{1 \leqslant i \leqslant n}\big(1 - \frac{1}{m_i}\big)$ is non-positive.

The following definition is useful to understand the structure of Kähler groups (see [ABCKT]).

Definition 6. *A compact Kähler manifold X fibers, if there exists a pair (Σ, F) where $\Sigma = (S; \{(q_1, m_1), \dots, (q_n, m_n)\})$, is a hyperbolic 2-orbifold, and $F : X \to \Sigma$ an holomorphic map with connected fibers. Two such maps $F : X \to \Sigma$, $F' : X' \to \Sigma'$ are equivalent if the fibers of F and F' are the same and images in Σ and Σ' of singular fibers have the same order. In this case there exists an holomorphic isomorphism from S to S' which maps singular points of S to singular points of S' preserving the multiplicity.*

Let $\pi : X \to S$ be an holomorphic map from a compact complex surface to a curve. If $q \in S$ is a singular value of π, the analytic set $\pi^{-1}(q)$ can be decomposed in a finite union of irreducible sets, (D_i). Away from a set of complex dimension $n-2$ in D_i, hence of complex codimension 2 in X, the map p can by written $\pi(z_1, \dots, z_n) = z_1^{d_i}$, where d_i is the multiplicity of D_i. The multiplicity of the fiber $\pi^{-1}(q)$ is by definition $m = \mathrm{pgcd}(d_i)$. Let Σ be the orbifold whose underlying space is S, singular points are singular values of π with corresponding multiplicity.

Lemma 3. *$\pi : X \to \Sigma$ is holomorphic.*

By construction, locally in the neighborhood of a point of $\pi^{-1}(q)$, $\pi(x) = f_1^{d_1} \dots f_k^{d_k} + \mathrm{cte}$, with $m | \operatorname{pgcd} d_i$. □

The following finiteness theorem is well known, and implicit in the literature at several places.

Theorem 2. *Let X be a compact complex manifold. There exists, up to equivalence, a finite set of pair (Σ_i, F_i) where $\Sigma_i$is a complex hyperbolic 2-orbifold, $F_i : X \to \Sigma_i$ is holomorphic with connected fibers.* □

Let us give a proof of this (well-known) fact based on the Kobayashi-hyperbolicity of a hyperbolic 2 orbifold: there exists no holomorphic map from $\mathbb{C}$ to an hyperbolic 2-orbifold as there exists no holomorphic map from $\mathbb{C}$ to the unit disk. Thus, by the Bloch principle, as X is compact there exists a uniform bound on the differential of an holomorphic map $F : X \to \Sigma$. Therefore the set of pairs (F, Σ) is *compact* (two such orbifold are ε-close if they are close for the Gromov–Hausdorff topology, i.e. there exists a map between them which is isometric up to an error of ε). But this compact space has only isolated points: if $F_1 : X \to \Sigma_1$ is given, and the (Gromov–Hausdorff) distance of F to F_1 is smaller than the diameter of Σ_1 (for instance $\leqslant 1/2 \operatorname{diam}(X)$ where X is endowed the Kobayashi pseudo-metric) all the fibers of F_1 are sent by F inside an open subsurface of Σ_1 therefore to a constant by the maximum principle; in other words F factorizes through F_1 and induces an isomorphism between Σ and Σ_1. □

The following is well known (see [Si1], [CaKO]).

Theorem 3. *Let $F : X \to S$ be an holomorphic map with connected fibers from the complex manifold X to a complex curve S. Let Σ be the orbifold whose singular points are the singular values of p and multiplicity the multiplicity of the corresponding fiber. Let $Y = F^{-1}(b)$ be the fiber of a non-singular point of S. Let $\pi_1'(Y)$the image in $\pi_1(X)$ of $\pi_1(Y)$. One has the exact sequence*

$$1 \to \pi_1'(Y) \to \pi_1(X) \to \pi_1^{\mathrm{orb}}(\Sigma) \to 1\,,$$

in particular the kernel of $F_* : \pi_1(X) \to \pi_1^{\mathrm{orb}}(\Sigma)$ is finitely generated. □

4.2 Valuations. The next result is a reformulation of a fibration theorem of Gromov–Schoen [GrS] and Simpson [Si3] in terms of the exceptional set in the sense of Bieri–Neumann–Strebel; see also [D] for a more general study of the BNS invariant of a Kähler group, where $\omega \in H^1(\Gamma, \mathbb{R})$ rather than $H^1(\Gamma, \mathbb{Z})$. Let Γ be the fundamental group of a compact Kähler manifold X.

Theorem 4. *Let $\omega \in H^1(\Gamma, \mathbb{Z})$. Then ω is exceptional iff there exists a hyperbolic orbifold Σ and an holomorphic map $F : X \to \Sigma$ such that $\omega \in F^* H^1(\Sigma, \mathbb{Z})$.*

Let η be a closed holomorphic $(1,0)$ form whose real part is the harmonic representative of ω. Let $\tilde{X}$ the universal cover of X, and $F : \tilde{X} \to \mathbb{R}$ a primitive of $\mathrm{Re}\,\eta$. From the definition (Remark 5) of $\mathcal{E}^1$ we know that $F \geqslant 0$ is not connected; [Si3] applies. One can also apply the proof of Corollary 9.2 of [GrS] to the foliation defined by the complex valued closed $(1,0)$ form whose real part is the harmonic representative of ω.

To prove the converse (which will not be used), one remarks that for every $w \in H^1(\Sigma, \mathbb{Z})$, its pull back to $H^1(\Sigma, \mathbb{Z})$ is exceptional, as $\pi_1^{\mathrm{orb}}(\Sigma)$ is hyperbolic, and the kernel of $\pi_1^{\mathrm{orb}}(\Sigma) \to \mathbb{Z}$ cannot be finitely generated. □

4.3 The Green–Lazarsfeld set of a Kähler group. Let K be a field. Recall that a character $\chi : \Gamma \to K^*$ is called *integral* in the sense of Bass [B] if $\chi(\Gamma) \subset O$, the ring of algebraic integers of K. In the case where $\mathrm{char}\, K = p > 0$, an algebraic integer is a root of unity, and an integral character is torsion.

Proposition 2. *Let X be a compact Kähler manifold, $\chi \in E^1(\Gamma, K^*)$ be a character. If χ is not integral, X fibers over a 2-orbifold Σ such that $\chi \in F^* E^1(\pi_1^{\mathrm{orb}}(\Sigma), K^*)$.*

Proof. Let $\theta \in H^1(\Gamma,\chi)$ be a non-trivial co-cycle. Consider the subfield K_1 of K generated by the coefficient of the matrices $\begin{pmatrix} \chi(g_i) & \theta(g_i) \\ 0 & 1 \end{pmatrix}_{1\leqslant i\leqslant r}$, g_i being a finite generating system of Γ. This field is finitely generated over its prime field. Therefore, if χ is not integral, one can find a valuation v on K_1 such that $\omega = v\circ\chi \neq 0$ (see, for instance, [B, Lem. 6.8]). Let Γ acts on the Bruhat–Tits tree T_v. By Proposition 1 this action is exceptional. Applying Theorem 4 we get a pair (F,Σ) such that $\omega \in F^*H^1(\Sigma,\mathbb{Z})$. From the exact sequence of Theorem 3, we see that $\pi_1'(Y)$ is a finitely generated normal subgroup of Γ made up with elliptic elements. As $\pi_1'(Y)$ is *finitely generated*, the subtree of T_v made up with fixed points of $\pi_1'(F)$ is not empty [S, Cor. 3, p. 90]. As $\pi_1'(Y)$ is normal, it is invariant by the action of Γ. Therefore the boundary of this tree contains at least 3 distinct elements. Thus acting on $P^1(K)$, $\pi_1'(Y)$ fixes three different points and is the identity: $\pi_1'(Y)\subset\ker\rho$, and ρ descends to some character on $\pi_1^{\rm orb}(\Sigma)$. □

The following proposition is a reformulation of a result by Beauville [Be, Cor. 3.6], it will be used to study the cohomology class of $v\circ\chi$, for the archimedean valuation $v(z)=\ln|z|$ of a character $\chi:\Gamma\to\mathbb{C}^*$.

PROPOSITION 3. *Let X be a compact Kähler manifold, $\chi\in E^1(\Gamma,\mathbb{C}^*)$ be character. If $|\chi|\neq 1$, there exists an holomorphic map $F:X\to\Sigma$ from X to a 2-orbifold Σ such that $\chi\in F^*E^1(\pi_1^{\rm orb}(\Sigma),K^*)$.*

Combining Propositions 2 and 3, we get the description of the GL set of a Kähler manifold in terms of its fibering over hyperbolic 2-orbifolds. This generalizes results by Green and Lazarsfeld [GL1], Beauville [Be], Simpson [Si2], Campana [C], Pink and Roessler [PR], who studied the case where the field K is the field of complex numbers.

Theorem 5. *Let Γ be the fundamental group of a compact Kähler manifold X, $(F_i,\Sigma_i)_{1\leqslant i\leqslant n}$ the family of fibration of X over hyperbolic 2-orbifolds. Let K be a field of characteristic p, $\bar F_p\subset K$ the algebraic closure of F_p in K, or $\mathbb{Q}$ in K if $p=0$. Then $E^1(\Gamma,K)$ is made with a finite set of torsion characters (contained in $E^1(\Gamma,\bar F_p)$) and the union of $F_i^*\operatorname{Hom}(\pi_1^{\rm orb}(\Sigma_i),K^*)$.*

Proof. We shall prove that a character χ which is not in the union $\bigcup F_i^*\operatorname{Hom}(\pi_1^{\rm orb}(\Sigma_i),K^*)$ must be a torsion character of bounded order. Let us fix such a character χ.

From Theorem 1, we know that there exists a finite number of fields K_ν and characters ξ_ν such that $H^1(\Gamma,\xi_\nu)\neq 0$, and for every $\chi\in E^1(\Gamma,K)$ there exists an index ν for which $\ker\xi_\nu\subset\ker\chi$. If ξ_ν is not integral, there exists a 2-orbifold Σ and a holomorphic map $F:X\to\Sigma$ such that $\ker F_*\subset\ker\xi_v$: therefore $\ker F_*\subset\ker\chi$ and $\chi\in F^*E^1(\pi_1^{\rm orb}(\Sigma))$.

Thus, as $\chi \notin \bigcup F_i^* \operatorname{Hom}(\pi_1^{\rm orb}(\Sigma_i), K^*)$, χ is integral.

Let us first discuss the case of positive characteristic. If ξ_ν is integral, then $\xi_\nu(\Gamma)$ is made with roots of unity of K_ν. But K_ν is finitely generated over F_p so admits only a finite number of roots of unity (see [B, Lem. 6.8(3)] for instance); let d_ν the order of the group of roots of unity in K_ν. We see that χ is a torsion character of order d dividing d_ν.

Suppose now that $\operatorname{char} K = 0$, and ξ_ν is integral. Thus K_ν is a number field, and $\xi_\nu(\Gamma)$ is contained in the ring O_ν of integers of ξ_ν. If $|\xi_\nu| \neq 1$, or if one of its conjugates $\sigma(\xi_\nu)$ has $|\sigma(\xi_\nu)| \neq 1$, as $H^1(\Gamma, \xi_\nu) \neq 0$ we know (Proposition 3) that there exists a 2-orbifold Σ and a holomorphic map $F : X \to \Sigma$ such that $\ker F_* \supset \ker \xi_v$; the previous argument apply and proves that $\chi \in F^* E^1(\pi_1^{\rm orb}(\Sigma))$. Therefore, by a theorem of Kronecker, χ must be a torsion character. It is furthermore of bounded degree, as the degree of the n-th cyclotomic polynomial goes to infinity with n, and as d divides the degree of K_ν. The rest of the argument is unchanged. □

Thus, the Theorem 5 reduces the computation of $E^1(\Gamma, K^*)$ to the case where Γ is the fundamental group of a 2-orbifold (see also [Be]).

Proposition 4. *Let $\Gamma = \pi_1^{\rm orb}(\Sigma)$, for $\Sigma = (S; (q_i, m_i)_{1 \leqslant i \leqslant n})$ a hyperbolic 2-orbifold then, $E^1(\pi_1^{\rm orb}(\Sigma), K^*) = \operatorname{Hom}(\pi_1^{\rm orb}(\Sigma, K^*))$ unless $g = 1$ and, for all i, $m_i \not\equiv 0 (\operatorname{char} K)$.*

If $g = 1$ and, for all i, $m_i \not\equiv 0(\operatorname{char} K)$, $E^1(\pi_1^{\rm orb}(\Sigma), K^*)$ is finite, made of torsion characters.

Let $\chi : \pi^{\rm orb}(\Sigma) \to K^*$ be a representation. If $\chi = 1, H^1(\pi_1^{\rm orb}(\Sigma), K^*) = \operatorname{Hom}(\pi_1^{\rm orb}(\Sigma), K^*) \neq 0$. If $g > 1$, consider a simple closed curve c on S such that c is homologous to 0 and separates S into two compact surfaces of positive genus S_1, S_2, with common boundary c and such that all singular points are in S_2. We may assume $g(S_1) = 1$; if $g = 1$ consider a curve c, which bounds a disk $\bar{D}$ on S containing all singular points q_i, and let $S_1 = S \setminus \operatorname{int}(D)$ be the other component. One consider a representation $\chi : \pi_1^{\rm orb}(\Sigma) \to K^*$, and note that $\chi(c) = 1$ as c is homologous to 0. We think of χ as a local system on Σ and we will use a Mayer–Vietoris exact sequence.

First note that if $\chi|_{\pi(S_1)}$ and $\chi|_{\pi^{\rm orb}(\Sigma_2)}$ are not 1, then $H^1(\pi_1^{\rm orb}(\Sigma), K^*) \neq 0$: let $x_0 \in K$, there exists a unique 1-cocycle c such that $c(g) = x_0(1 - \chi(g))$ is $g \in \pi_1(S_1)$, $c(g) = 0$ if $g \in S_2$.

If $\chi|_{S_1} = 1$, as $H^1(S_1, \partial S_1, K) = K^2$, one can find a 1-cocycle c whose restriction on S_2 or D is 0, and whose restriction on S_1 is not trivial. In the case where $g(S) = 1$ such a character is torsion.

We are left to the case $\chi|_{S_2}$ or $\chi|_D = 1$. If $g(S_2) > 0 H^1(\pi_1^{\rm orb}(\Sigma_2, \partial\Sigma_2), K) \twoheadrightarrow K^{2g}$ and the previous argument apply while interchanging the role of S_1 and S_2.

The remaining case is $g = 1$, $\chi|_{\pi_1^{\rm orb}(D)} = 1$, $\chi|_{\pi_1(S_1)} \neq 1$. One has $H^1(\pi_1^{\rm orb}(\Sigma_2), K) = \{(z_1, \dots, z_n) \in K / m_i z_i = 0\}$. This space is 0 unless $m_i \equiv 0$ $(\operatorname{char} K)$ for some i. On the other hand, if $\rho|_{\pi_1(S_1)} \neq 0$ the homomorphism $H^1(\pi_1(\Sigma_1), \rho) \to K$ which sends θ to $\theta(c)$ is an isomorphism. Using the exact sequence of Mayer–Vietoris, we see that $H^1(\pi_1^{\rm orb}(\Sigma), \chi) \neq 0$ iff $g > 1$ or $g = 1$ and for some i, m_i divides the characteristic of K. □

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