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DISPLACING REPRESENTATIONS AND ORBIT MAPS

FOR BOB ZIMMER, WITH ADMIRATION

Contents

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1. Introduction

Let γ be an isometry of a metric space X. We recall that the *displacement* of γ is

$$d_X(\gamma) = \inf_{x \in X} d(x, \gamma(x)).$$

In the case of the Cayley graph of a group Γ with set of generators S and word length $\| \ \|_S$, the displacement function is called the *translation length*—or the *stable translation length*—and is denoted by $\ell_S : \ell_S(\gamma) = \inf_\eta \|\eta\gamma\eta^{-1}\|_S$. We finally say the action by isometries on X of a group Γ is *displacing*, if given a set S of generators of Γ, there exist positive constants A and B such that

$$d_X(\gamma) \geqslant A\ell_S(\gamma) - B.$$

This definition does not depend on the choice of S. As first examples, it is easy to check that cocompact groups are displacing, as well as convex cocompact whenever X is *Hadamard* (i.e., complete, nonpositively curved, and simply connected). We recall that a cocompact action is by definition a properly discontinuous action whose quotient is compact, and a convex cocompact is an action such that there exists a convex invariant on which the action is cocompact.

The notion naturally arose in [7] where it is shown that for displacing representations of surface groups, the energy functional is proper on Teichmüller space and that, moreover, a large class of representations of surface groups are displacing.

This definition is a cousin to a more well-known one. Assume Γ acts by isometries on a space X. We say the *orbit maps are quasi-isometric embedding*—or in short the action is QI—if for every x in X there exist constants A and B so that we have

$$\forall \gamma \in \Gamma, \ \ d(x, \gamma(x)) \geq A\|\gamma\|_S - B.$$

The purpose of this modest note is to collect some elementary observations about the relation between the two notions in order to complete the circle of ideas discussed in [7]. For hyperbolic groups the two notions turn out to be equivalent. In general, however, this relation is slightly more involved than expected.

Does displacing imply QI? In general, the answer is no: this follows immediately from the existence proved by Osin in [9] of infinite groups with finitely many conjugacy classes. However, we isolate a class of groups for which displacing implies QI. We say a finitely generated group is *undistorted in its conjugacy classes* (see Section 2)—or satisfies the *U*-property if there exists finitely many elements $g_1, \ldots, g_p$ of Γ, positive constants A and B such that

$$\forall \gamma \in \Gamma, \ \ \|\gamma\|_S \leqslant A \sup_{1 \leqslant i \leqslant n} \ell_S(g_i \gamma) + B.$$

We prove in Theorem 2.1.1 that some class of linear groups—in particular lattices—enjoy the U-property. We also prove that hyperbolic groups have the U-property. For all groups enjoying the U-property displacing implies QI.

Does QI imply displacing? In general, again, the answer is no: in Corollary 3.2.3, we show that for a residually finite group (see Section 3.1) any linear that representation contains a unipotent is not displacing. It follows that $SL(n, \mathbb{Z})$ acting on $SL(n, \mathbb{R})/SO(n, \mathbb{R})$ is QI but not displacing. However, again a simple argument using the stable length shows that for hyperbolic groups QI implies displacing (see Section 4).

We expand in the article the discussion of this introduction and start by discussing the U-property.

2. Groups Whose Displacing Actions Have Orbit Maps That Are Quasi-Isometric Embeddings

We say a finitely generated group is *undistorted in its conjugacy classes*—in short, has the *U-property*,—if there exists finitely many elements $g_1, \ldots, g_p$ of Γ, positive constants A and B such that

$$\forall \gamma \in \Gamma, \ \ \|\gamma\| \leqslant A \sup_{1 \leqslant i \leqslant n} \ell(g_i \gamma) + B.$$

REMARKS:

- This property is clearly independent of S. (Hence we omitted the subscript S.)
- This property is satisfied by free groups and commutative groups. On the other hand, the groups constructed by D. Osin [9] described in the above paragraph do not have this U-property.
- Note also that this property is very similar to the statement of Abels, Margulis, and Soifer's result [1, theorem 4.1] and it is no surprise that their result plays a role in the proof of Theorem 2.1.1.
- Finally, by the conjugacy invariance of the translation length, $\ell(g_i \gamma) = \ell(\gamma g_i)$, so we will indifferently write this property with left or right multiplication by the finite family (g_i).

LEMMA 2.0.1. *If Γ has the U-property, then every displacing action has orbit maps that are quasi-isometric embeddings.*

Proof. Indeed, assume that Γ acts on X by isometries and that the action is displacing. In particular, we have positive constants α and β so that

$$\forall x \in X, \quad \gamma \in \Gamma, \quad d(x, \gamma(x)) \geqslant \alpha \ell_S(\gamma) - \beta.$$

Moreover, there exists finitely many elements $g_1, \ldots, g_p$ of Γ, positive constants A and B such that

$$\forall \gamma \in \Gamma, A\|\gamma\| - B \leqslant \sup_{1 \leqslant i \leqslant n} \ell(g_i \gamma).$$

Hence, let $x \in X$, then

$$\begin{aligned} d(x, \gamma(x)) &\geqslant \sup_{1 \leqslant i \leqslant n} d(x, g_i.\gamma(x)) - \sup_{1 \leqslant i \leqslant n} d(x, g_i(x)) \\ &\geqslant \alpha \sup_{1 \leqslant i \leqslant n} \ell(g_i \gamma) - \beta - \sup_{1 \leqslant i \leqslant n} d(x, g_i(x)) \\ &\geqslant \alpha.A\|\gamma\| - B\alpha - \beta - \sup_{1 \leqslant i \leqslant n} d(x, g_i(x)). \end{aligned}$$

Hence, the orbit map is a quasi-isometric embedding. □

We will prove in the next section,

THEOREM 2.0.2. *Every uniform lattice—and nonuniform lattice in higher rank—in characteristic 0 has the U-property. In particular, every surface group has the U-property.*

Moreover, we show that hyperbolic groups have the *U*-property.

2.1. Linear Groups Having the U-Property

We prove the following result that implies Theorem 2.0.2.

THEOREM 2.1.1. *Let Γ be a finitely generated group and $\mathbf{G}$ a reductive group defined over a field F, suppose that*

- *there exists a homomorphism $\Gamma \to \mathbf{G}(F)$ with Zariski-dense image, and*
- *there are a finitely many field homomorphisms $(i_v)_{v \in S}$ of F in local fields F_v such that the diagonal embedding $\Gamma \to \Pi_{v \in S} \mathbf{G}(F_v)$ is a quasi-isometric embedding.*

Then the group Γ has the U-property.

Since lattices are Zariski dense (Borel theorem) and that higher-rank irreducible lattices in characteristic 0 are quasi-isometrically embedded [8], this theorem implies Theorem 2.0.2 for higher-rank lattices. The same holds for all uniform lattices. A specific corollary is the following:

COROLLARY 2.1.2. *Let Γ be a finitely generated group that is quasi-isometrically embedded and Zariski dense in a reductive group $G(F)$ where F is a local field. Then Γ has the U-property.*

2.1.1. GENERALITIES FOR THE *U*-PROPERTY We prove two lemmas for the *U*-property.

LEMMA 2.1.3. *Let Γ be a finitely generated group. Let $\Gamma_0 \lhd \Gamma$ be a normal subgroup of finite index. If Γ_0 has the U-property, so has Γ.*

We do not know whether the converse statement holds, in other words, whether the *U*-property is a property of commensurability classes.

Proof. We first observe that every finite-index subgroup of a finitely generated group is finitely generated. Let S_0 be a generating set for Γ_0 and write Γ as the union of left cosets for Γ_0

$$\Gamma = \bigcup_{t \in T} \Gamma_0 \cdot t.$$

We assume that T is symmetric. Clearly $S = S_0 \cup T$ is a generating set for Γ.

We denote $\|\cdot\|_{\Gamma_0}$ and ℓ_{Γ_0} the word and translation lengths, respectively, for Γ_0.

We observe that Γ_0 is quasi-isometrically embedded in Γ. Hence there exist positive constants α and β such that

$$\forall \gamma \in \Gamma_0, \;\; \|\gamma\|_{\Gamma_0} \geqslant \|\gamma\|_{\Gamma} \geqslant \alpha \|\gamma\|_{\Gamma_0} - \beta. \tag{1}$$

For any γ in Γ, we write $\gamma = \gamma_0 t_0$ with $t_0 \in T$ and $\gamma_0 \in \Gamma_0$. Hence

$$\|\gamma\|_{\Gamma} \leqslant \|\gamma_0\|_{\Gamma_0} + 1 \leqslant A \sup \ell_{\Gamma_0}(\gamma_0 g_i) + B + 1$$

since Γ_0 has the *U*-property.

Next, we need to compare ℓ_Γ and ℓ_{Γ_0}. Let δ in Γ_0, then

$$\begin{aligned} \ell_\Gamma(\delta) &= \inf_{t \in T, \eta \in \Gamma_0} \|t\eta t\delta\eta^{-1}t^{-1}\|_\Gamma \\ &\geqslant \inf_{\eta \in \Gamma_0} \|\eta\delta\eta^{-1}\|_\Gamma - 2 \end{aligned}$$

$$\geqslant \alpha \inf_{\eta \in \Gamma_0} \|\eta\delta\eta^{-1}\|_{\Gamma_0} - \beta - 2$$

$$\geqslant \alpha \ell_{\Gamma_0}(\delta) - \beta - 2. \tag{2}$$

Finally, combining Inequality (2) and the one preceding, we have

$$\|\gamma\|_\Gamma \leqslant \frac{A}{\alpha} \sup_{t \in T, i \in I} \ell_\Gamma(\gamma t^{-1}\beta_i) + B + 1 + \frac{\beta + 2}{\alpha}.$$

This is exactly the U-property for Γ. □

Also,

LEMMA 2.1.4. *Let Γ be a finitely generated group. Suppose that $\Gamma \to \Gamma_0$ is onto with finite kernel. Then the group Γ has the U-property if and only if Γ_0 has the U-property.*

Proof. We choose a generating set S for Γ that contains the kernel of $\Gamma \to \Gamma_0$. We choose the generating set S_0 for Γ_0 to be the image of S. Then we have, using surjectivity, for all γ projecting to γ_0

$$\|\gamma\|_\Gamma \geqslant \|\gamma_0\|_{\Gamma_0} \geqslant \|\gamma\|_\Gamma - 1,$$

$$\ell_\Gamma(\gamma) \geqslant \ell_{\Gamma_0}(\gamma_0) \geqslant \ell_\Gamma(\gamma) - 1.$$

These two inequalities enable us to transfer the U-property from Γ to Γ_0 and vice versa. □

2.1.2. PROXIMALITY We recall the notion of proximality and a result of Abels, Margulis, and Soifer.

Let k be a local field. Let V be a finite-dimensional k-vector space equipped with a norm. Let d be the induced metric on $\mathbb{P}(V)$. Let r and ϵ be positive numbers such that

$$r > 2\epsilon.$$

An element g of $\mathrm{SL}(V)$ is said to be *(r, ε)-proximal*, if there exist a point x_+ in $\mathbb{P}(V)$ and a hyperplane H in V such that

- $d(x_+, \mathbb{P}(H)) \geqslant r$, and
- $\forall x \in \mathbb{P}(V), \quad d(x, \mathbb{P}(H)) \geqslant \varepsilon \implies d(g \cdot x, x_+) \leqslant \varepsilon.$

In particular, a proximal element has a unique eigenvalue of highest norm. Conversely, if an element g admits a unique eigenvalue of highest norm, then some power of g is proximal (for some (r, ϵ)).

We cite the needed result from [1] and [2].

THEOREM 2.1.5. ([1], THEOREM 5.17) *Let* $\mathbf{G}$ *a semisimple group over a field* F. *Let* $(i_v)_{v\in V}$ *be finitely many field homomorphisms of* F *in local fields* F_v. *Let* $\rho_v : \mathbf{G}(F_v) \to \mathrm{GL}(n_v, F_v)$ *be an irreducible representation of* $\mathbf{G}(F_v)$ *for each* v.

Suppose that Γ *is a Zariski-dense subgroup of* $\mathbf{G}(F)$. *Suppose that for every* v, $\rho_v(\Gamma)$ *contains proximal elements. Then there exist*

- $r > 2\varepsilon > 0$, *and*
- *a finite subset* $\Delta \subset \Gamma$,

such that for every γ *in* Γ *there is some* δ *in* Δ *such that* $\rho_v(\gamma s)$ *is* (r, ε)*-proximal for every* v *in* V.

This result is usually stated with *one* local field but the proof of the above extension and the following is straightforward.

We shall also need the following Lemma.

LEMMA 2.1.6. ([2], COROLLAIRE P.13) *Let* Γ *be a Zariski-dense subgroup in* $\mathbf{G}(k)$, $\mathbf{G}$ *a reductive group over a local field* k. *Then* Γ *is unbounded if and only if there exists an irreducible representation of* $\mathbf{G}(k)$, *such that* $\rho(\Gamma)$ *contains a proximal element.*

The special case of $k = \mathbb{R}$ was proved in [3].

2.1.3. PROXIMAL ELEMENTS AND TRANSLATION LENGTHS

We recall some facts on length and translation length in $\mathbf{G}(k)$ where $\mathbf{G}$ is a semisimple group over a local field k.

Let K be the maximal compact subgroup of $\mathbf{G}(k)$. This defines a norm $\|g\|_G = d_{G/K}(K, gK)$ in $\mathbf{G}(k)$, which satisfies

$$\|gh\|_G \leqslant \|g\|_G + \|h\|_G.$$

We also consider the translation length ℓ_G in $\mathbf{G}(k)$:

$$\ell_G(g) = \inf_{h\in G} \|hgh^{-1}\|_G.$$

Observe that the translation norm is actually independent of the choice of the maximal compact subgroup since they are all conjugated. The translation length and norm of (r, ϵ)-proximal elements can be compared:

LEMMA 2.1.7. (COMPARE [2] §4.5) *Let* $\mathbf{G}$ *be a semisimple group over a local field* k. *Let* $\rho : \mathbf{G}(k) \to \mathrm{GL}(n, k)$ *be an irreducible representation. Let* $\varepsilon > 0$.

Then there exist positive constants α and β such that if $\rho(g)$ is (r,ε)-proximal, then

3 $$\ell_G(g) \geqslant \alpha\|g\|_G - \beta.$$

Proof. We use classical notation and refer to [2, pp. 6–8] for precise definitions.

Any g in $\mathbf{G}(k)$ is contained in a unique double coset $K\mu(g)K$. The element $\mu(g) \in A^+ \subset A$ is called the Cartan projection of g. Here we see A^+ as a subset of a cone $A^\times$ in some $\mathbf{R}$*-vector space.*

For some integer n ($n = 1$ in the Archimedean case) the element g^n admits a Jordan decomposition $g^n = g_e g_h g_u$ with g_e, g_h, g_u commuting, g_e elliptic, g_u unipotent, and g_h hyperbolic, that is, conjugated to a unique element $a \in A^+$. We set $\lambda(g) = \frac{1}{n}a \in A^\times$.

If we fix some norm on the vector space containing the cone $A^\times$, then (up to quasi-isometry constants) the norm of $\mu(g)$ is $\|g\|_G$ in $\mathbf{G}(k)$ and the norm of $\lambda(g)$ is $\ell_G(g)$.

Then by a result of Y. Benoist [2], there exists a compact subset N_ε of the vector space containing $A^\times$ such that for every g such that $\rho(g)$ is (r,ε)-proximal we have

$$\lambda(g) - \mu(g) \in N_\varepsilon. \qquad \square$$

The lemma follows.

2.1.4. PROOF OF THEOREM 2.1.1 By taking a finite-index normal subgroup and projecting (Lemmas 2.1.3 and 2.1.4) we can make the hypothesis that $\mathbf{G}$ is the product of a semisimple group $\mathbf{S}$ and a torus $\mathbf{T}$ and Γ is a subgroup of $\mathbf{S}(F) \times \mathbf{T}(F)$.

Moreover, since length and translation length for elements in $\mathbf{T}(F_\nu)$ are equal, we only need to work with the semisimple part $\mathbf{S}$.

Finally, it suffices to prove the existence of a finite family $F \subset \Gamma$ and constants A, B such that, for any $\gamma \in \Gamma$

4 $$\|\gamma\|_S \leqslant A \sup_{f\in F} \ell_S(\gamma f) + B,$$

where $S = \Pi_{\nu\in V}\mathbf{S}(F_\nu)$. Indeed, since Γ is quasi-isometrically embedded in G, $\|\gamma\|_\Gamma$ is less than $\|\gamma\|_G = \|\gamma\|_T + \|\gamma\|_S$ and $\ell_G(\gamma) = \ell_T(\gamma) + \ell_S(\gamma) = \|\gamma\|_T + \ell_S(\gamma)$ is less than $\ell_\Gamma(\gamma)$—up to quasi-isometry constants—and Inequality (4) implies that Γ has the *U*-property.

Note that we can forget any completion F_ν where the subgroup $\Gamma \subset \mathbf{S}(F_\nu)$ is bounded without changing the fact that Γ is quasi-isometrically embedded. So by Lemma 2.1.6, for each v there is an irreducible representation $\rho_v : \mathbf{S}(F_v) \to \mathrm{GL}(n_v, F_v)$ such that $\rho_v(\Gamma)$ contains proximal elements.

Applying Theorem 2.1.5 we find a finite family $F \subset \Gamma$ and r, ε such that for every γ in Γ there is some $f \in F$ such that $\rho_v(\gamma f)$ is (r, ε)-proximal for every v. Hence, as a consequence of Lemma 2.1.7, we have for such γ and f:

$$\|\gamma f\|_S \leqslant A\ell_S(\gamma f) + B.$$

This implies Inequality 4 and concludes the proof.

2.2. Hyperbolicity and the U-Property

Let Γ be a finitely generated group and S a set of generators. Let d be its word distance and $\|g\| = d(e, g)$. We denote by

$$\langle g, h\rangle_u = \frac{1}{2}(d(g, u) + d(h, u) - d(g, h)),$$

the *Gromov product*—based at U—on Γ. We abbreviate $\langle g, h\rangle_e$ by $\langle g, h\rangle$. Observe that

5 $$\langle gu, hu\rangle_u = \langle g, h\rangle_e.$$

Recall that Γ is called δ*-hyperbolic* if for all g, h, k in Γ we have

6 $$\langle g, k\rangle \geqslant \inf\,(\langle g, h\rangle; \langle h, k\rangle) - \delta$$

and Γ is called *hyperbolic* if it is δ-hyperbolic for some δ. A hyperbolic group is called *nonelementary* if it is not finite and does not contain $\mathbb{Z}$ as a subgroup of finite index.

Then,

PROPOSITION 2.2.1. *Hyperbolic groups have the U-property.*

We recall the *stable translation length of an element* g:

$$[g]_\infty = \lim_{n\to\infty} \frac{\|g^n\|}{n}.$$

We remark that obviously

$$[g]_\infty \leqslant \ell(g).$$

We shall actually prove

PROPOSITION 2.2.2. *Let Γ be hyperbolic. There exist a pair $u, v \in \Gamma$ and a constant α such that for every g one has*

$$\|g\| \leqslant 3\sup\,([g]_\infty, [gu]_\infty, [gv]_\infty) + \alpha.$$

In particular Γ has the U-property.

REMARK:

- Let Γ be a free group generated by some elements $u, v, w_1, \ldots, w_n$. Then G is 0 hyperbolic. If $[g] \neq \|g\|$ the first letter of g must be equal to the inverse of the last one. Multiplying either by u or by v we find a new element that is cyclically reduced; for this element the stable translation length and the length are equal. The proof of Proposition 2.2.2 is a generalization of this remark.

2.2.1. ALMOST CYCLICALLY REDUCED ELEMENTS An element g in Γ is said to be *almost cyclically reduced* if $\langle g, g^{-1}\rangle \leqslant \frac{\|g\|}{3} - \delta$. We prove in this paragraph

LEMMA 2.2.3. *If g is almost cyclically reduced, then*

$$[g]_\infty \geqslant \frac{\|g\|}{3}.$$

The following result [5, lemma 1.1] will be useful.

LEMMA 2.2.4. *Let (x_n) be a finite or infinite sequence in G. Suppose that*

$$d(x_{n+2}, x_n) \geqslant \sup\,(d(x_{n+2}, x_{n+1}), d(x_{n+1}, x_n)) + a + 2\delta,$$

or equivalently that

$$\langle x_{n+2}, x_n\rangle_{x_{n+1}} \leqslant \frac{1}{2}\inf\,(d(x_{n+2}, x_{n+1}), d(x_{n+1}, x_n)) - \frac{a}{2} - \delta.$$

Then

$$d(x_n, x_p) \geqslant |n-p|a.$$

This implies Lemma 2.2.3.

Proof. Let $x_n = g^n$. By left invariance and since g is almost cyclically reduced,

$$\langle x_{n+2}, x_n\rangle_{x_{n+1}} = \langle g, g^{-1}\rangle \leqslant \frac{\|g\|}{2} - \frac{a}{2} - \delta,$$

for $a = \frac{\|g\|}{3}$. By Lemma 2.2.4,

$$\|g^n\| \geqslant n\frac{\|g\|}{3}.$$ □

The result follows:

2.2.2. PING-PONG PAIRS A *ping-pong pair* in Γ is a pair of elements u, v such that

1) $\inf(\|u\|, \|v\|) \geqslant 100\delta$,
2) $\langle u^{\pm 1}, v^{\pm 1}\rangle \leqslant \frac{1}{2}\inf(\|u\|, \|v\|) - 20\delta$, and
3) $\langle u, u^{-1}\rangle \leqslant \frac{\|u\|}{2} - 20\delta$ and $\langle v, v^{-1}\rangle \leqslant \frac{\|v\|}{2} - 20\delta$.

REMARKS:

- A ping-pong pair generates a free subgroup. This is an observation from [5]. To prove this, consider a reduced word w on the letter u, v, u^{-1}, v^{-1}. If x_n is the prefix of length n of w, the sequence x_n satisfies the hypothesis of Lemma 2.2.4.
- In the present proof, the third property will not be used.

We shall prove

LEMMA 2.2.5. *If Γ is hyperbolic nonelementary, there exists a ping-pong pair.*

Proof. In [6] explicit ping-pong pairs are constructed. Here is a construction whose idea goes back to F. Klein. Let f be some hyperbolic element (an element of infinite order). Replacing f by a conjugate of some power, we may assume that

$$\|f\| = [f] > 1000\delta.$$

As Γ is not elementary, there exists a generator a of Γ that does not fix the pair of fixed points f^+, f^- of f on the boundary $\partial\Gamma$: otherwise, since the action of Γ is topologically transitive, $\partial\Gamma$ would be reduced to these two points and Γ would be elementary. Now, let us prove that for some integer N, $(f, af^N a^{-1})$ is a ping-pong pair. We have

$$\lim_{N\to+\infty} f^N = f^+ \neq af^+ = \lim_{N\to+\infty} af^N a^{-1}.$$

It follows that the Gromov product $\langle f^N, af^N a^{-1}\rangle$ remains bounded, by the very definition of the boundary. Hence, for N large enough, we have

$$(f, af^N a^{-1}) \leqslant \frac{1}{2}\inf(\|f^N\|, \|af^N a^{-1}\|) - 20\delta.$$

A similar argument also yields that $\langle f^N, af^{-N} a^{-1}\rangle$ remains bounded. Therefore, $(f^N, af^N a^{-1})$ is a ping-pong pair for $N \gg 1$. □

2.2.3. PROOF OF PROPOSITION 2.2.2 We first reduce this proof to the following lemma.

LEMMA 2.2.6. *Let (u, v) be a ping-pong pair. Let $g \in \Gamma$ such that*

$$\|g\| \geqslant 3 \sup(\|u\|, \|v\|) + 100\delta.$$

Then one of the three elements g, gu, gv is almost cyclically reduced.

We observe at once that Proposition 2.2.2 follows from Lemmas 2.2.3, 2.2.5, and 2.2.6; choose a ping-pong pair u, v and take

$$\alpha = 3 \sup(\|u\|, \|v\|) + 100\delta.$$

Proof. Assume g is not almost cyclically reduced. Then

7 $$\langle g, g^{-1} \rangle \geqslant \frac{\|g\|}{3} - \delta \geqslant \sup(\|u\|, \|v\|) + 30\delta.$$

Moreover, one of the following pair of inequalities holds:

8 $$\langle g^{-1}, u^{\pm 1} \rangle \leqslant \frac{\|u\|}{2} - 10\delta,$$

or

9 $$\langle g^{-1}, v^{\pm 1} \rangle \leqslant \frac{\|v\|}{2} - 10\delta.$$

Otherwise, by the definition of hyperbolicity we would have for some $\varepsilon, \varepsilon' \in \{\pm 1\}$,

$$\langle u^{\varepsilon}, v^{\varepsilon'} \rangle \geqslant \frac{1}{2} \inf(\|u\|, \|v\|) - 10\delta - \delta,$$

contradicting the second property of the definition of ping-pong pairs.

So we may assume that Inequality (8) holds. We will show that gu is almost cyclically reduced. Let $k = gu$. By the triangle inequality, $\langle u^{-1}, g \rangle \leqslant \|u\|$. Thus from Inequality (7) we deduce that

$$\inf(\langle g, u^{-1} \rangle, \langle g, g^{-1} \rangle) = \langle g, u^{-1} \rangle.$$

Then, by the definition of hyperbolicity, we get

10 $$\langle g, u^{-1} \rangle \leqslant \langle u^{-1}, g^{-1} \rangle + \delta.$$

Using Inequality (8) now, we have

11 $$\langle g, u^{-1} \rangle \leqslant \frac{\|u\|}{2} - 9\delta.$$

Note that

$$\langle g, k \rangle = \frac{1}{2}(\|g\| + \|gu\| - \|u\|) \geqslant \frac{1}{2}(2\|g\| - 2\|u\|) \geqslant 2\|u\| + 100\delta.$$

By the triangle inequality again,

$$\langle k, u^{-1}\rangle \leqslant \|u\|.$$

Therefore,

$$\inf\,(\langle g, k\rangle, \langle k, u^{-1}\rangle) = \langle k, u^{-1}\rangle.$$

From hyperbolicity and Inequality (11), we have

12 $$\langle k, u^{-1}\rangle \leqslant \langle g, u^{-1}\rangle + \delta \leqslant \frac{\|u\|}{2} - 8\delta.$$

Applying successively Inequalities (8) and (12), we get that

13 $$\langle k^{-1}, u^{-1}\rangle = \|u\| - \langle u, g^{-1}\rangle \geqslant \frac{\|u\|}{2} + 10\delta \geqslant \langle k, u^{-1}\rangle + 18\delta.$$

By hyperbolicity,

$$\inf\,(\langle k, k^{-1}\rangle, \langle k^{-1}, u^{-1}\rangle) \leqslant \langle k, u^{-1}\rangle + \delta.$$

Therefore, Inequalities (12) and (13) imply that

14 $$\langle k, k^{-1}\rangle \leqslant \langle k, u^{-1}\rangle + \delta \leqslant \frac{\|u\|}{2} - 7\delta.$$

Since $\|k\| \geqslant \|g\| - \|u\| \geqslant 3\|u\| - \|u\| + 100\delta$, we finally obtain that k is almost cyclically reduced. □

3. Nondisplacing Actions Whose Orbit Maps are Quasi-Isometric Embeddings

We prove in particular

PROPOSITION 3.0.7. *The action of* $\mathrm{SL}(n, \mathbb{Z})$ *on* $X_n = \mathrm{SL}(n, \mathbb{R})/SO(n, \mathbb{R})$ *is not displacing, although, for* $n \geqslant 3$, *the orbit maps are quasi-isometric embeddings.*

The second part of this statement is a theorem of Lubotzky, Mozes, and Ragunathan [8]. Note that for the action of $\mathrm{SL}(2, \mathbb{Z})$ on the hyperbolic plane $\mathbb{H}_2 = X_2$ the orbit maps are not quasi-isometries so it is obviously not displacing since $\mathrm{SL}(2, \mathbb{Z})$ is a hyperbolic group (see Corollary 4.0.6).

3.1. Infinite Contortion

We say a group Γ has *infinite contortion*, if the set of conjugacy classes of powers of every nontorsion element is infinite. In other words, for every nontorsion element γ, for every finite family $g_1, \ldots, g_q$ of conjugacy classes of elements of Γ, there exists $k > 0$ such that

$$\forall i \in \{1, \ldots, q\}, \gamma^k \notin g_i.$$

We prove

LEMMA 3.1.1. *Every residually finite group has infinite contortion.*

Proof. Let γ be a nontorsion element. Let $g_1, \ldots, g_n$ be finitely many conjugacy classes. We want to prove that there exists $k > 0$ such that γ^k belongs to no g_i. Since γ is not a torsion element, we can assume that all the g_i are nontrivial. Let $h_i \in g_i$. Since all h_i are nontrivial, by residual finiteness there exist a homomorphism ϕ in a finite group H, such that

$$\forall i, \varphi(h_i) \neq 1.$$

Let $k = \|H\|$, hence $\varphi(\gamma^k) = 1$. This implies that $\gamma^k \notin g_i$. □

3.2. Displacement Function and Infinite Contortion

We will prove

LEMMA 3.2.1. *Assume Γ has infinite contortion. Assume Γ acts cocompactly and properly discontinuoulsy by isometry on a space X. Assume furthermore that every closed bounded set in X is compact. Then, for every nontorsion element γ in Γ, we have*

$$\limsup_{p \to \infty} d_X(\gamma^p) = \infty.$$

REMARKS:

- We should notice that the conclusion immediately fails if Γ does not have infinite contortion. Indeed there exists an element γ such that its powers describe only finitely many conjugacy classes of elements $g_1, \ldots, g_q$, and hence

$$\limsup_{p \to \infty} d_X(\gamma^p) \leq \sup_{i \in \{1, \ldots, q\}} d_X(g_i) < \infty.$$

- It is also interesting to notice that there are groups with infinite contortion that possess elements γ such that

$$\liminf_{p \to \infty} d_X(\gamma^p) < \infty.$$

  Indeed, there are finitely generated linear groups that contain elements γ that are conjugated to infinitely many of its powers. Hence, for such γ

we have

$$\liminf_{p\to\infty} d_X(\gamma^p) \leqslant d_X(\gamma).$$

Here is a simple example. We take $\Gamma = \mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$ and

$$\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then for all n, γ^{p^n} is conjugated to γ.

- However, in Section 3.4, we shall give a condition—*bounded depth roots* (satisfied, for example, by any group commensurable to a subgroup of $\mathrm{SL}(n, \mathbb{Z})$) so that together with the hypothesis of the previous lemma,

$$\lim_{p\to\infty} d_X(\gamma^p) = \infty.$$

Proof. We want to prove that

$$\limsup_{p\to\infty} \inf_{x\in X} d(x, \gamma^p x) = \infty.$$

Assume the contrary, then there exists

- a constant R, and
- a sequence of points x_i of points in X,

such that for every p,

$$d(x_i, \gamma^p x_i) \leqslant R.$$

Now let K be a compact in X such that $\Gamma.K = X$. Let $f_i \in \Gamma$ such that $y_i = f_i^{-1}(x_i) \in K$. Then

$$d(y_i, f_i^{-1}\gamma^p f_i(y_i)) \leqslant R.$$

Let $K_R = \{z \in X, d(z, K) \leqslant R\}$. It follows that

$$\forall p,\ f_i^{-1}\gamma^p f_i(K_R) \cap K_R \neq 0.$$

Observe that K_R is compact. By the properness of the action of Γ, we conclude that the family $\{g_i^{-1}\gamma_i^p g_i\}$ is finite. Hence the family of conjugacy classes of the sequence γ^p is finite. But this contradicts infinite contortion for Γ. □

COROLLARY 3.2.2. *Assume Γ has infinite contortion. Let C be its Cayley graph, then for γ a nontorsion element*

$$\limsup_{p\to\infty} \ell(\gamma^p) = \infty.$$

COROLLARY 3.2.3. *Assume* Γ *has infinite contortion. Let* ρ *be a representation of dimension n. Assume* $\rho(\Gamma)$ *contains a nontrivial unipotent, then* ρ *is not displacing on* $X_n = \mathrm{SL}(n, \mathbb{R})/SO(n, \mathbb{R})$.

Proof. Assume ρ is displacing. Let γ such that $\rho(\gamma)$ is a nontrivial unipotent. Then for all p, $d_{X_n}(\gamma^p) = 0$. However, γ is not a torsion element. We obtain the contradiction using the previous lemma. □

3.3. Nonuniform Lattices

LEMMA 3.3.1. *For* $n \geqslant 3$, *the action of* $\mathrm{SL}(n, \mathbb{Z})$ *on* X_n *is such that the orbit maps are quasi-isometric embeddings. But it is not displacing.*

Proof. The group $\mathrm{SL}(n, \mathbb{Z})$ is residually finite. Hence it has infinite contortion by Lemma 3.1.1. The standard representation ρ contains a nontrivial unipotent; hence it is not displacing by Corollary 3.2.3.

By a theorem of Lubotzky, Mozes, and Ragunathan [8] irreducible higher-rank lattices Λ are quasi-isometrically embedded in the symmetric space. □

3.4. Bounded Depth Roots

This section is complementary. We say that group Γ has *bounded depth roots property*, if for every γ in Γ is a nontorsion element, there exists some integer p, such that we have

$$q \geqslant p, \eta \in \Gamma \implies \eta^q \neq \gamma.$$

Observe that $\mathrm{SL}(2, \mathbb{Z}[\frac{1}{p}])$ does not have bounded depth roots. Note that this property is well behaved by taking subgroups and is a property of commensurability (see Lemma 3.4.4).

We prove

PROPOSITION 3.4.1. *The following groups have bounded depth roots property:*

- *the group* $\mathrm{SL}(n, \mathbb{Z})$,
- $\mathrm{SL}(n, \mathcal{O})$ *where* $\mathcal{O}$ *is the ring of integers of a number field F,*
- *any subgroup of a group having bounded depth roots property or any group commensurable to a group having this property, and*
- *in particular, any arithmetic lattice in an Archimedian Lie group.*

LEMMA 3.4.2. (BOUNDED DEPTH ROOT) *Let* Γ *be a group with bounded depth roots property. Assume* Γ *acts cocompactly and properly discontinuously by* *isometry on a space X. Assume furthermore that every closed bounded set in X is compact. Then, for every nontorsion element γ in Γ, we have*

$$\lim_{p\to\infty} d_X(\gamma^p) = \infty.$$

For the proof see Section 3.4.3. The lemma and proposition above again imply that the action of $\mathrm{SL}(n,\mathbb{Z})$ on X_n is not displacing.

3.4.1. BOUNDED DEPTH ROOTS PROPERTY FOR SL($n,\mathbb{Z}$)

We prove the above proposition.

LEMMA 3.4.3. $\mathrm{SL}(n,\mathbb{Z})$ *has bounded depth roots.*

Proof. Let $A \in \mathrm{SL}(n,\mathbb{Z})$. Let $B \in \mathrm{SL}(n,\mathbb{Z})$. We assume there exists k such that $B^k = A$. Let $\{\lambda_j^A\}$ and $\{\lambda_j^B\}$ be the eigenvalues of A and B, respectively. Let

$$K = \sup_j |\lambda_j^A|.$$

Then,

$$\sup_j |\lambda_j^B| \leqslant K^{\frac{1}{k}} \leqslant K.$$

Hence, all the coefficients of the characteristic polynomial of B have a bound K_1 that depends only on A. Therefore, since these coefficients only take values in $\mathbb{Z}$, it follows the characteristic polynomials of B that belongs to the finite family

$$\mathcal{P} = \{P(x) = x^n + \sum_{k=0}^{k=n-1} a_k x^k \ : \ a_k \in \mathbb{Z}, |a_k| \leqslant K_1\}.$$

Since $\mathcal{P}$ is finite, there exists a constant $b > 1$ such that for every root λ of a polynomial $P \in \mathcal{P}$,

$$|\lambda| > 1 \implies |\lambda| \geqslant b.$$

Let $q \in N$ be such that b^q is greater than K. It follows that if $B^q = A$, then all eigenvalues of B have complex modulus 1. Therefore, the same holds for A.

It follows from this discussion that we can reduce to the case where

$$\forall i,j, |\lambda_j^A| = 1.$$

We say such an element has *trivial hyperbolic part*. Note that necessarily also B has trivial hyperbolic part.

We first prove that there exists an integer M depending only on n, such that if $C \in \mathrm{SL}(n, \mathbb{Z})$ has a trivial hyperbolic part, then C^M is unipotent. The same argument as above shows that the characteristic polynomials of elements with a trivial hyperbolic part belong to a finite family of the form

$$\mathcal{P} = \{P(x) = x^n + \sum_{k=0}^{k=n-1} a_k x^k \ : \ a_k \in \mathbb{Z}, |a_k| \leqslant K_2\},$$

where K_2 depends only on n. Note that roots of polynomials belonging to $\mathcal{P}$ that are of complex modulus 1 are roots of unity. Thus we may take M to be a common multiple of the orders of those roots of unity and deduce that C^M is unipotent if $C \in \mathrm{SL}(n, \mathbb{Z})$ has a trivial hyperbolic part.

Returning to our setting we can replace A by A^M and B by B^M and consider $A = B^k$ where both A and B are unipotents. There is some rational matrix $g_0 \in \mathrm{SL}(n, \mathbb{Q})$ depending on A such that $A_0 = g_0 A g_0^{-1}$ is in a Jordan form. We claim that $B_0 = g_0 B g_0^{-1}$ is made of blocks that correspond to the Jordan blocks of A_0 and each such block of B_0 is an upper-triangular unipotent matrix (this follows from observing that for unipotent matrices a matrix and its powers have the same invariant subspaces). Moreover, note that the denominators of the entries of B_0 are bounded by some $L \in \mathbb{N}$ depending only on g_0 (and thus on A). By considering (some of) the entries just above the main diagonal it is easily seen that $k \leq L$. □

3.4.2. COMMENSURABILITY We observe

LEMMA 3.4.4. *If Γ is commensurable to a subgroup of a group that has bounded depth roots, then Γ has bounded depth root.*

Proof. By definition a subgroup of a group having bounded depth roots has bounded depth roots. Let G be a group and H a subgroup having finite index k. Observe that for every element g of G, we have $g^k \in H$. It follows that if H has bounded depth roots, then G has bounded depth roots. □

3.4.3. PROOF OF LEMMA 3.4.2.

Proof. Let K be a compact in X. We first prove that

$$\lim_{p \to \infty} \inf_{x \in K, \eta \in \Gamma} d(x, \eta^{-1} \gamma^p \eta x) = \infty.$$

Assume the contrary, then there exists

- a constant R,
- a sequence of integers p_i going to infinity,
- a sequence of points x_i in K, and
- a sequence of elements η_i of Γ,

such that $d(x_i, \eta_i^{-1}\gamma^{p_i}\eta_i x_i) \leqslant R$. It follows that

$$\forall i, (\eta_i^{-1}\gamma\eta_i)^{p_i} K_R \cap K_R \neq 0.$$

By the properness of the action of Γ, we conclude that the family $\{(\eta_i^{-1}\gamma\eta_i)^{p_i}\}$ is finite. But this contradicts the bounded depth root property.

We now choose the compact K such that $\Gamma.K = X$. It follows that

$$\lim_{p\to\infty} \inf_{x\in X} d(x, \gamma^p x) = \lim_{p\to\infty} \inf_{x\in K,\eta\in\Gamma} d(\eta x, \gamma^p \eta x) = \infty.$$

This is what we wanted to prove. □

4. Stable Norm, Quasi-Isometric Embedding of Orbits, and Displacing Action

If a group Γ acts by isometries on a metric space X, we define the *stable norm with respect to* X by

$$[g]_\infty^X = \lim_{n\to\infty} \frac{1}{n} d(x_0, g^n(x_0)).$$

We observe that this quantity does not depend on the choice of the base point x_0. The *stable norm* $[g]_\infty$ is the stable norm with respect to the Cayley graph of Γ.

We now prove the following easy result

PROPOSITION 4.0.5. *Let Γ be a group. Assume that there exists $\alpha > 0$ such that*

$$\forall g \in \Gamma,\ \ [g]_\infty \geqslant \alpha.\ell(g).$$

Then every action of Γ on (X, d) for which the orbit map is a quasi-isometric embedding is displacing.

Proof. By definition, if the orbit map is a quasi-isometric embedding, for every $x \in X$ there exists some constant A and B such that

$$A\|\gamma\| + B \geqslant d(x, \gamma(x)) \geqslant A^{-1}\|\gamma\| - B.$$

It follows that

$$A[\gamma]_\infty \geqslant [\gamma]_\infty^X \geqslant A^{-1}[\gamma]_\infty.$$

Now we remark that

$$[\gamma]_\infty^X \leqslant d_X(\gamma). \qquad \square$$

The result follows:

REMARKS:

- We will show later that this inequality fails for $SL(n,\mathbb{Z})$ for $n \geqslant 3$.
- On the other hand, we observe following [4, p. 119], that for Γ a hyperbolic group, the stable norm of an element coincides up to a constant with its translation length; there exists a constant K such that $|\ell(g) - [g]_\infty| \leqslant K$.

Therefore, we have

COROLLARY 4.0.6. *Let Γ be a hyperbolic group. If an isometric action is such that the orbit maps are quasi-isometries, then this action is displacing.*

5. Infinite Groups Whose Actions Are Always Displacing

We have

PROPOSITION 5.0.7. *There exists infinite finitely generated groups whose actions are always displacing. Hence there exists action for which the orbit maps are not quasi-isometric embeddings, but that are displacing.*

Proof. Denis Osin [9] has constructed infinite finitely generated groups with exactly n conjugacy classes. Any action of such a group is displacing. For the second part, we just take the trivial action on a point. $\square$

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