Filtered distributive laws

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A motivating side story

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Jensen–McCammond–Meier (2006): computation of the cohomology algebra of $\mathbb{G}_n$ (Brownstein–Lee conjecture). It is an anticommutative algebra generated by degree one elements y_{ij} subject to quadratic relations

$$y_{ij}y_{ji} = 0,$$
$$y_{kj}y_{ji} = (y_{kj} - y_{ij})y_{ki}.$$

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Configurations of n unlinked circles in $\mathbb{R}^3$ in that story are usually replaced by configurations of little disks inside a big disk, and this gives an operad structure.

An operadic approximation to our question: there exists an operad intimately related to the groups of loops, whose underlying $\mathbb{S}$-module is isomorphic to $\mathrm{Perm} \circ \mathrm{PreLie}[1]$.

Fouxe-Rabinovitch groups

A natural generalisation of pure symmetric automorphisms:
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Griffin 2010: suppose that $Y_i = K(H_i, 1)$ are classifying spaces for H_i. There exists a functorial construction of $K(\mathrm{FR}(H_1 * \cdots * H_n), 1)$ from $Y_1, \ldots, Y_n$.

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so his classifying space construction is called the space of **Y**-cacti ($\mathbf{Y} = (Y_1, \ldots, Y_n)$).

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The operadic part of the story: let $(C, \Delta, \epsilon, \gamma)$ be a graded augmented cocommutative coalgebra. The operad $\mathrm{BC{\small ACT}}_C$ of *based C-cacti* is generated by binary operations $C \otimes \Bbbk S_2$; these operations satisfy the relations

$$c' \circ_1 c''.(23) = (-1)^{|c'||c''|} c'' \circ_1 c' \quad \text{(for homogeneous } c', c'' \in C),$$
$$c \circ_2 \mathbb{1} = \sum c_{(1)} \circ_1 c_{(2)} \quad \text{(for } c \in C),$$

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These relations give a rewriting system allowing to move the operation $\mathbb{1}$ towards the top level of compositions.

A toy example of a distributive law

The Poisson operad is generated by a symmetric binary operation $\cdot \star \cdot$ and a skew-symmetric binary operation $[\cdot, \cdot]$ that satisfy the relations

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0,$$
$$[a \star b, c] = a \star [b, c] + [a, c] \star b,$$
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Moreover, it is possible to prove that all possible commutative products of Lie monomials are linearly independent, so form a basis. Rewriting rules with this property are called *distributive laws*.

A warning example

The nil-Poisson operad is generated by a symmetric binary operation $\cdot \star \cdot$ and a skew-symmetric binary operation $[\cdot, \cdot]$ that satisfy the relations

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It is built from the nilpotent operad Nil and the operad Com via the Leibniz rule that allows to rewrite every expression as a product of brackets.

However, commutative products of brackets are not independent any more: expanding $[a_1, [a_2, a_3 \star a_4]] = 0$, we obtain

$$[a_1, a_4] \star [a_2, a_3] + [a_1, a_3] \star [a_2, a_4] = 0.$$

A motivating example

An unconventional presentation of the associative operad (Livernet–Loday). The operad generated by a symmetric binary operation $\cdot\star\cdot$ and a skew-symmetric binary operation $[\cdot,\cdot]$ that satisfy the relations

$$[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0,$$
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is isomorphic to the associative operad As.
It looks like a deformation of the Poisson operad, but the product is no longer associative. Can we still think of As as built from Com and Lie by some procedure?

Filtered distributive laws

$\mathscr{A} = \mathscr{F}(\mathscr{V})/(\mathscr{R})$ and $\mathscr{B} = \mathscr{F}(\mathscr{W})/(\mathscr{S})$ are two quadratic operads. For two $\mathbb{S}$-module mappings

$$s\colon \mathscr{R} \to \mathscr{W} \bullet \mathscr{V} \oplus \mathscr{V} \bullet \mathscr{W} \oplus \mathscr{W} \bullet \mathscr{W}$$

and

$$d\colon \mathscr{W} \bullet \mathscr{V} \to \mathscr{V} \bullet \mathscr{W} \oplus \mathscr{W} \bullet \mathscr{W},$$

one can define a quadratic operad $\mathscr{E}$ with generators $\mathscr{U} = \mathscr{V} \oplus \mathscr{W}$ and relations $\mathscr{T} = \mathscr{Q} \oplus \mathscr{D} \oplus \mathscr{S}$, where

$$\mathscr{Q} = \{x - s(x) \mid x \in \mathscr{R}\}, \quad \mathscr{D} = \{x - d(x) \mid x \in \mathscr{W} \bullet \mathscr{V}\}.$$

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Here $\mathscr{V} \bullet \mathscr{W}$ is the span of all elements $\phi \circ_i \psi$ with $\phi \in \mathscr{V}$, $\psi \in \mathscr{W}$.

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Usual rewriting rules for distributive laws (Markl 1994): $s = 0$, $d\colon \mathscr{W} \bullet \mathscr{V} \to \mathscr{V} \bullet \mathscr{W}$ (no lower terms anywhere).

Examples — 1

The operad PostLie is generated by a skew-symmetric operation $[\cdot,\cdot]$ and an operation $\cdot\circ\cdot$ without any symmetries that satisfy the relations

$$[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0,$$
$$a\circ[b,c]=(a\circ b)\circ c-a\circ(b\circ c)-(a\circ c)\circ b+a\circ(c\circ b),$$
$$[a,b]\circ c=[a\circ c,b]+[a,b\circ c].$$

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It is based on a rewriting rule between the operads Lie and Mag.

Examples — 2

The Koszul dual operad $\mathrm{PostLie}^! = \mathrm{ComTrias}$ of commutative trialgebras is generated by a symmetric operation $\cdot \bullet \cdot$ and an operation $\cdot \star \cdot$ without any symmetries; the identities between them can be expressed as follows:

$$
\begin{aligned}
(a \star b) \star c &= a \star (b \bullet c),\\
a \star (b \star c) &= a \star (b \bullet c),\\
a \star (c \star b) &= a \star (b \bullet c),\\
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It is based on a rewriting rule between the operads Nil and Com.

Examples — 3

The operad CTD of commutative tridendriform algebras is generated by a symmetric operation $\cdot \star \cdot$ and an operation $\cdot \prec \cdot$ without any symmetries that satisfy the relations

$$
\begin{aligned}
(a \prec b) \prec c &= a \prec (b \prec c + c \prec b + b \star c), \\
a \star (b \prec c) &= (a \star b) \prec c, \\
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It is based on a rewriting rule between the operads Zinb and Com.

Examples — 4

The operad $\mathrm{CTD}^!$ is generated by a skew-symmetric operation $[\cdot, \cdot]$ and an operation $\cdot \bullet \cdot$ without any symmetries that satisfy the relations

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0,$$
$$a \bullet [b, c] = a \bullet (b \bullet c),$$
$$[a, b] \bullet c = [a \bullet c, b] + [a, b \bullet c],$$
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Examples — 4

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It is based on a rewriting rule between the operads Lie and Leib.

Examples — 5

The operad of *based two-point cacti* is generated by the operations $\cdot_0$ and $\cdot_1$ without any symmetries that satisfy the relations

$$(a \cdot_0 b) \cdot_0 c = a \cdot_0 (b \cdot_0 c) = (-1)^{|b||c|}(a \cdot_0 c) \cdot_0 b,$$
$$(a \cdot_0 b) \cdot_1 c = (-1)^{|b||c|}(a \cdot_1 c) \cdot_0 b,$$
$$a \cdot_1 (b \cdot_0 c) = (a \cdot_0 b) \cdot_0 c,$$
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The operation $a, b \mapsto a \cdot_0 b$ is permutative, the operation $a, b \mapsto a \cdot_1 b$ is nonassociative permutative, and the remaining defining relations give a rewriting rule between the operads Perm and NAP.

Examples — 6

The operad of *based S^1-cacti* is generated by the operations $\cdot$ and $\bullet$ without any symmetries that satisfy the relations

$$(a \cdot b) \cdot c = a \cdot (b \cdot c),$$
$$(a \cdot b) \cdot c = (-1)^{|b||c|}(a \cdot c) \cdot b,$$
$$(a \cdot b) \bullet c = (-1)^{|b||c|}(a \bullet c) \cdot b,$$
$$a \bullet (b \cdot c) = (a \cdot b) \bullet c + (a \bullet b) \cdot c,$$
$$(a \bullet b) \bullet c = (-1)^{1+|b||c|}(a \bullet c) \bullet b.$$

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The operation $a, b \mapsto a \cdot b$ is permutative, the operation $a, b \mapsto a \bullet b$ is nonassociative permutative *of degree* 1, and the defining relations give a rewriting rule between the operads Perm and NAP[1].

Back to filtered distributive laws

$\mathscr{A} = \mathscr{F}(\mathscr{V})/(\mathscr{R})$ and $\mathscr{B} = \mathscr{F}(\mathscr{W})/(\mathscr{S})$ are two quadratic operads. For two $\mathbb{S}$-module mappings

$$s\colon \mathscr{R} \to \mathscr{W} \bullet \mathscr{V} \oplus \mathscr{V} \bullet \mathscr{W} \oplus \mathscr{W} \bullet \mathscr{W}$$

and

$$d\colon \mathscr{W} \bullet \mathscr{V} \to \mathscr{V} \bullet \mathscr{W} \oplus \mathscr{W} \bullet \mathscr{W},$$

one can define a quadratic operad $\mathscr{E}$ with generators $\mathscr{U} = \mathscr{V} \oplus \mathscr{W}$ and relations $\mathscr{T} = \mathscr{Q} \oplus \mathscr{D} \oplus \mathscr{S}$, where

$$\mathscr{Q} = \{x - s(x) \mid x \in \mathscr{R}\}, \quad \mathscr{D} = \{x - d(x) \mid x \in \mathscr{W} \bullet \mathscr{V}\}.$$

The filtered distributive laws criterion

Assume that the natural projection of $\mathbb{S}$-modules $\pi\colon \mathscr{E} \twoheadrightarrow \mathscr{A}$ splits (for example, it is always true in characteristic zero, or in arbitrary characteristic whenever the relations of $\mathscr{A}$ remain undeformed, including the case of usual distributive laws). Then the composite of natural mappings

$$\mathscr{F}(\mathscr{V}) \circ \mathscr{F}(\mathscr{W}) \hookrightarrow \mathscr{F}(\mathscr{V} \oplus \mathscr{W}) \twoheadrightarrow \mathscr{F}(\mathscr{V} \oplus \mathscr{W})/(\mathscr{T})$$

gives rise to a surjection of $\mathbb{S}$-modules

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$$\xi\colon \mathscr{A} \circ \mathscr{B} \twoheadrightarrow \mathscr{E}.$$

Definition

We say that the mappings s and d define a filtered distributive law between the operads $\mathscr{A}$ and $\mathscr{B}$ if $\pi\colon \mathscr{E} \twoheadrightarrow \mathscr{A}$ splits, and the restriction of ξ to weight 3 elements

$$\xi_3\colon (\mathscr{A} \circ \mathscr{B})_{(3)} \to \mathscr{E}_{(3)}$$

The filtered distributive laws criterion

Theorem (V.D., 2007)

Assume that the operads $\mathscr{A}$ and $\mathscr{B}$ are Koszul, and that the mappings s and d define a filtered distributive law between them. Then the operad $\mathscr{E}$ is Koszul, and the $\mathbb{S}$-modules $\mathscr{A} \circ \mathscr{B}$ and $\mathscr{E}$ are isomorphic.

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Koszulness of the operad of cacti

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For any choice of an augmented graded cocommutative coalgebra C, the operad $\mathrm{BC}\textsc{act}_C$ is Koszul, and as $\mathbb{S}$-modules,

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$$\mathrm{BC}\textsc{act}_C \simeq \mathsf{Perm} \circ \mathsf{NAP}_{\overline{C}} .$$

Here $\mathsf{NAP}_{\overline{C}}$ is the operad of NAP-*algebras enriched in the graded vector space* $\overline{C}$. It is based on rooted trees whose edges are decorated by $\overline{C}$.