Shuffle operads

Vladimir Dotsenko

Trinity College Dublin

partly joint with A. Khoroshkin (ETH Zürich)

New developments in noncommutative algebra and its applications

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Isle of Skye, June 30, 2011

Operads: who cares?

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- if we have several linear operators acting on a vector space, it is natural to formulate and answer questions about this action in terms associative algebras and their (possibly derived) categories of modules.
- what about operations with several arguments?
- a very convenient language for that is given by operads. Informally, for all algebras of some type, there exists one "higher algebra" (an operad) for which all these algebras are modules.

The Cheshire Cat analogy

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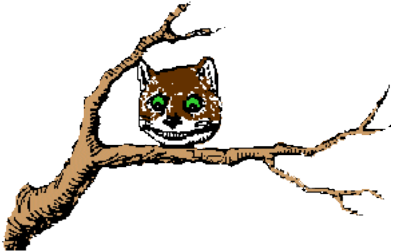

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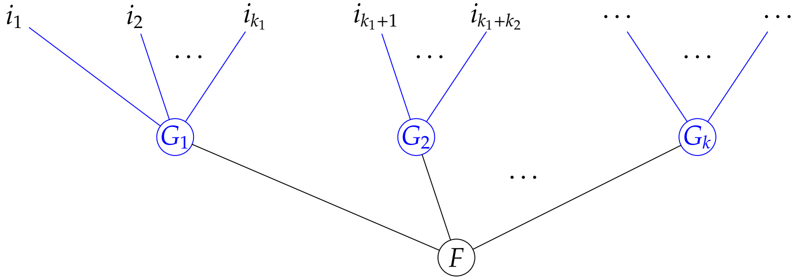
"An operad is the grin of a given algebra [all operations that can act on that algebra]." (S. Merkulov)

Compositions via surjections

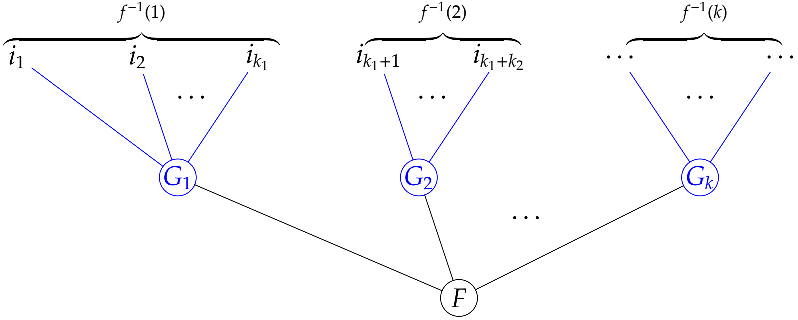
From a typical composition of operations



we can extract a surjection $f\colon I \twoheadrightarrow [k]$, so that
$f(i_1) = \ldots = f(i_{k_1}) = 1$, $f(i_{k_1+1}) = \ldots = f(i_{k_1+k_2}) = 2$, $\ldots$

Compositions via surjections

In other words,



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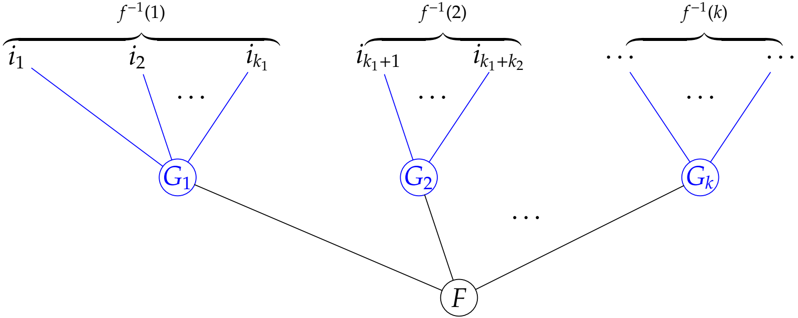

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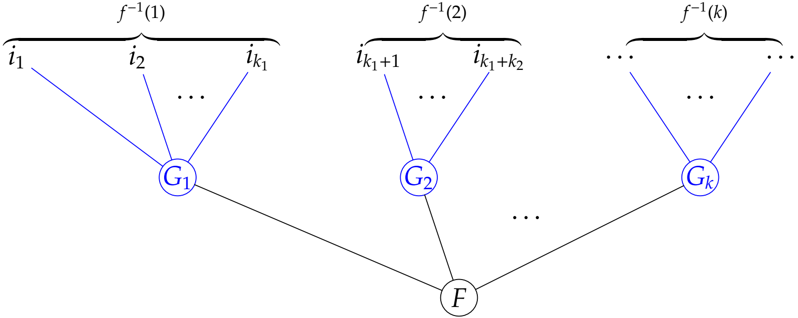

which somewhat justifies the following definition. The category *Fin* has nonempty finite sets as objects, and bijections as morphisms. The category of symmetric collections is the category of functors from *Fin* to *Vect*. It has a monoidal structure:

$$F \circ_{sym} G(I) = \bigoplus_k F([k]) \otimes_{S_k} \bigoplus_{f \in \mathrm{Surj}(I,[k])} G(f^{-1}(1)) \otimes \ldots \otimes G(f^{-1}(k)).$$

Symmetric operads

Definition: a (symmetric) operad is an "associative algebra" with respect to the symmetric composition, that is a symmetric collection O with an associative product $O \circ_{sym} O \to O$.

EXAMPLE: THE OPERAD *Lie*

A naive definition. The operad *Lie* of Lie algebras consists of all Lie operations:

$$a_1 \mapsto a_1,$$
$$a_1, a_2 \mapsto [a_1, a_2],$$
$$a_1, a_2, a_3 \mapsto [[a_1, a_2], a_3], [[a_2, a_3], a_1], [[a_3, a_1], a_2],$$
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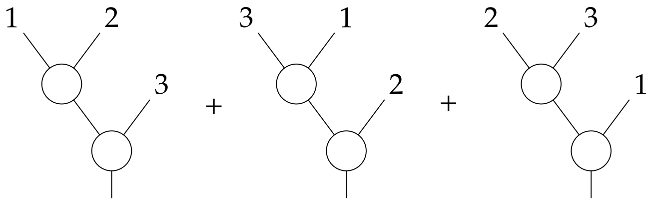
– all other operations you can think of can be expressed in terms of these, e.g. $[a_2, [a_1, a_3]] = [[a_3, a_1], a_2]$;
– the three operations in the third line are linearly dependent (Jacobi identity), thus the space of Lie operations with three arguments is two-dimensional.

EXAMPLE: THE OPERAD *Lie*

A formal definition. The operad *Lie* of Lie algebras is the quotient of the free operad with one skew-symmetric binary generator modulo the ideal generated by the element



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In a sense, symmetries get in the way, contrary to the usual philosophy telling us that symmetries are helpful!

Shuffle compositions

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Definition. The shuffle composition of two nonsymmetric collections is

$$F \circ_{sh} G(I) = \bigoplus_k F([k]) \otimes \bigoplus_{f \in \mathrm{Surj}_{sh}(I,[k])} G(f^{-1}(1)) \otimes \ldots \otimes G(f^{-1}(k)),$$

where the allowed *shuffle* surjections satisfy the condition

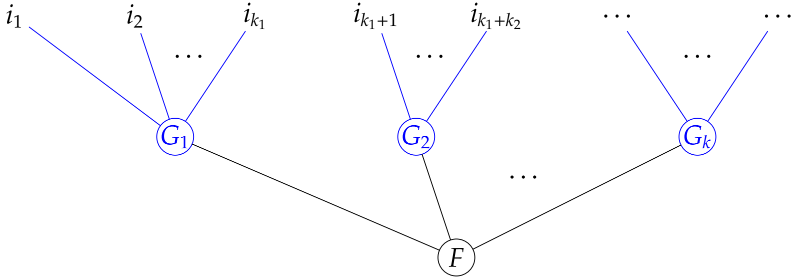
$$\min f^{-1}(1) < \min f^{-1}(2) < \ldots < \min f^{-1}(k).$$

Shuffle operads

Definition: a shuffle operad is an "associative algebra" with respect to the shuffle composition, that is a nonsymmetric collection O with an associative product $O \circ_{sh} O \to O$.

Why shuffle?

The word "shuffle" reflects the combinatorics of allowed compositions: in the composition

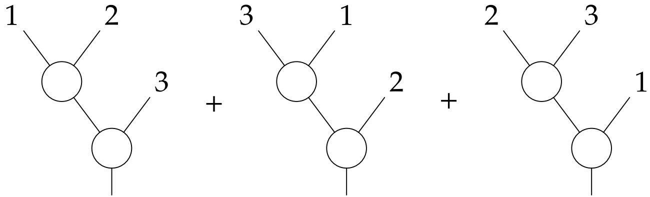


we are only allowed to use sequences I for which

$$i_1 < i_2 \dots < i_{k_1}, \quad i_{k_1+1} < \dots < i_{k_1+k_2}, \quad \dots,$$
$$i_1 < i_{k_1+1} < i_{k_1+k_2+1} < \dots$$

Back to the operad *Lie*

Note that while the Jacobi identity



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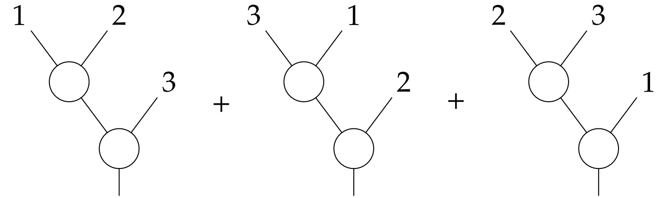
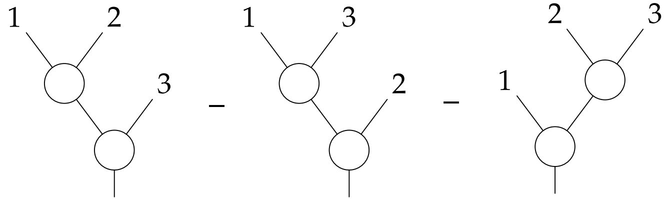


did not belong in the shuffle world,its not-so-symmetric version



consists of tree monomials obtained from the original binary generator via shuffle compositions.

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Also, in the shuffle world the identity $[[a_1, a_2], a_3] = 0$ does not imply $[[a_1, a_3], a_2] = 0$ anymore!

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In fact, for shuffle operads it is possible to define Gröbner bases, and therefore every shuffle operad has a monomial replacement.

Forgetful functor is monoidal

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Proposition. We have

$$(F \circ_{sym} G)^f \simeq F^f \circ_{sh} G^f.$$

Forgetful functor is monoidal

Proof. Compare

$$F \circ_{sh} G(I) = \bigoplus_k F([k]) \otimes \bigoplus_{f \in \mathrm{Surj}_{sh}(I,[k])} G(f^{-1}(1)) \otimes \ldots \otimes G(f^{-1}(k)),$$

and

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  - we have $\mathbf{B}(P)^f \simeq \mathbf{B}(P^f)$ as shuffle dg-cooperads.

Therefore, if we can formulate a question about operads without mentioning symmetries, we can (choose to) solve this question "in the shuffle world" instead!

Koszul duality

Ginzburg–Kapranov '94: Koszul duality theory for operads parallel to the Koszul duality theory for associative algebras. Simplest pair of Koszul operads *Lie* and *Com* explain Quillen's duality between dg Lie algebras and dg commutative algebras.

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Question. Find natural examples of Koszul operads without quadratic Gröbner bases ("Sklyanin operads"?).

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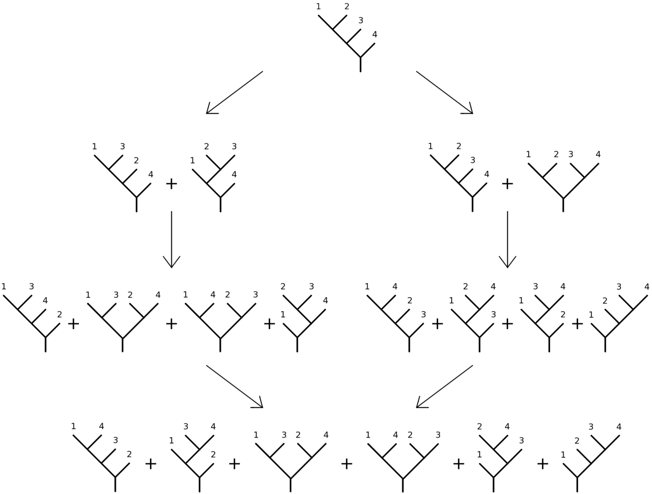
Proof. Indeed, this operad has a quadratic Gröbner basis:



Operads from commutative algebras

Let A be a commutative associative graded algebra, $A = \bigoplus_{n \geq 0} A_n$.

We define an operad O_A by $O_A(I) = A_{|I|-1}$ with the composition maps

$$O_A([n]) \otimes O_A(I_1) \otimes \ldots \otimes O_A(I_n) = A_{n-1} \otimes A_{|I_1|-1} \otimes \ldots \otimes A_{|I_n|-1} \to$$
$$\to A_{|I_1|+\ldots+|I_n|-1} = O_A(I_1 \sqcup \ldots \sqcup I_n)$$

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coming from the product in A.

Theorem. If the algebra A is Koszul, then the operad O_A is Koszul as well.

The symmetrised pre-Lie product.

A pre-Lie algebra is a vector space V with a binary operation $a, b \mapsto a \circ b$ such that

$$(a \circ b) \circ c - a \circ (b \circ c) = (a \circ c) \circ b - a \circ (c \circ b)$$

(example: vector fields on a manifold with the "half-commutator" $a\partial_i \circ b\partial_j = a\partial_i(b)\partial_j$).

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Theorem. (Bergeron–Loday, D.) The symmetrised pre-Lie product $a \cdot b := a \circ b + b \circ a$ does not satisfy any identities.

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Conjecture. (Bergeron–Loday) Free pre-Lie algebras are free algebras with respect to the symmetrised pre-Lie product.

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Theorem. (D.) Free pre-Lie algebras are free algebras with respect to the symmetrised pre-Lie product.

Two open questions

Question 1 (growth): What is the "right" definition of the GK dimension for operads? More generally, what are possible growth rates of dimensions of components for shuffle operads? What replaces rationality for monomial shuffle operads?

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Question 2 (Noether property): Which of the "natural" operads are Noetherian? Kemer's proof (1985) of the Specht conjecture (circa 1950) states that the associative operad is Noetherian in char 0. What about positive characteristic?

That's all

Thank you for your patience!