Two languages for connections in principal bundles: proofs of basic results

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Recall that we have two approaches to connections, a geometric and an algebraic one. Geometrically, for the total space E of a principal G-bundle $E \overset{\pi}{\to} B$ we have the vertical distribution, that is the tangent distribution defined by the foliation of E into G-orbits, $(T_eE)^v := T_e(e.G)$. This distribution is *smooth*, that is, spanned by vector fields, *G-invariant*, that is,

$$(T_{e.g}E)^v = (R_g)_*(T_eE)^v,$$

and is *integrable*, that is locally tangent to a foliation, by its very construction. A *connection* in a principal bundle is a choice of a complementary G-invariant (horisontal) distribution, that is a decomposition

$$T_eE = (T_eE)^v \oplus (T_eE)^h$$

with $(T_{e.g}E)^h = (R_g)_*(T_eE)^h$. By definition, a connection is *flat* if the horisontal distribution is integrable.

Algebraically, we note that defining a direct sum decomposition

$$T_eE = (T_eE)^v \oplus (T_eE)^h$$

at a point $e \in E$ is exactly the same as defining a projection

$$p\colon T_eE \twoheadrightarrow (T_eE)^v \simeq \mathfrak{g}.$$

Indeed, from such a projection the horisontal subspace is reconstructed as $(T_eE)^h := \ker(p)$. To make these subspaces depend smoothly on a point, it is natural to require that the projection p is built out of 1-forms on E, that is defined via a $\mathfrak{g}$-valued differential form $\omega \in \Omega^1(E) \otimes \mathfrak{g}$. To formulate conditions on such a differential form that ensure that it defines a connection, we recall that the canonical Maurer–Cartan form $\omega_G \in \Omega^1(G) \otimes \mathfrak{g}$ on a Lie group G is defined as the left invariant $\mathfrak{g}$-valued differential form corresponding to the identity operator

$$\mathrm{Id}_{\mathfrak{g}} \in \mathrm{End}(\mathfrak{g}) \simeq \mathfrak{g}^* \otimes \mathfrak{g} \simeq \Omega^1(G)^{inv} \otimes \mathfrak{g}.$$

In more down to earth terms, if for $\alpha \in \mathfrak{g}$ we denote by X_α the respective left invariant vector field on G, the form ω_G is defined by $\omega_G(X_\alpha) = \alpha$. Then the following result holds:

Theorem 1. *A differential form $\omega \in \Omega^1(E) \otimes \mathfrak{g}$ defines a connection on E if and only if*

- $(R_e)^*(\omega) = \omega_G$*;*
- $(R_g)^*(\omega) = (Ad(g^{-1})_*(\omega)$*.*

Here for $e \in E$ the map $R_e \colon G \to E$ is defined by $R_e(g) = e.g$, for $g \in G$ the map $R_g \colon E \to E$ is defined by $R_g(e) = e.g$, pullbacks $(R_e)^*$ and $(R_g)^*$ act on the differential forms only and do not change $\mathfrak{g}$, and $Ad(g^{-1})_* = (L_{g^{-1}})_*(R_g)_*$ comes from the adjoint action of the Lie group on its Lie algebra (and so is pointwise in E).

Proof. First, suppose that a differential form ω satisfies the two given conditions. For each $e \in E$, ω_e is a map from T_eE to $\mathfrak{g}$, and we can put $(T_eE)^h := \ker(\omega_e)$. Since the pullback of ω by R_e is ω_G, ω_e is surjective, therefore, $\dim(T_eE)^h = \dim(E) - \dim(G)$, so we have a distribution. This distribution is smooth, as the kernel distribution of a differential form. Let us show that it gives a direct sum decomposition. Since $\ker(\omega_e)$ and $(T_eE)^v \simeq \mathfrak{g}$ have complementary dimension, it is enough to show that their sum is direct, that is $\ker(\omega_e) \cap (T_eE)^v = \{0\}$. Assume that $v \in \ker(\omega_e) \cap (T_eE)^v$. Since $R_e \colon G \to \pi^{-1}(\pi(e))$ is a diffeomorphism, the map

$$(R_e)_* \colon \mathfrak{g} \to T_e(\pi^{-1}(\pi(e))) = (T_eE)^v$$

is an isomorphism. Hence $v = (R_e)_*(\alpha)$ for some $\alpha \in \mathfrak{g}$. We have

$$0 = \omega_e(v) = \omega_e((R_e)_*(\alpha)) = (R_e)^*(\omega)(\alpha) = \omega_G(\alpha) = \alpha,$$

which proves that the sum is direct. Furthermore, let $u \in \ker(\omega_e)$ be an arbitrary vector. We have

$$\omega_{e.g}((R_g)_*(u)) = (R_g)^*(\omega_e)(u) = (Ad(g^{-1})_*(\omega_e(u)) = 0,$$

since Ad_* is a linear action and therefore $Ad(g^{-1})_*(0) = 0$. Therefore $(R_g)_*(T_eE)^h \subset (T_{e.g}E)^h$, and the equality of dimensions implies that these two subspaces coincide.

The other way around, suppose that we have a connection on E. We define a differential form ω as the composition $T_eE \twoheadrightarrow (T_eE)^v \simeq \mathfrak{g}$, $\omega(X) = (R_e)_*^{-1}(X^v)$. Since the action of G on E as well as the vertical and the horisontal distributions are smooth, this gives a differential form. Note that the following identities hold for the actions of G on E and on itself:

$$R_{e.g_1}(g_2) = (e.g_1).g_2 = e.(g_1.g_2) = R_e \circ L_{g_1}(g_2),$$
$$R_e \circ R_{g_1}(g_2) = e.(g_2.g_1) = (e.g_2).g_1 = R_{g_1} \circ R_e(g_2).$$

As a consequence, for $X \in T_eE$ we have

$$\begin{aligned}(R_g)^*(\omega_e)(X) &= \omega_{e.g}((R_g)_*(X)) = \\ &= \omega_{e.g}((R_g)_*(X^v) + (R_g)_*(X^h)) = (R_{e.g})_*^{-1} \circ R_g(X^v) = \\ &= (L_g)_*^{-1} \circ (R_e)_*^{-1} \circ R_g(X^v) = (L_{g^{-1}})_* \circ R_g \circ (R_e)_*^{-1}(X^v) = \\ &= (Ad(g^{-1})_*((R_e)_*^{-1}(X^v)) = (Ad(g^{-1})_*(\omega_e(X))\end{aligned}$$

and for $g \in G$, $\alpha \in \mathfrak{g}$ we have

$$\begin{aligned}(R_e)^*(\omega)_g(\alpha) &= \omega_{e.g}((R_e)_*(\alpha)) = \\ &= (R_{e.g})_*^{-1} \circ (R_e)_*(\alpha) = (L_g)_*^{-1} \circ (R_e)_*^{-1} \circ (R_e)_*(\alpha) = \\ &= (L_{g^{-1}})(\alpha) = (\omega_G)_g(\alpha).\end{aligned}$$

This completes the proof. □

We now want to formulate in the language of differential forms the flatness. By definition, for a $\mathfrak{g}$-valued differential 1-form ω we denote by $d\omega$ its exterior differential (as a differential form, that is not interacting with $\mathfrak{g}$), and by $[\omega, \omega]$ the 2-form defined as

$$[\omega, \omega](X, Y) = [\omega(X), \omega(Y)] - [\omega(Y), \omega(X)] = 2[\omega(X), \omega(Y)],$$

so that we compute the pointwise commutator in $\mathfrak{g}$. The *curvature* of ω is made up from these two forms,

$$\Xi = d\omega + \frac{1}{2}[\omega, \omega].$$

Theorem 2. *The connection defined by a differential form ω is flat if and only if its curvature vanishes.*

The equation $d\omega + \frac{1}{2}[\omega, \omega] = 0$ is called the *Maurer–Cartan equation* and is of crucial importance for differential geometry and mathematical physics.

Proof. According to the Cartan's formula for the exterior derivative, we have

$$d\omega(X, Y) = -\omega([X, Y]) + X(\omega(Y)) - Y(\omega(X)).$$

Suppose X and Y are horisontal vector fields. Then they belong to the kernel of ω, so $d\omega(X, Y) = -\omega([X, Y])$, and $[\omega, \omega](X, Y) = 0$. Therefore $\Xi(X, Y) = -\omega([X, Y])$. This shows that if $\Xi = 0$ then horisontal vector fields are closed under commutator, and hence define an involutive distribution which, by Frobenius's theorem, is integrable, so our connection is flat. Also, we see that if our connection is flat, that is the horisontal distribution is integrable, then it of course is involutive, and therefore the condition $\Xi(X, Y) = 0$ is satisfied on horisontal vector fields. It remains to check that condition when one of the vector fields is vertical. Let us consider, for each $\alpha \in G$ the vector field $X_\alpha := (R_e)_*(\alpha)$ on E. We denote by $\exp(t\alpha)$ the integral curve for the left invariant vector field on G corresponding to α with $\exp(0) = 1 \in G$. The integral curve of the vector field X_α in E is clearly $R_{\exp(t\alpha)}(e)$. The Lie derivative of ω along the vector field X_α is

$$\mathcal{L}_{X_\alpha}(\omega) = \frac{d}{dt}\left((R_{\exp(t\alpha)})^*(\omega)\right)\Big|_{t=0} = \\ = \frac{d}{dt}\left((Ad(\exp(t\alpha)^{-1})_*)(\omega)\right)\Big|_{t=0} = ad(-\alpha)(\omega) = -[\alpha, \omega].$$

Alternatively, $\mathcal{L}_{X_\alpha} = i_{X_\alpha} \circ d + d \circ i_{X_\alpha}$, so we have

$$\mathcal{L}_{X_\alpha}(\omega) = i_{X_\alpha} \circ d\omega + d \circ i_{X_\alpha}(\omega) = \\ = i_{X_\alpha} \circ d\omega + d(\omega((R_e)_*(\alpha))) = i_{X_\alpha} \circ d\omega + d((R_e)^*(\omega)(\alpha)) = \\ = i_{X_\alpha} \circ d\omega + d(\omega_G(\alpha)) = i_{X_\alpha} \circ d\omega + d(\alpha) = i_{X_\alpha} \circ d\omega.$$

Therefore,

$$i_{X_\alpha}\Xi = i_{X_\alpha}(d\omega + \frac{1}{2}[\omega, \omega]) = i_{X_\alpha}(d\omega) + [\omega(X_\alpha), \omega] = \\ = -[\alpha, \omega] + [\alpha, \omega] = 0.$$

Also, at each point e the operator $(R_e)_*$ is an isomorphism, so for every vertical vector field X, we have $X(e) = X_\alpha(e)$ for some $\alpha \in \mathfrak{g}$. Thus, Ξ vanishes when one of the arguments is vertical, and overall we conclude that Ξ vanishes identically. This completes the proof. □