# UNIVERSITY OF DUBLIN

XMA1131

## TRINITY COLLEGE

Faculty of Science

school of mathematics

**JF Mathematics**
**JF Theoretical Physics**
**JF Two Subject Mod**

**Trinity Term 2009**

Course 113

??? ??? ???

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For each task, the number of points you can get for a complete solution of that task is printed next to it.

All vector spaces unless otherwise specified are over complex numbers.

You may use all statements proved in class and in home assignments; when using some statement, you should formulate it clearly, e.g. "in class, we proved that if $A$ is invertible, then the reduced row echelon form of $A$ is the identity matrix".

Non-programmable calculators are permitted for this examination.

1. Denote by $A$ the matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix}$ and by $b$ the vector $\begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix}$.

   (a) (5 points) List all minors and all cofactors of $A$, and write down the expansion of $\det(A)$ along the second row and along the third column.

   (b) (5 points) Show how to use the Cramer's rule to solve the system $Ax = b$.

2. (10 points) Describe all possible values of $i$, $j$, $k$ and $l$ for which the term

$$a_{4k}a_{35}a_{il}a_{67}a_{j1}a_{23}a_{14}$$

   occurs in the expansion of a $7 \times 7$ determinant with coefficient $-1$.

3. (a) (5 points) Prove that for two square matrices $A$ and $B$ of the same size we always have $\mathrm{tr}(AB) = \mathrm{tr}(BA)$.

   (b) (10 points) How many *distinct* numbers can there be among the six traces

$$\mathrm{tr}(ABC), \mathrm{tr}(ACB), \mathrm{tr}(BCA), \mathrm{tr}(BAC), \mathrm{tr}(CBA), \mathrm{tr}(CAB)?$$

   for different choices of square matrices $A$, $B$, $C$ of the same size? For each variant of the answer, give an example.

4. (12 points) Is the subspace $U$ of $\mathbb{R}^4$ spanned by $\begin{pmatrix} 1 \\ 1 \\ 4 \\ -2 \end{pmatrix}$ and $\begin{pmatrix} -2 \\ -1 \\ -1 \\ 1 \end{pmatrix}$ an invariant subspace of the operator $A$ whose matrix relative to the standard basis is

$$\begin{pmatrix} 0 & 3 & -3 & -1 \\ 1 & 3 & -1 & 0 \\ 7 & 12 & 2 & 3 \\ -3 & -6 & 0 & -1 \end{pmatrix}?$$

   Explain your answer.

5. (a) (6 points) Find all eigenvalues and eigenvectors of the matrix

$$B = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}.$$

(b) (7 points) Find the Jordan normal form of the matrix $B$, and a matrix $C$ which is the transition matrix of some Jordan basis of $B$.

(c) (8 points) Find a formula for $B^n$, and use it to find a closed formula for the $n^{\text{th}}$ terms of the sequences $\{x_m\}$, $\{y_m\}$ defined recursively as follows:

$$x_0 = 1, y_0 = -5,$$

$$x_{k+1} = x_k - y_k, \quad y_{k+1} = x_k + 3y_k.$$

6. (a) (5 points) Which bases of a Euclidean space $V$ are called orthogonal? orthonormal?

(b) (5 points) Show that the $f_1 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $f_2 = \begin{pmatrix} 0 \\ -2 \\ 3 \end{pmatrix}$, and $f_3 = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}$ form a basis of $\mathbb{R}^3$.

(c) (7 points) Find the orthogonal basis of $\mathbb{R}^3$ which is the output of the Gram-Schmidt orthogonalisation applied to the basis from the previous question. (The inner product on the $\mathbb{R}^3$ is the standard one.)

7. (a) (5 points) Write down the definition of a bilinear form on a real vector space. Which symmetric bilinear forms are said to be positive definite?

(b) (10 points) Consider the vector space $V$ of all polynomials in $t$ of degree at most $2$. The bilinear form $\psi_a$ on $V$ (depending on a [real] parameter $a$) is defined by the formula

$$\psi_a(f(t), g(t)) = \int_{-1}^{1} f(t)g(t)(t-a)\,dt.$$

Determine all values of $a$ for which $\psi_a$ is positive definite.