UNIVERSITY OF DUBLIN

XMA1131

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

JF Mathematics
JF Theoretical Physics
JF Two Subject Mod

Trinity Term 2009

COURSE 113, A SAMPLE PAPER

Dr. Vladimir Dotsenko

For each task, the number of points you can get for a complete solution of that task is printed next to it.

All vector spaces unless otherwise specified are over complex numbers.

You may use all statements proved in class and in home assignments; when using some statement, you should formulate it clearly, e.g. "in class, we proved that if A is invertible, then the reduced row echelon form of A is the identity matrix".

Non-programmable calculators are permitted for this examination.

1. (10 points) For the matrix $A = \begin{pmatrix} 4 & 0 & 1 & 2 & 1 \\ 3 & 1 & 2 & 0 & -1 \\ 5 & -1 & 0 & 4 & 2 \end{pmatrix}$, compute its reduced row echelon form. Which variables are leading for the system $Ax = 0$, and which are free? Describe all solutions to that system, and all solutions to the system $Ax = \begin{pmatrix} 7 \\ 2 \\ 0 \end{pmatrix}$.

2. (10 points) Describe all possible values of i, j, k and l for which the term

$$a_{41}a_{3l}a_{64}a_{j5}a_{i6}a_{1k}a_{73}$$

occurs in the expansion of a 7×7 determinant with coefficient -1.

3. (12 points) Prove that if for two $n \times n$-matrices A and B we have $AB - BA = A$, then A is not invertible.

4. (15 points) Consider two following subspaces of $\mathbb{R}^5$: the subspace U is the linear span of the vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \\ -4 \\ 2 \end{pmatrix}$, $\begin{pmatrix} -4 \\ 1 \\ 3 \\ -1 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 5 \\ -1 \\ -1 \\ 14 \end{pmatrix}$, and subspace W is the linear span of the vectors $\begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ -1 \\ -2 \\ -3 \\ -1 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \\ -2 \\ -2 \\ 2 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$. Compute $\dim U$ and $\dim W$, and find some basis for the intersection of $U \cap W$.

5. (a) (6 points) Find all eigenvalues and eigenvectors of the matrix

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 3 \end{pmatrix}$$

(b) (7 points) Find the Jordan normal form of the matrix B, and a matrix C which is the transition matrix of some Jordan basis of B.

(c) (8 points) Find a formula for B^n, and use it to find a closed formula for the n^{th} term of the sequence $\{x_m\}$ defined recursively as follows:

$$x_0 = 0, x_1 = 1$$

$$x_{k+2} = 3x_{k+1} - x_k.$$

6. (a) (5 points) Which bases of a Euclidean space V are called orthogonal? orthonormal?

(b) (5 points) Show that the polynomials $f_1 = 1 + t$, $f_2 = 1 + 2t + t^2$, and $f_3 = 2 - t^2$ form a basis of the space of all polynomials in t of degree at most 2.

(c) (7 points) For the space of all polynomials in t of degree at most 2, find the orthogonal basis which is the output of the Gram-Schmidt orthogonalisation applied to the basis from the previous question. (The inner product is the standard one $(f(t), g(t)) = \int_{-1}^{1} f(t)g(t)\,dt$.)

7. (a) (5 points) Write down the definition of a inner product on a real vector space.

(b) (10 points) The function $f_a\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ (depending on a real parameter a) is defined by the formula

$$\begin{aligned} f_a(x_1e_1 + x_2e_2 + x_3e_3, y_1e_1 + y_2e_2 + y_3e_3) = \\ = x_1y_1 + (a-1)(x_1y_2 + x_2y_1) + x_2y_2 + \\ + (a+1)(x_1y_3 + x_3y_1) + 4(x_2y_3 + x_3y_2) + 20x_3y_3 \end{aligned}$$

(here e_1, e_2, e_3 is a basis of $\mathbb{R}^3$). Determine all values of a for which f_a determines an inner product.