UNIVERSITY OF DUBLIN

XMA1132

TRINITY COLLEGE

FACULTY OF SCIENCE

SCHOOL OF MATHEMATICS

JF Mathematics
JF Theoretical Physics
JF Two Subject Mod

A sample Easter paper

COURSE 113

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This sample paper is provided for training purposes only; you are not supposed to hand in your solutions. Please, do attempt all questions by Monday March 2: both lectures on that day will be mostly about discussing this paper.

For each task, the number of points you can get for a complete solution of that task is printed next to it.

You may use all statements proved in class and in home assignments; when using some statement, you should formulate it clearly, e.g. "in class, we proved that if A is invertible, then the reduced row echelon form of A is the identity matrix".

All vector spaces unless otherwise specified are over complex numbers.

Non-programmable calculators are permitted for this examination.

1. (a) (6 points) Under which condition a system of vectors of a vector space V is called linearly independent? Prove that if a system of vectors $\{v_1, \dots, v_k\}$ is linearly independent, then for every $p = 1, \dots, k$ it remains linearly independent after removing the vector v_p from it.

   (b) (8 points) Assume that the system of vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ (all belonging to the same vector space V) is linearly independent. Prove that then the system of vectors $\mathbf{u}' = \mathbf{u} + \mathbf{v}$, $\mathbf{v}' = \mathbf{u} - \mathbf{w}$, $\mathbf{w}' = 2\mathbf{v} + \mathbf{w}$ is also linearly independent. What are possible values of $\dim V$ in this situation? Explain your answer.

2. (a) (5 points) Define the rank of a linear operator.

   (b) (9 points) Consider the vector space V of all 2×2-matrices (with obvious addition and multiplication by scalars). Show that for every 2×2-matrix A the mapping $L_A \colon V \to V$ given by the formula $L_A(X) = AX - XA$, is a linear operator. In the case $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, write down the matrix of L_A relative to the basis $E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and compute $\operatorname{rk}(L_A)$.

   (c) (12 points) Let V and W be vector spaces. Show that for every two linear operators $A, B \colon V \to W$ we have

$$\operatorname{rk}(A + B) \leq \operatorname{rk}(A) + \operatorname{rk}(B).$$

3. Consider the matrices

$$A = \begin{pmatrix} -2 & -4 & 16 \\ 0 & 2 & 0 \\ -1 & -1 & 6 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

   (a) (7 points) Describe all eigenvalues and eigenvectors of A and B.

   (b) (16 points) Describe the Jordan normal form of A and find a Jordan basis for A.

   (c) (8 points) Is A similar to B? Explain your answer.

(d) (9 points) Find a closed formula for A^n.

4. (a) (5 points) Show that if for square matrices A and B it is known that A is similar to B, then A^T is similar to B^T (here X^T, as usual, denotes the transpose matrix of X).

(b) (15 points) Show that (over complex numbers) every square matrix A is similar to A^T.