Solutions to the sample Christmas exam paper

1. (a) A system of vectors $v_1, \ldots, v_k$ is said to be linearly independent if the only linear combination

$$c_1v_1 + c_2v_2 + \ldots + c_kv_k$$

which is equal to zero is the trivial one ($c_1 = \ldots = c_k = 0$). Assume that the system $v_1, \ldots, v_k$ is linearly independent, and let us remove the vector v_p from it. Furthermore, take a linear combination of vectors of the resulting system which is equal to 0. This combination can be thought of as a combination of $v_1, \ldots, v_k$, where the coefficient of v_p is equal to 0. Since our original system was linearly independent, we see that all coefficients should be equal to 0, which is what we need.

(b) Assume that $a\mathbf{u}' + b\mathbf{v}' + c\mathbf{w}' = 0$. Substituting into that $\mathbf{u}' = \mathbf{u} + \mathbf{v}$, $\mathbf{v}' = \mathbf{u} - \mathbf{w}$, $\mathbf{w}' = 2\mathbf{v} + \mathbf{w}$, we get

$$a(\mathbf{u} + \mathbf{v}) + b(\mathbf{u} - \mathbf{w}) + c(2\mathbf{v} + \mathbf{w}) = 0,$$

which can be rewritten as

$$(a + b)\mathbf{u} + (a + 2c)\mathbf{v} + (-b + c)\mathbf{w}) = 0.$$

Since $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent, we have $a + b = 0$, $a + 2c = 0$, $-b + c = 0$. This system of 3 equations with 3 unknowns has only the trivial solution $a = b = c = 0$, which is what we need. Since our vector space contains 3 linearly independent vectors, its dimension is at least 3, and, clearly, can be any number greater than 3 (we can take first three vectors of any basis as an example).

2. (a) For a linear operator A from a vector space V to the vector space W the rank of A, by definition, is equal to the dimension of the image of A.

(b) First of all, L_A is a linear operator: $L_A(X + Y) = A(X + Y) - (X + Y)A = (AX - XA) + (AY - YA) = L_A(X) + L_A(Y)$, $L_A(cX) = A(cX) - (cX)A = c(AX - XA) = cL_A(X)$. Furthermore, for $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, we have

$$L_A(E_{11}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = E_{12} + E_{21},$$

$$L_A(E_{12}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -E_{11} + E_{22},$$

$$L_A(E_{21}) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -E_{11} + E_{22},$$

$$L_A(E_{22}) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -E_{12} - E_{21},$$

so the matrix of L_A relative to our basis is

$$\begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

This matrix, obviously, has two linearly independent columns, so its rank is equal to 2.

(c) We have

$$\mathrm{Im}(A+B) = \{(A+B)(v) \mid v \in V\} \subset \{A(v_1) + B(v_2) \mid v_1, v_2 \in V\} = \\ = \mathrm{Im}(A) + \mathrm{Im}(B).$$

Thus

$$\mathrm{rk}(A+B) = \dim \mathrm{Im}(A+B) \leq \dim(\mathrm{Im}(A) + \mathrm{Im}(B)) \leq \\ \leq \dim(\mathrm{Im}(A)) + \dim(\mathrm{Im}(B)) = \mathrm{rk}(A) + \mathrm{rk}(B).$$

(as we proved in class, $\dim(U_1+U_2) \leq \dim(U_1)+\dim(U_2)$ for every two subspaces U_1, U_2).

3. (a) We have

$$\det(A - tI) = \det \begin{pmatrix} -2-t & -4 & 16 \\ 0 & 2-t & 0 \\ -1 & -1 & 6-t \end{pmatrix} = \\ = (2-t) \det \begin{pmatrix} -2-t & 16 \\ -1 & 6-t \end{pmatrix} = (2-t)(t^2 - 4t + 4) = (2-t)^3$$

(the 3×3-determinant is expanded along the second row). Also, B is triangular, so $\det(B - tI) = (2-t)^3$. We see that all eigenvalues of A and B are equal to 2. Also, solving the system $A \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, we see that it reduces to the equation $4z = x + y$ (we have $A - 2I = \begin{pmatrix} -4 & -4 & 16 \\ 0 & 0 & 0 \\ -1 & -1 & 4 \end{pmatrix}$). Thus, eigenvectors of A are of the form $\begin{pmatrix} 4v-u \\ u \\ v \end{pmatrix}$, where u and v are arbitrary parameters. Also, it is easy to see that all eigenvectors of B are of the form $\begin{pmatrix} s \\ 0 \\ 0 \end{pmatrix}$, where s is a parameter.

(b) We have already seen that $\dim \operatorname{Ker}(A-2I)=2$. Also, $(A-2I)^2=0$, so $\operatorname{Ker}(A-2I)^2=V$. To find a Jordan basis, we first take a nonzero vector in V outside $\operatorname{Ker}(A-2I)$ (the difference of dimensions is $3-2=1$, so the relative basis consists of one vector). For such a vector we can take $f=\begin{pmatrix}1\\0\\0\end{pmatrix}$; we have $(A-2I)f=\begin{pmatrix}-4\\0\\-1\end{pmatrix}$. It remains to extend $(A-2I)f$ to a basis of $\operatorname{Ker}(A-2I)$. The latter subspace is 2-dimensional, so we need just one more vector which is not proportional to $(A-2I)f$. If we put $u=1$, $v=0$ in the parametrisation of $\operatorname{Ker}(A-2I)$, we get the vector $e=\begin{pmatrix}-1 & 1 & 0\end{pmatrix}$ which is definitely not proportional to $(A-2I)f$. Overall, we get a Jordan basis $(A-2I)f, f, e$, and the Jordan normal form of A is $\begin{pmatrix}2&1&0\\0&2&0\\0&0&2\end{pmatrix}$. Indeed,

$$A(A-2I)f=(A-2I+2I)(A-2I)f=((A-2I)^2+2(A-2I))f=2(A-2I)f,$$
$$Af=(A-2I+2I)f=(A-2I)f+2f,$$
$$Ae=(A-2I+2I)e=(A-2I)e+2e=2e.$$

(c) No, A has two linearly independent eigenvectors, and B — only one, so they cannot be matrices of the same linear operator. More generally, their Jordan forms are different, so they are not similar.

(d) We have

$$C^{-1}AC=\begin{pmatrix}2&1&0\\0&2&0\\0&0&2\end{pmatrix}=J,$$

where $C=\begin{pmatrix}-4&1&-1\\0&0&1\\-1&0&0\end{pmatrix}$ is the matrix whose columns are the vectors of the Jordan basis. We have $J^n=\begin{pmatrix}2^n&n2^{n-1}&0\\0&2^n&0\\0&0&2^n\end{pmatrix}$ and $C^{-1}=\begin{pmatrix}0&0&-1\\1&1&-4\\0&1&0\end{pmatrix}$, so

$$A^n=CJ^nC^{-1}=\begin{pmatrix}2^n-n2^{n+1}&-n2^{n+1}&n2^{n+3}\\0&2^n&0\\-n2^{n-1}&-n2^{n-1}&2^n+n2^{n+1}\end{pmatrix}.$$

4. (a) Recall that $(PQ)^T=Q^TP^T$ for every two matrices A and B whose product is defined. Thus, is C is invertible, we see that C^T is invertible, and $(C^T)^{-1}=$

$(C^{-1})^T$. Indeed, $(C^{-1})^T C^T = (CC^{-1})^T = I^T = I$, and $C^T(C^{-1})^T = (C^{-1}C)^T = I^T = I$. Finally, if B is similar to A, that is $B = C^{-1}AC$, then

$$B^T = C^T A^T (C^{-1})T = C^T A^T (C^T)^{-1},$$

which proves that B^T is similar to A^T.

(b) First of all, every square matrix A is similar to a Jordan matrix, so it is enough to show that a Jordan matrix is similar to its transpose. Also, it is enough to show that for a single Jordan block, since if it is true for every block, we can then collect matrices C_i for different blocks into a block diagonal matrix which will be the transition matrix proving the similarity for A and A^T. Finally, for a single block J transposition corresponds to reversing the order of the basis vectors, so J and J^T do define the same operator in two bases which differ by the ordering of vectors; two matrices which define the same operator in different bases are similar.

Another solution: we know that sizes of Jordan blocks are defined by numbers $\mathrm{rk}(A - \lambda I)^k$ for various λ, k. Clearly,

$$\mathrm{rk}(A^T - \lambda I)^k = \mathrm{rk}((A - \lambda I)^T)^k = \mathrm{rk}((A - \lambda I)^k)^T = \mathrm{rk}((A - \lambda I)^k)$$

(the rank does not change when we replace a matrix by its transpose), so A and A^T have the same Jordan form and are similar.