# Jordan normal forms: some examples

From this week's lectures, one sees that for computing the Jordan normal form and a Jordan basis of a linear operator $A$ on a vector space $V$, one can use the following plan:

- Find all eigenvalues of $A$ (that is, compute the characteristic polynomial $\det(A - tI)$ and determine its roots $\lambda_1, \ldots, \lambda_k$).
- For each eigenvalue $\lambda$, form the operator $B_\lambda = A - \lambda I$ and consider the increasing sequence of subspaces
$$\{0\} \subset \operatorname{Ker} B_\lambda \subset \operatorname{Ker} B_\lambda^2 \subset \ldots$$
and determine where it stabilizes, that is find $k$ which is the smallest number such that $\operatorname{Ker} B_\lambda^k = \operatorname{Ker} B_\lambda^{k+1}$. Let $U_\lambda = \operatorname{Ker} B_\lambda^k$. The subspace $U_\lambda$ is an invariant subspace of $B_\lambda$ (and $A$), and $B_\lambda$ is nilpotent on $U_\lambda$, so it is possible to find a basis consisting of several "threads" of the form $f, B_\lambda f, B_\lambda^2 f, \ldots$, where $B_\lambda$ shifts vectors along each thread (as in the previous tutorial).
- Joining all the threads (for different $\lambda$) together (and reversing the order of vectors in each thread!), we get a Jordan basis for $A$. A thread of length $p$ for an eigenvalue $\lambda$ contributes a Jordan block $J_p(\lambda)$ to the Jordan normal form.

*Example 1.* Let $V = \mathbb{R}^3$, and $A = \begin{pmatrix} -2 & 2 & 1 \\ -7 & 4 & 2 \\ 5 & 0 & 0 \end{pmatrix}$.

The characteristic polynomial of $A$ is $-t + 2t^2 - t^3 = -t(1-t)^2$, so the eigenvalues of $A$ are $0$ and $1$.

Furthermore, $\operatorname{rk}(A) = 2$, $\operatorname{rk}(A^2) = 2$, $\operatorname{rk}(A - I) = 2$, $\operatorname{rk}(A - I)^2 = 1$. Thus, the kernels of powers of $A$ stabilise instantly, so we should expect a thread of length 1 for the eigenvalue 0, whereas the kernels of powers of $A - I$ do not stabilise for at least two steps, so that would give a thread of length at least 2, hence a thread of length $2$ because our space is $3$-dimensional.

To determine the basis of $\operatorname{Ker}(A)$, we solve the system $Av = 0$ and obtain a vector $f = \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix}$.

To deal with the eigenvalue $1$, we see that the kernel of $A - I$ is spanned by the vector $\begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$, the kernel of $(A - I)^2 = \begin{pmatrix} 0 & 0 & 0 \\ 10 & -5 & -3 \\ -20 & 10 & 6 \end{pmatrix}$ is spanned by the vectors $\begin{pmatrix} 1/2 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 3/10 \\ 0 \\ 1 \end{pmatrix}$. Reducing the latter vectors using the former one, we end up with the vector $e = \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix}$, which gives rise to a thread $e, (A - I)e = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix}$. Overall, a Jordan basis is given by $f, (A - I)e, e$, and the Jordan normal form has a block of size $2$ with $1$ on the diagonal, and a block of size $1$ with $0$ on the diagonal.

*Example 2.* Let $V = \mathbb{R}^4$, and $A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 11 & 6 & -4 & -4 \\ 22 & 15 & -8 & -9 \\ -3 & -2 & 1 & 2 \end{pmatrix}$.

The characteristic polynomial of $A$ is $1-2t^2+t^4 = (1+t)^2(1-t)^2$, so the eigenvalues of $A$ are $-1$ and $1$.

To avoid unnecessary calculations (similar to avoiding computing $(A-I)^3$ in the previous example), let us compute the ranks for both eigenvalues simultaneously. For $\lambda = -1$ we have $A+I = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 11 & 7 & -4 & -4 \\ 22 & 15 & -7 & -9 \\ -3 & -2 & 1 & 3 \end{pmatrix}$, $\text{rk}(A+I) = 3$, $(A+I)^2 = \begin{pmatrix} 12 & 8 & -4 & -4 \\ 12 & 8 & -4 & -4 \\ 60 & 40 & -20 & -24 \\ -12 & -8 & 4 & 8 \end{pmatrix}$, $\text{rk}((A+I)^2) = 2$. For $\lambda = 1$ we have $A - I = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 11 & 5 & -4 & -4 \\ 22 & 15 & -9 & -9 \\ -3 & -2 & 1 & 1 \end{pmatrix}$, $\text{rk}(A-I) = 3$, $(A-I)^2 = \begin{pmatrix} 12 & 4 & -4 & -4 \\ -32 & -16 & 12 & 12 \\ -28 & -20 & 12 & 12 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $\text{rk}((A-I)^2) = 2$. Thus, each of these eigenvalues gives rise to a thread of length at least $2$, and since our vector space is 4-dimensional, each of the threads should be of length $2$, and in each case the stabilisation happens on the second step.

In the case of the eigenvalue $-1$, we first determine the kernel of $A+I$, solving the system $(A+I)v = 0$; this gives us a vector $\begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$. The equations that determine the kernel of $(A+I)^2$ are $t = 0, 3x + 2y = z$ so $y$ and $z$ are free variables, and for the basis vectors of that kernel we can take $\begin{pmatrix} 1/3 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} -2/3 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Reducing the basis vectors of $\text{Ker}(A+I)^2$ using the basis vector of $\text{Ker}(A+I)$, we end up with a relative basis vector $e = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}$, and a thread $e, (A+I)e = \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix}$.

In the case of the eigenvalue $1$, we first determine the kernel of $A - I$, solving the system $(A-I)v = 0$; this gives us a vector $\begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$. The equations that determine the kernel of $(A-I)^2$ are $4x = z+t, 4y = z+t$ so $z$ and $t$ are free variables, and for the basis vectors of that kernel we can take $\begin{pmatrix} 1/4 \\ 1/4 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1/4 \\ 1/4 \\ 0 \\ 1 \end{pmatrix}$. Reducing the basis vectors of $\text{Ker}(A-I)^2$ using the basis vector of $\text{Ker}(A+I)$, we end up with a relative basis vector

$f = \begin{pmatrix} 1/4 \\ 1/4 \\ 0 \\ 1 \end{pmatrix}$, and a thread $e, (A - I)e = \begin{pmatrix} 0 \\ 0 \\ 1/4 \\ -1/4 \end{pmatrix}$.

Finally, the vectors $(A + I)e, e, (A - I)f, f$ form a Jordan basis for $A$; the Jordan normal form of $A$ is $\begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

*Example 3.* $V$ is arbitrary, all eigenvalues of $V$ are different.

In this case, for every eigenvalue we get at least one thread of length $1$ which altogether is already enough to form a basis. Thus, we recover our old result: the eigenvectors form a Jordan basis, and the Jordan normal form consists of blocks of size $1$, so the corresponding Jordan matrix is not just block-diagonal but really diagonal.

*Example 4.* How to use Jordan normal forms to compute something with matrices? There are two main ideas: (1) to multiply block-diagonal matrices, one can multiply the corresponding blocks, and (2) for a Jordan block $J_p(\lambda)$ we have

$$J_p(\lambda)^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} & \binom{n}{2}\lambda^{n-2} & \dots & \binom{n}{p-1}\lambda^{n-p+1} \\ 0 & \lambda^n & n\lambda^{n-1} & \dots & \binom{n}{p-2}\lambda^{n-p+2} \\ \vdots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & \lambda^n & n\lambda^{n-1} \\ 0 & 0 & 0 & \dots & \lambda^n \end{pmatrix}.$$

In particular, $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}^n = \begin{pmatrix} \lambda^n & n\lambda^{n-1} \\ 0 & \lambda^n \end{pmatrix}$.

For example, to compute the $n^{\text{th}}$ power of the matrix from Example 1 in closed form, we notice that $C^{-1}AC = J$, where $J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ is its Jordan normal form, and $C = \begin{pmatrix} 0 & 1 & 0 \\ -1 & -1 & 3 \\ 2 & 5 & -5 \end{pmatrix}$ is the transition matrix to the Jordan basis (its columns form the Jordan basis). Thus, we have $C^{-1}A^nC = J^n$, and $A^n = CJ^nC^{-1}$. From the above formula, $J^n = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & n \\ 0 & 0 & 1 \end{pmatrix}$, so we get

$$A^n = \begin{pmatrix} -3n + 1 & 2n & n \\ 3n - 10 & -2n + 6 & -n + 3 \\ -15n + 20 & 10n - 10 & 5n - 5 \end{pmatrix}.$$