MA1212: Notes for January 24–25, 2011

Let V be a vector space. Recall that the *linear span* of a set of vectors $v_1, \dots, v_k \in V$ is the set of all linear combinations $c_1v_1 + \dots + c_kv_k$. It is denoted by $\mathrm{span}(v_1, \dots, v_k)$. Vectors $v_1, \dots, v_k$ are linearly independent if and only if they form a basis of their linear span. Our next definition provides a generalization of what is just said, dealing with subspaces, and not vectors.

Definition. Let $V_1, \dots, V_k$ be subspaces of V. Their *sum* $V_1 + \dots + V_k$ is defined as the set of vectors of the form $v_1 + \dots + v_k$, where $v_1 \in V_1, \dots, v_k \in V_k$. The sum of the subspaces $V_1, \dots, V_k$ is said to be *direct* if $0 + \dots + 0$ is the only way to represent $0 \in V_1 + \dots + V_k$ as a sum $v_1 + \dots + v_k$. In this case, it is denoted by $V_1 \oplus \dots \oplus V_k$.

Lemma. *$V_1 + \dots + V_k$ is a subspace of V.*

Proof. It is sufficient to check that $V_1 + \dots + V_k$ is closed under addition and multiplication by numbers. Clearly,

$$(v_1 + \dots + v_k) + (v_1' + \dots + v_k') = ((v_1 + v_1') + \dots + (v_k + v_k'))$$

and

$$c(v_1 + \dots + v_k) = ((cv_1) + \dots + (cv_k)),$$

and the lemma follows, since each V_i is a subspace and hence closed under the vector space operations. □

Examples.

1. Consider the subspaces U_n and U_m of $\mathbb{R}^{n+m}$, the first one being the linear span of the first n standard unit vectors, and the second one being the linear span of the last m standard unit vectors. We have $\mathbb{R}^{n+m} = U_n + U_m = U_n \oplus U_m$.
2. For a collection of nonzero vectors $v_1, \dots, v_k \in V$, consider the subspaces $V_1, \dots, V_k$, where V_i consists of all multiples of v_i. Then, clearly, $V_1 + \dots + V_k = \mathrm{span}(v_1, \dots, v_k)$, and this sum is direct if and only if the vectors v_j are linearly independent.
3. For two subspaces V_1 and V_2, their sum is direct if and only if $V_1 \cap V_2 = \{0\}$. Indeed, if $v_1 + v_2 = 0$ is a nontrivial representation of 0, $v_1 = -v_2$ is in the intersection, and vice versa.

Theorem. *If V_1 and V_2 are subspaces of V, we have*

$$\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2) - \dim(V_1 \cap V_2).$$

In particular, the sum of V_1 and V_2 is direct if and only if $\dim(V_1 + V_2) = \dim(V_1) + \dim(V_2)$.

Proof. Let us pick a basis $e_1, \dots, e_k$ of the intersection $V_1 \cap V_2$, and extend this basis to a bigger set of vectors in two different ways, one way obtaining a basis of V_1, and the other way — a basis of V_2. Let $e_1, \dots, e_k, f_1, \dots, f_l$ and $e_1, \dots, e_k, g_1, \dots, g_m$ be the resulting bases of V_1 and V_2 respectively. Let us prove that

$$e_1, \dots, e_k, f_1, \dots, f_l, g_1, \dots, g_m$$

is a basis of $V_1 + V_2$. It is a complete system of vectors, since every vector in $V_1 + V_2$ is a sum of a vector from V_1 and a vector from V_2, and vectors there can be represented as linear combinations of $e_1, \dots, e_k, f_1, \dots, f_l$ and $e_1, \dots, e_k, g_1, \dots, g_m$ respectively. To prove linear independence, let us assume that

$$a_1e_1 + \dots + a_ke_k + b_1f_1 + \dots + b_lf_l + c_1g_1 + \dots + c_mg_m = 0.$$

Rewriting this formula as $a_1e_1 + \dots + a_ke_k + b_1f_1 + \dots + b_lf_l = -(c_1g_1 + \dots + c_mg_m)$, we notice that on the left we have a vector from V_1 and on the right a vector from V_2, so both the left hand side and the right hand side is a vector from $V_1 \cap V_2$, and so can be represented as a linear combination of $e_1, \dots, e_k$ alone. However, the vectors on the right hand side together with e_i form a basis of V_2, so there is no nontrivial linear combination of these vectors that is equal to a linear combination of e_i. Consequently, all coefficients c_i are equal to zero, so the left hand side is zero. This forces all coefficients a_i and b_i to be equal to zero, since $e_1, \dots, e_k, f_1, \dots, f_l$ is a basis of V_1. This completes the proof of the linear independence of the vectors $e_1, \dots, e_k, f_1, \dots, f_l, g_1, \dots, g_m$.

Summing up, $\dim(V_1) = k + l$, $\dim(V_2) = k + m$, $\dim(V_1 + V_2) = k + l + m$, $\dim(V_1 \cap V_2) = k$, and our theorem follows. □

In practice, it is important sometimes to determine the intersection of two subspaces, each presented as a linear span of several vectors. This question naturally splits into two different questions.

First, it makes sense to find a basis of each of these subspaces. To determine a basis for a linear span of given vectors, there are two ways to go. If we form a matrix whose columns are the given vectors, and find its reduced row echelon form, then the *original* vectors corresponding to columns with leading 1's form a basis of our subspace. Alternatively, we may find the reduced column echelon form (like the reduced row echelon form, but with elementary operations on columns). Nonzero columns of the *result* form a basis of our subspace. In practice, this second method is a little bit better, because it produces basis vectors with "many zero entries".

Once we know a basis $v_1, \ldots, v_k$ for the first subspace, and a basis $w_1, \ldots, w_l$ for the second one, the question reduces to solving the linear system $c_1v_1 + \ldots + c_kv_k = d_1w_1 + \ldots + d_lw_l$. For each solution to this system, the vector $c_1v_1 + \ldots + c_kv_k$ is in the intersection, and vice versa.

Let us introduce another bit of linear algebra vocabulary. It will prove extremely useful for various results we are going to prove in the coming weeks.

Definition. Let V be a vector space, and let U be a subspace of V.

A set of vectors $v_1, \ldots, v_k$ is said to be complete relative to U, if every vector in V can be represented in the form $c_1v_1 + \ldots + c_kv_k + u$, where $u \in U$.

A set of vectors $v_1, \ldots, v_k$ is said to be linearly independent relative to U, if $c_1v_1 + \ldots + c_kv_k = u$ with $u \in U$ implies $c_1 = \ldots = c_k = 0$ (and, consequently, $u = 0$).

A set of vectors $v_1, \ldots, v_k$ is said to form a basis of V relative to U, if it is complete and linearly independent relative to U.

Examples.
1. A system of vectors is complete (linearly independent, forms a basis) in the usual sense if and only if it is complete (linearly independent, forms a basis) relative to the zero subspace $\{0\}$.
2. A system of vectors $v_1, \ldots, v_k$ is complete relative to U if and only if $U + \operatorname{span}(v_1 \ldots, v_k) = V$.
3. A system of vectors $v_1, \ldots, v_k$ is linearly independent relative to U if and only if it is linearly independent in the usual sense, and the sum $U + \operatorname{span}(v_1 \ldots, v_k)$ is direct.
4. A system of vectors forms a basis relative to U if and only if it is linearly independent and $U \oplus \operatorname{span}(v_1 \ldots, v_k) = V$ (in particular, that sum should be direct). Thus, computing a relative basis amounts to finding a "complement" of U in V.
5. If $f_1, \ldots, f_k$ is a basis of U, then picking a basis of V relative to U is the same as extending $f_1, \ldots, f_k$ to a basis of V.

To compute a basis of $\mathbb{R}^n$ relative to the linear span of several vectors, one may compute the reduced column echelon form for the corresponding matrix (like when looking for a basis, as before), and pick the standard unit vectors corresponding to "missing leading 1's", that is to the rows of the reduced column echelon form which do not have leading 1's in them.

More generally, if we are required to determine a basis of a vector space V relative to its subspace U, we can proceed as follows. Let A be a matrix whose columns form a basis of U, C — a matrix whose columns form a basis of V. We can find the reduced column echelon form B for A. Write C next to B, and "reduce" it using the leading 1's of B; making sure that all rows that contain leading 1's of B do not contain any other nonzero elements. Then it remains to find the reduced column echelon form of the resulting matrix C'. Its nonzerocolumns form a relative basis.

Examples.
1. Assume that we want to find a basis of $\mathbb{R}^4$ relative to the linear span of the vectors $u_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$. The reduced column echelon form of the matrix whose columns are these vectors is $\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \\ 0 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}$, so the missing leading 1's correspond to the second and the fourth row,

and the vectors $\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ form a relative basis.

2. Furthermore, assume that we want to find a basis of the linear span of the vectors $v_1 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix}$, $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$ relative to the linear span of the vectors $u_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $u_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$ (note that $\operatorname{span}(u_1, u_2)$ is a subspace of $\operatorname{span}(v_1, v_2, v_3)$ because $u_1 = -v_1 - v_2$, $u_2 = -v_1 - v_3$). The reduced column echelon form of the matrix whose columns are u_1 and u_2 is $\begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & 0 \\ 0 & 1 \\ -\frac{1}{2} & -1 \end{pmatrix}$. Now we adjoin the columns equal to v_1, v_2, and v_3, obtaining the matrix $\begin{pmatrix} 1 & 0 & 1 & 1 & 1 \\ -\frac{1}{2} & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ -\frac{1}{2} & -1 & 0 & 0 & -1 \end{pmatrix}$. Reducing its three last columns using the first two columns gives the matrix $\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{2} & -1 & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$. Removing the part corresponding to the $\operatorname{span}(u_1, u_2)$ leaves us with the matrix $\begin{pmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$ whose reduced column echelon form is $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$, so the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$ forms a relative basis.