

## The notion of linearity, on the example of dot and cross products

The goal of this note is to explain how one of the key notions of linear algebra — that of *linearity* — can be used to replace dealing with *infinite* data by only handling a *finite* number of cases. Recall that for dot and cross products of 3d vectors we have the following properties satisfied:

$$\begin{aligned}(\mathbf{u}_1 + \mathbf{u}_2) \cdot \mathbf{v} &= \mathbf{u}_1 \cdot \mathbf{v} + \mathbf{u}_2 \cdot \mathbf{v}, \\ \mathbf{u} \cdot (\mathbf{v}_1 + \mathbf{v}_2) &= \mathbf{u} \cdot \mathbf{v}_1 + \mathbf{u} \cdot \mathbf{v}_2, \\ (c\mathbf{u}) \cdot \mathbf{v} &= \mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v}), \\ (\mathbf{u}_1 + \mathbf{u}_2) \times \mathbf{v} &= \mathbf{u}_1 \times \mathbf{v} + \mathbf{u}_2 \times \mathbf{v}, \\ \mathbf{u} \times (\mathbf{v}_1 + \mathbf{v}_2) &= \mathbf{u} \times \mathbf{v}_1 + \mathbf{u} \times \mathbf{v}_2, \\ (c\mathbf{u}) \times \mathbf{v} &= \mathbf{u} \times (c\mathbf{v}) = c(\mathbf{u} \times \mathbf{v}).\end{aligned}$$

In other words, if we have a (dot or cross) product of two vectors, and we replace one of the vectors by a sum of two vectors, then the result is the same as when you first compute the product with each of the two summands and then add the results, and the same holds for re-scaling: if in a product you multiply one of the vectors by some number, the result is the same as when you compute the product and then multiply the result by that number.

In general, a function  $f$  of several vectors  $\mathbf{u}, \mathbf{v}, \dots$  is said to be *linear in all its arguments* (or *multilinear*) if

$$\begin{aligned}f(\mathbf{u}_1 + \mathbf{u}_2, \mathbf{v}, \dots) &= f(\mathbf{u}_1, \mathbf{v}, \dots) + f(\mathbf{u}_2, \mathbf{v}, \dots), \\ f(c\mathbf{u}, \mathbf{v}, \dots) &= cf(\mathbf{u}, \mathbf{v}, \dots), \\ f(\mathbf{u}, \mathbf{v}_1 + \mathbf{v}_2, \dots) &= f(\mathbf{u}, \mathbf{v}_1, \dots) + f(\mathbf{u}, \mathbf{v}_2, \dots), \\ f(\mathbf{u}, c\mathbf{v}, \dots) &= cf(\mathbf{u}, \mathbf{v}, \dots), \\ &\dots\end{aligned}$$

Using dot and cross products, it is easy to build many multilinear functions; for example,

$$\mathbf{u}, \mathbf{v}, \mathbf{w} \mapsto \mathbf{u} \times (\mathbf{v} \times \mathbf{w})$$

is a multilinear function, which immediately follows from the linearity of cross products.

**Claim.** Every multilinear function  $f(\mathbf{u}, \mathbf{v}, \dots)$  is determined by its values assumed when each of the vectors  $\mathbf{u}, \mathbf{v}, \dots$  is one of the standard unit vectors  $\mathbf{i}, \mathbf{j}, \mathbf{k}$ .

*Proof.* We have

$$\begin{aligned}f(\mathbf{u}, \mathbf{v}, \dots) &= f(u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}, \mathbf{v}, \dots) = u_1f(\mathbf{i}, \mathbf{v}, \dots) + u_2f(\mathbf{j}, \mathbf{v}, \dots) + u_3f(\mathbf{k}, \mathbf{v}, \dots) = \\ &= u_1f(\mathbf{i}, v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}, \dots) + u_2f(\mathbf{j}, v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}, \dots) + u_3f(\mathbf{k}, v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}, \dots) = \\ &= u_1v_1f(\mathbf{i}, \mathbf{i}, \dots) + u_1v_2f(\mathbf{i}, \mathbf{j}, \dots) + \dots\end{aligned}$$

which makes the statement very clear.

Now, to prove statements like

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w},$$

it is enough to consider the function

$$f(\mathbf{u}, \mathbf{v}, \mathbf{w}) = \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) - (\mathbf{u} \cdot \mathbf{w})\mathbf{v} + (\mathbf{u} \cdot \mathbf{v})\mathbf{w},$$

and check that it assumes the value  $\mathbf{0}$  whenever each of the vectors  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  is one of the standard unit vectors. After that, the claim above guarantees that this function is identically zero.