1111: Linear Algebra I

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Lecture 17

Linear independence, span, basis

By definition of a vector space, we can form arbitary linear combinations: if $v_1, \dots, v_k$ are vectors and $c_1, \dots, c_k$ are scalars, then $c_1v_1 + \dots + c_kv_k$ is a vector which is called the linear combination of $v_1, \dots, v_k$ with coefficients $c_1, \dots, c_k$.

All the definitions that we gave in the case of $\mathbb{R}^n$ proceed in the same way. Below we assume that V is a vector space over real numbers (but one can use any other field if necessary).

Definition 1. A system of vectors $v_1, \dots, v_k \in V$ is said to be linearly independent if the only linear combination of these vectors that is equal to zero is the combination where all the coefficients are equal to zero.

Note that the property stating that if $c \cdot v = 0$ then $c = 0$ or $v = 0$ can be rephrased as follows: one non-zero vector is always linearly independent.

Definition 2. A system of vectors $v_1, \dots, v_k \in V$ is said to be *complete*, or to *span* V, if every vector in V is equal to a linear combination of those vectors.

Definition 3. A system of vectors $v_1, \dots, v_k \in V$ is said to form a *basis* of V, if it is linearly independent and spans V.

Remark 1. In case a system of vectors is infinite, the same definitions apply, but we always use *finite* linear combinations: a system is linearly independent if no non-trivial finite linear combination is zero, a system is complete if every vector can be represented as their finite linear combination.

Example 1. The spanning set that we constructed for the solution set of an arbitrary system of linear equations was, as we remarked, linearly independent, so in fact it provided a basis of that vector space.

Example 2. The *monomials* x^k, $k \geqslant 0$, form a basis in the space of polynomials in one variable. Note that this basis is infinite, but we nevertheless only consider finite linear combinations at all stages.

Dimension

Note that in $\mathbb{R}^n$ we proved that a linearly independent system of vectors consists of at most n vectors, and a complete system of vectors consists of at least n vectors. In a general vector space V, there is no *a priori* n that can play this role. Moreover, the previous example shows that sometimes, no n bounding the size of a linearly independent system of vectors may exist. It however is possible to prove a version of those statements which is valid in every vector space.

Theorem 1. *Let V be a vector space, and suppose that $e_1, \dots, e_k$ is a linearly independent system of vectors and that $f_1, \dots, f_m$ is a complete system of vectors. Then $k \leqslant m$.*

Proof. Assume the contrary; without loss of generality, $k > m$. Since $f_1, \ldots, f_m$ is a complete system, we can find coefficients a_{ij} for which

$$\begin{aligned} e_1 &= a_{11}f_1 + a_{21}f_2 + \cdots + a_{m1}f_m, \\ e_2 &= a_{12}f_1 + a_{22}f_2 + \cdots + a_{m2}f_m, \\ &\ldots \\ e_k &= a_{1k}f_1 + a_{2k}f_2 + \cdots + a_{mk}f_m. \end{aligned}$$

Let us look for linear combinations $c_1e_1 + \cdots + c_kv_k$ that are equal to zero (since these vectors are assumed linearly independent, we should not find any nontrivial ones). Such a combination, once we substitute the expressions above, becomes

$$c_1(a_{11}f_1+a_{21}f_2+\cdots+a_{m1}f_m)+c_2(a_{12}f_1+a_{22}f_2+\cdots+a_{m2}f_m)+\ldots+c_k(a_{1k}f_1+a_{2k}f_2+\cdots+a_{mk}f_m) = \\ = (a_{11}c_1 + a_{12}c_2 + \cdots + a_{1k}c_k)f_1 + \cdots + (a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mk}c_k)f_m.$$

This means that if we ensure

$$\begin{aligned} a_{11}c_1 + a_{12}c_2 + \cdots + a_{1k}c_k &= 0, \\ &\ldots \\ a_{m1}c_1 + a_{m2}c_2 + \cdots + a_{mk}c_k &= 0, \end{aligned}$$

then this linear combination is automatically zero. But since we assume $k > m$, this system of linear equations has a nontrivial solution $c_1, \ldots, c_k$, so the vectors $e_1, \ldots, e_k$ are linearly dependent, a contradiction. □

This result leads, indirectly, to an important new notion.

Definition 4. We say that a vector space V is *finite-dimensional* if it has a basis consisting of finitely many vectors. Otherwise we say that V is *infinite-dimensional*.

Example 3. Clearly, $\mathbb{R}^n$ is finite-dimensional. The space of all polynomials is infinite-dimensional: finitely many polynomials can only produce polynomials of bounded degree as linear combinations.

Exercise. Let V be a finite-dimensional vector space. Then every basis of V consists of the same finite number of vectors.
Solution. Indeed, having a basis consisting of n elements implies, in particularly, having a complete system of n vectors, so by our theorem, it is impossible to have a linearly independent system of more than n vectors. Thus, every basis has finitely many elements, and for two bases $e_1, \ldots, e_k$ and $f_1, \ldots, f_m$ we have $k \leqslant m$ and $m \leqslant k$, so $m = k$.

Definition 5. For a finite-dimensjonal vector V, the number of vectors in a basis of V is called the *dimension* of V, and is denoted by $\dim(V)$.

Coordinates

Let V be a finite-dimensional vector space, and let $e_1, \ldots, e_n$ be a basis of V.

Definition 6. For a vector $v \in V$, the scalars $c_1, \ldots, c_n$ for which

$$v = c_1e_1 + c_2e_2 + \cdots + c_ne_n$$

are called the *coordinates of* v *with respect to the basis* $e_1, \ldots, e_n$.

Lemma 1. *The above definition makes sense: each vector has (unique) coordinates.*

Proof. Existence follows from the spanning property of a basis, uniqueness — from linear independence. □