Last time: hints about importance ranking of webpages.

(1) Very naive: a page is important if there are many links to it

(2) Less naive: a page is important if there are many links to it from important pages

Pages $1, 2, \ldots, k$
importance ranking vector $\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}$

$$x_i = \sum \frac{1}{n_j} x_j$$

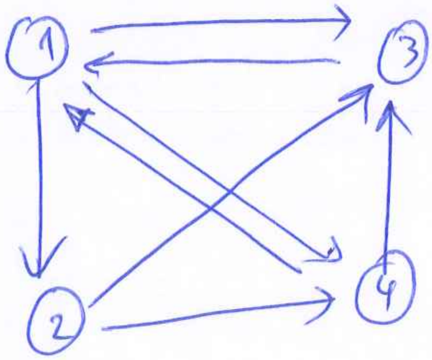
all j such that there is a link from page j to page i,
n_j = ~~total~~ number of links from page j.

Informally, each page j has one vote. it gives $\frac{1}{n_j}$ of it to each page it links to.

Another view: let x_p be the probability to be on page p.
Then $\sum_{j \to i} \frac{1}{n_j} x_j$ is the probability of ending on page i in one click!

And our equation just describes the stable probability distribution after many clicks.

Example.



Naïvely : 1 has two backlinks
2 has one backlink
3 has 3 backlinks
4 has two backlinks.

Our "less naive" approach:

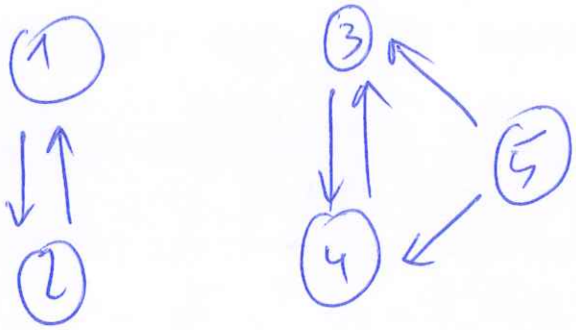
$$A = \begin{pmatrix} 0 & 0 & 1 & 1/2 \\ \frac{1}{3} & 0 & 0 & 0 \\ 1/3 & \frac{1}{2} & 0 & 1/2 \\ \frac{1}{3} & 1/2 & 0 & 0 \end{pmatrix}$$

because 4 links to 1 and 3.

for eigenvalue 1,
every eigenvector proportional to $\begin{pmatrix} 12 \\ 4 \\ 9 \\ 6 \end{pmatrix}$

normalizing, get $\begin{pmatrix} \frac{12}{31} \\ \frac{4}{31} \\ \frac{9}{31} \\ \frac{6}{31} \end{pmatrix} \simeq \begin{pmatrix} 0.387 \\ 0.129 \\ 0.290 \\ 0.194 \end{pmatrix}$ so page 1 is the most important

Example



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$\begin{pmatrix} 1/2 \\ 1/2 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 0 \\ 1/2 \\ 1/2 \\ 0 \end{pmatrix}$ are eigen-vectors

non-uniqueness $\leftrightarrow$ disconnectedness of the network

Modification

$$M = (1-p)A + p \boxed{\frac{1}{K}\begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & & 1 \\ & & & & \\ 1 & 1 & \cdots & & 1 \end{pmatrix}} \qquad 0<p<1$$

with probability p incorporate chaos (move to another page randomly)

Usually, $p \sim 0.15$

M has positive matrix elements

M is column-stochastic: numbers in each column add up to 1.

Some theory.

1. Always have eigenvalue 1.

$\{\text{Eigenvalues of } A\} = \{\text{Eigenvalues of } A^T\}$

$$\det(A - cI_n) = \det(A^T - cI_n)$$

And A^T has eigenvalue 1 because $\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$ is an eigenvector.

2. if all entries of A are positive, then every eigenvector with eigenvalue 1 has all coordinates of the same sign. Otherwise

$$|x_i| = \left|\sum_j A_{ij}x_j\right| < \sum_j A_{ij}|x_j|$$

$$\sum_i |x_i| = \sum_i \left|\sum_j A_{ij}x_j\right| < \sum_i \sum_j A_{ij}|x_j| = \boxed{\sum_i |x_i|}.$$

contradiction

3. Up to proportionality, just one eigenvector for eigenvalue 1.

(Idea: If there are two linearly independent eigenvectors for eigenvalue 1, then can find a combination of these that has both positive and negative coefficients, contradicting 2.)